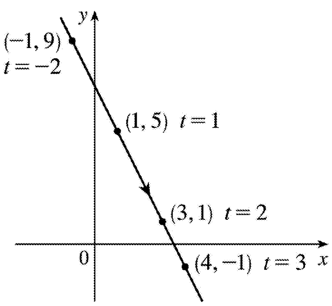
1. $x=1+t$, $y=5-2t$, $-2\leq t\leq 3$

(a)

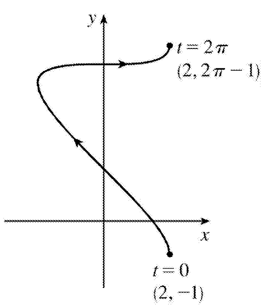
t	-2	-1	0	1	2	3
x	-1	0	1	2	3	4
y	9	7	5	3	1	-1

(b) $x=1+t\Rightarrow t=x-1\Rightarrow y=5-2(x-1)$, so $y=-2x+7$, $-1\leq x\leq 4$.



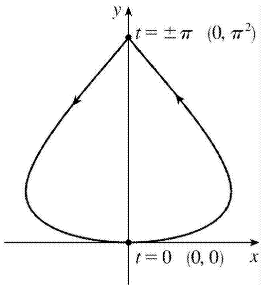
2. $x=2\cos t$, $y=t-\cos t$, $0\leq t\leq 2\pi$

t	0	$\pi/2$	π	$3\pi/2$	2π
x	2	0	-2	0	2
y	-1	$\pi/2$ 1.57	$\pi+1$ 4.14	$3\pi/2$ 4.71	$2\pi-1$ 5.28



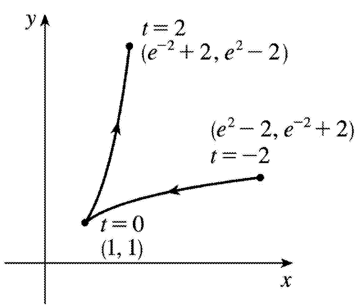
3. $x=5\sin t$, $y=t^2$, $-\pi\leq t\leq\pi$

t	$-\pi$	$-\pi/2$	0	$\pi/2$	π
x	0	-5	0	5	0
y	π^2 9.87	$\pi^2/4$ 2.47	0	$\pi^2/4$ 2.47	π^2 9.87



4. $x=e^{-t}+t$, $y=e^{t}-t$, $-2\leq t\leq 2$

t	-2	-1	0	1	2
x	$e^{2}-2$ 5.39	$e-1$ 1.72	1	$e^{-1}+1$ 1.37	$e^{-2}+2$ 2.14
y	$e^{-2}+2$ 2.14	$e^{-1}+1$ 1.37	1	$e-1$ 1.72	$e^{2}-2$ 5.39

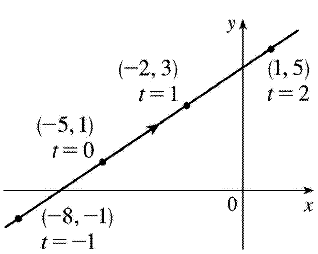


5. $x=3t-5$, $y=2t+1$

(a)

t	-2	-1	0	1	2	3	4
x	-11	-8	-5	-2	1	4	7
y	-3	-1	1	3	5	7	9

(b) $x=3t-5\Rightarrow 3t=x+5\Rightarrow t=\frac{1}{3}(x+5)\Rightarrow y=2\cdot\frac{1}{3}(x+5)+1$, so $y=\frac{2}{3}x+\frac{13}{3}$.

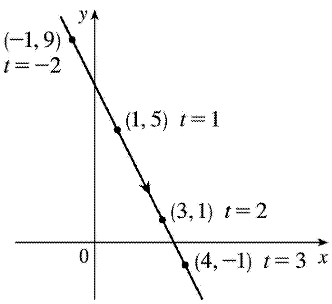


6. $x=1+t$, $y=5-2t$, $-2\leq t\leq 3$

(a)

t	–2	–1	0	1	2	3
x	–1	0	1	2	3	4
y	9	7	5	3	1	–1

(b) $x=1+t \Rightarrow t=x-1 \Rightarrow y=5-2(x-1)$, so $y=-2x+7$, $-1 \leq x \leq 4$.

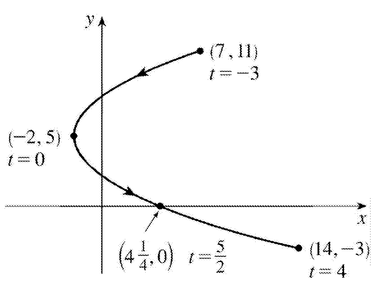


7. $x=t^2-2$, $y=5-2t$, $-3 \leq t \leq 4$

(a)

t	–3	–2	–1	0	1	2	3	4
x	7	2	–1	–2	–1	2	7	14
y	11	9	7	5	3	1	–1	–3

(b) $y=5-2t \Rightarrow 2t=5-y \Rightarrow t=\frac{1}{2}(5-y) \Rightarrow x=\left[\frac{1}{2}(5-y)\right]^2-2$, so $x=\frac{1}{4}(5-y)^2-2$, $-3 \leq y \leq 11$.

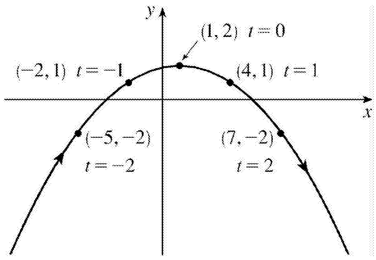


8. $x=1+3t$, $y=2-t^2$

(a)

t	–3	–2	–1	0	1	2	3
x	–8	–5	–2	1	4	7	10
y	–7	–2	1	2	1	–2	–7

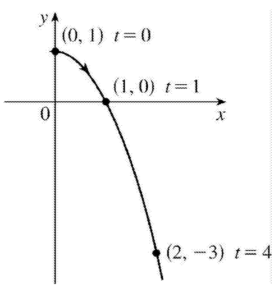
(b) $x=1+3t \Rightarrow t=\frac{1}{3}(x-1) \Rightarrow y=2-\left[\frac{1}{3}(x-1)\right]^2$, so $y=-\frac{1}{9}(x-1)^2+2$.



9. **(a)** $x=\sqrt{t}$, $y=1-t$

t	0	1	2	3	4
x	0	1	1.414	1.732	2
y	1	0	–1	–2	–3

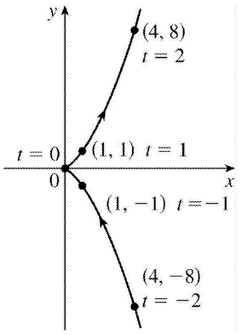
(b) $x=\sqrt{t}$, $\Rightarrow t=x^2 \Rightarrow y=1-t=1-x^2$.
Since $t\geq 0$, $x\geq 0$



10. **(a)** $x=t^2$, $y=t^3$

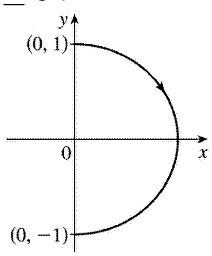
t	–2	–1	0	1	2
x	4	1	0	1	4
y	–8	–1	0	1	8

(b) $y=t^3 \Rightarrow t=\sqrt[3]{y} \Rightarrow x=t^2=\left(\sqrt[3]{y}\right)^2=y^{2/3}$.
$t\in R$, $y\in R$, $x\geq 0$.



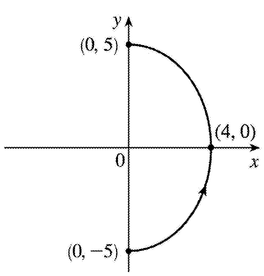
11. **(a)** $x = \sin\theta$, $y = \cos\theta$, $0 \le \theta \le \pi$. $x^2 + y^2 = \sin^2\theta + \cos^2\theta = 1$. Since $0 \le \theta \le \pi$, we have $\sin\theta \ge 0$, so $x \ge 0$.

(b)



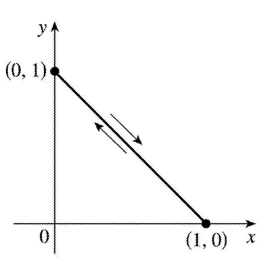
12. **(a)** $x = 4\cos\theta$, $y = 5\sin\theta$, $-\pi/2 \le \theta \le \pi/2$.
$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = \cos^2\theta + \sin^2\theta = 1$, which is an ellipse with x –intercepts $(\pm 4, 0)$ and y –intercepts $(0, \pm 5)$. We obtain the portion of the ellipse with $x \ge 0$ since $4\cos\theta \ge 0$ for $-\pi/2 \le \theta \le \pi/2$.

(b)



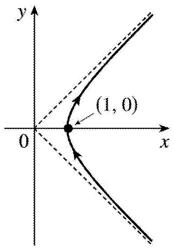
13. **(a)** $x = \sin^2\theta$, $y = \cos^2\theta$. $x + y = \sin^2\theta + \cos^2\theta = 1$, $0 \le x \le 1$. Note that the curve is at $(0,1)$ whenever $\theta = \pi n$ and is at $(1,0)$ whenever $\theta = \frac{\pi}{2} n$ for every integer n .

(b)



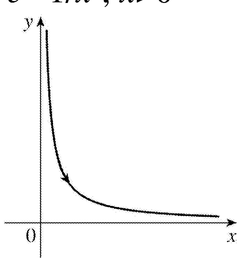
14. **(a)** $x=\sec\theta$, $y=\tan\theta$, $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$. $x^2-y^2=\sec^2\theta-\tan^2\theta=1$, $x\geq 1$,

or $x=\sqrt{y^2+1}$.

(b) 

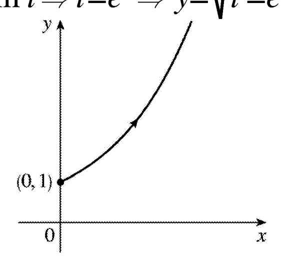
15. **(a)** $x=e^t$, $y=e^{-t}$.

$y=1/e^t=1/x$, $x>0$

(b) 

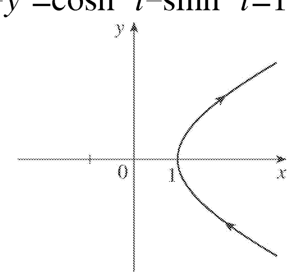
16. **(a)** $x=\ln t$, $y=\sqrt{t}$, $t\geq 1$.

$x=\ln t\Rightarrow t=e^x\Rightarrow y=\sqrt{t}=e^{x/2}$, $x\geq 0$.

(b) 

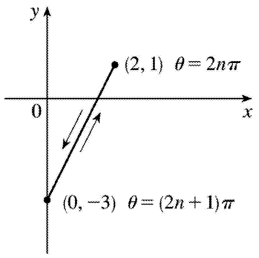
17. **(a)** $x=\cosh t$, $y=\sinh$ t,

$x^2-y^2=\cosh^2 t-\sinh^2 t=1$, $x\geq 1$

(b) 

18. **(a)** $x=1+\cos\theta \Rightarrow \cos\theta = x-1$.
$y=2\cos\theta - 1 = 2(x-1)-1 = 2x-3$, $0\le x\le 2$.

(b)



19. $x^2+y^2=\cos^2\pi t+\sin^2\pi t=1$, $1\le t\le 2$, so the particle moves counterclockwise along the circle $x^2+y^2=1$ from $(-1,0)$ to $(1,0)$, along the lower half of the circle.

20. $(x-2)^2+(y-3)^2=\cos^2 t+\sin^2 t=1$, so the motion takes place on a unit circle centered at $(2,3)$. As t goes from 0 to 2π , the particle makes one complete counterclockwise rotation around the circle, starting and ending at $(3,3)$.

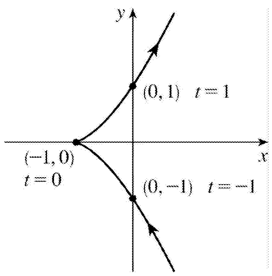
21. $\left(\frac{1}{2}x\right)^2+\left(\frac{1}{3}y\right)^2=\sin^2 t+\cos^2 t=1$, so the particle moves once clockwise along the ellipse $\frac{1}{4}x^2+\frac{1}{9}y^2=1$, starting and ending at $(0,3)$.

22. $x=\cos^2 t=y^2$, so the particle moves along the parabola $x=y^2$. As t goes from 0 to 4π , the particle moves from $(1,1)$ down to $(1,-1)$ (at $t=\pi$), back up to $(1,1)$ again (at $t=2\pi$), and then repeats this entire cycle between $t=2\pi$ and $t=4\pi$.

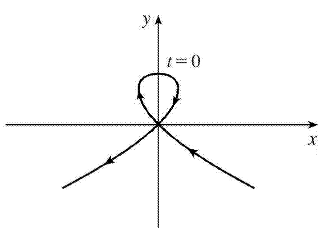
23. We must have $1\le x\le 4$ and $2\le y\le 3$. So the graph of the curve must be contained in the rectangle [1,4] by [2,3] .

24. **(a)** From the first graph, we have $1\le x\le 2$. From the second graph, we have $-1\le y\le 1$. The only choice that satisfies either of those conditions is III.
(b) From the first graph, the values of x cycle through the values from -2 to 2 four times. From the second graph, the values of y cycle through the values from -2 to 2 six times. Choice I satisfies these conditions.
(c) From the first graph, the values of x cycle through the values from -2 to 2 three times. From the second graph, we have $0\le y\le 2$. Choice IV satisfies these conditions.
(d) From the first graph, the values of x cycle through the values from -2 to 2 two times. From the second graph, the values of y do the same thing. Choice II satisfies these conditions.

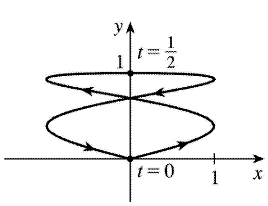
25. When $t = -1$, $(x,y)=(0,-1)$. As t increases to 0 , x decreases to -1 and y increases to 0 . As t increases from 0 to 1 , x increases to 0 and y increases to 1 . As t increases beyond 1 , both x and y increase. For $t<-1$, x is positive and decreasing and y is negative and increasing. We could achieve greater accuracy by estimating x – and y –values for selected values of t from the given graphs and plotting the corresponding points.



26. For $t<-1$, x is positive and decreasing, while y is negative and increasing (these points are in Quadrant IV). When $t=-1$, $(x,y)=(0,0)$ and, as t increases from -1 to 0 , x becomes negative and y increases from 0 to 1 .
At $t=0$, $(x,y)=(0,1)$ and, as t increases from 0 to 1 , y decreases from 1 to 0 and x is positive. At $t=1,(x,y)=(0,0)$ again, so the loop is completed. For $t>1$, x and y both become large negative. This enables us to draw a rough sketch. We could achieve greater accuracy by estimating x – and y –values for selected values of t from the given graphs and plotting the corresponding points.



27. When $t=0$ we see that $x=0$ and $y=0$, so the curve starts at the origin. As t increases from 0 to $\frac{1}{2}$, the graphs show that y increases from 0 to 1 while x increases from 0 to 1 , decreases to 0 and to -1 , then increases back to 0 , so we arrive at the point $(0,1)$. Similarly, as t increases from $\frac{1}{2}$ to 1 , y decreases from 1 to 0 while x repeats its

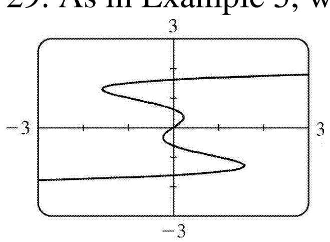


pattern, and we arrive back at the origin. We could achieve greater accuracy by estimating x – and y –

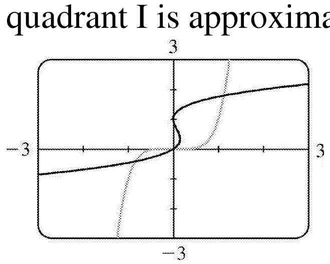
values for selected values of t from the given graphs and plotting the corresponding points.

28. **(a)** Note that as $t\to-\infty$, we have $x\to-\infty$ and $y\to\infty$, whereas when $t\to\infty$, both x and $y\to\infty$. This description fits only IV.
(b) Note that as $t\to\pm\infty$, $y\to-\infty$. This is only the case with VI.
(c) If $t=0$, then $(x,y)=(\sin 0,\sin 0)=(0,0)$. Also, $|x|=|\sin 3t|\le 1$ for all t , and $|y|=|\sin 4t|\le 1$ for all t . The only graph which includes the point $(0,0)$ and which has $|x|\le 1$ and $|y|\le 1$, is V.
(d) Note that as $t\to-\infty$, both x and $y\to-\infty$, and as $t\to\infty$, both x and $y\to\infty$. This description fits only III. (Also note that, since $\sin 2t$ and $\sin 3t$ lie between -1 and 1 , the curve never strays very far from the line $y=x$.)
(e) Note that both $x(t)$ and $y(t)$ are periodic with period 2π and satisfy $|x|\le 1$ and $|y|\le 1$. Now the only $y-$ intercepts occur when $x=\sin(t+\sin t)=0\Leftrightarrow t=0$ or π . So there should be two $y-$ intercepts: $y(0)=\cos 1\approx 0.54$ and $y(\pi)=\cos(\pi-1)\approx-0.54$. Similarly, there should be two $x-$ intercepts: $x\left(\frac{\pi}{2}\right)=\sin\left(\frac{\pi}{2}+1\right)\approx 0.54$ and $x\left(\frac{3\pi}{2}\right)=\sin\left(\frac{3\pi}{2}-1\right)\approx-0.54$. The only curve with these $x-$ and $y-$ intercepts is I.
(f) Note that $x(t)$ is periodic with period 2π , so the only $y-$ intercepts occur when $x=\cos t=0\Leftrightarrow$ $t=\frac{\pi}{2}$ or $\frac{3\pi}{2}$. Also, the graph is symmetric about the $x-$ axis, since $y(-t)=\sin(-t+\sin 5(-t))=\sin(-t-\sin 5t)=-\sin(t+\sin 5t)=-y(t)$, and $x(-t)=\cos(-t)=\cos t=x(t)$. The only graph which has only two $y-$ intercepts, and is symmetric about the $x-$ axis, is II.

29. As in Example 5, we let $y=t$ and $x=t-3t^3+t^5$ and use a t –interval of $[-2\pi,2\pi]$.



30. We use $x_1=t$, $y_1=t^5$ and $x_2=t(t-1)^2$, $y_2=t$ with $-2\pi\le t\le 2\pi$. There are 3 points of intersection; $(0,0)$ is fairly obvious. The point in quadrant III is approximately $(-0.8,-0.4)$ and the point in quadrant I is approximately $(1.1,1.8)$.



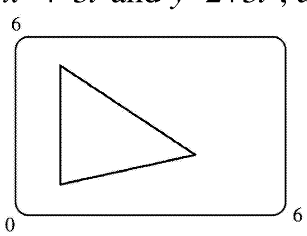
31. **(a)** $x=x_1+(x_2-x_1)t$, $y=y_1+(y_2-y_1)t$, $0\le t\le 1$. Clearly the curve passes through $P_1\left(x_1,y_1\right)$ when

$t=0$ and through $P_2(x_2,y_2)$ when $t=1$. For $0<t<1$, x is strictly between x_1 and x_2 and y is strictly between y_1 and y_2. For every value of t, x and y satisfy the relation $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$, which is the equation of the line through $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$.

Finally, any point (x,y) on that line satisfies $\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$; if we call that common value t, then the given parametric equations yield the point (x,y); and any (x,y) on the line between $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ yields a value of t in $[0,1]$. So the given parametric equations exactly specify the line segment from $P_1(x_1,y_1)$ to $P_2(x_2,y_2)$.

(b) $x=-2+[3-(-2)]t=-2+5t$ and $y=7+(-1-7)t=7-8t$ for $0\leq t\leq 1$.

32. For the side of the triangle from A to B, use $(x_1,y_1)=(1,1)$ and $(x_2,y_2)=(4,2)$. Hence, the equations are $x=x_1+(x_2-x_1)t=1+(4-1)t=1+3t$, $y=y_1+(y_2-y_1)t=1+(2-1)t=1+t$. Graphing $x=1+3t$ and $y=1+t$ with $0\leq t\leq 1$ gives us the side of the triangle from A to B. Similarly, for the side BC we use $x=4-3t$ and $y=2+3t$, and for the side AC we use $x=1$ and $y=1+4t$.



33. The circle $x^2+y^2=4$ can be represented parametrically by $x=2\cos t$, $y=2\sin t$; $0\leq t\leq 2\pi$. The circle $x^2+(y-1)^2=4$ can be represented by $x=2\cos t$, $y=1+2\sin t$; $0\leq t\leq 2\pi$. This representation gives us the circle with a counterclockwise orientation starting at $(2,1)$.

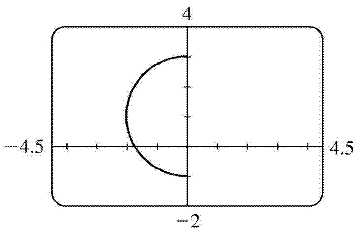
(a) To get a clockwise orientation, we could change the equations to $x=2\cos t$, $y=1-2\sin t$, $0\leq t\leq 2\pi$.

(b) To get three times around in the counterclockwise direction, we use the original equations $x=2\cos t$, $y=1+2\sin t$ with the domain expanded to $0\leq t\leq 6\pi$.

(c) To start at $(0,3)$ using the original equations, we must have $x_1=0$; that is, $2\cos t=0$. Hence, $t=\frac{\pi}{2}$. So we use $x=2\cos t$, $y=1+2\sin t$; $\frac{\pi}{2}\leq t\leq \frac{3\pi}{2}$.

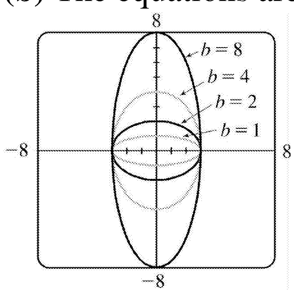
Alternatively, if we want t to start at 0, we could change the equations of the curve. For example, we could use $x=-2\sin t$, $y=1+2\cos t$, $0\leq t\leq \pi$.

34.



35. **(a)** Let $x^2/a^2 = \sin^2 t$ and $y^2/b^2 = \cos^2 t$ to obtain $x = a\sin t$ and $y = b\cos t$ with $0 \le t \le 2\pi$ as possible parametric equations for the ellipse $x^2/a^2 + y^2/b^2 = 1$.

(b) The equations are $x = 3\sin t$ and $y = b\cos t$ for $b \in \{1,2,4,8\}$.



(c) As b increases, the ellipse stretches vertically.

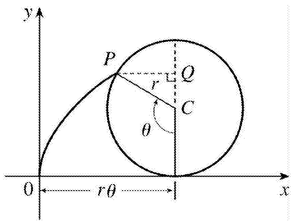
36. The possible parametrizations of the curve $y = x^3$ include

(1) $x = t$, $y = t^3$, $t \in R$

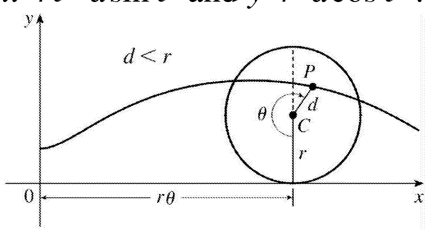
(2) $x = -t$, $y = -t^3$, $t \in R$

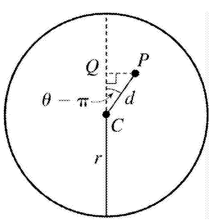
(3) $x = t+1$, $y = (t+1)^3$, $t \in R$

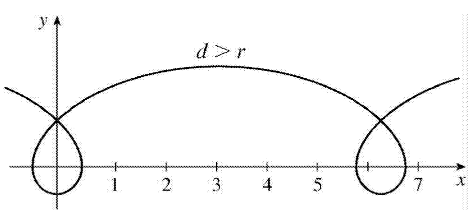
37. The case $\frac{\pi}{2} < \theta < \pi$ is illustrated. C has coordinates $(r\theta, r)$ as in Example 6, and Q has coordinates $(r\theta, r + r\cos(\pi - \theta)) = (r\theta, r(1 - \cos\theta))$, so P has coordinates $(r\theta - r\sin(\pi - \theta), r(1 - \cos\theta)) = (r(\theta - \sin\theta), r(1 - \cos\theta))$. Again we have the parametric equations $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.



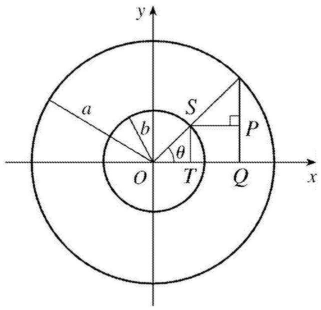
38. The first two diagrams depict the case $\pi<\theta<\frac{3\pi}{2}$, $d<r$. As in Example 6, C has coordinates $(r\theta,\mathrm{r})$.
Now Q (in the second diagram) has coordinates $(r\theta,\mathrm{r}+d\cos(\theta-\pi))=(r\theta,\mathrm{r}-d\cos\theta)$, so a typical point P of the trochoid has coordinates $(r\theta+d\sin(\theta-\pi),\mathrm{r}-d\cos\theta)$. That is, P has coordinates $(\mathrm{x,y})$, where $x=r\theta-d\sin\theta$ and $y=r-d\cos\theta$. When $d=r$, these equations agree with those of the cycloid.





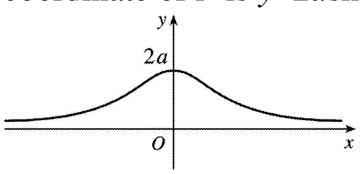


39. It is apparent that $x=|OQ|$ and $y=|QP|=|ST|$. From the diagram, $x=|OQ|=a\cos\theta$ and $y=|ST|=b\sin\theta$. Thus, the parametric equations are $x=a\cos\theta$ and $y=b\sin\theta$. To eliminate θ we rearrange: $\sin\theta=y/b\Rightarrow\sin^2\theta=(y/b)^2$ and $\cos\theta=x/a\Rightarrow\cos^2\theta=(x/a)^2$. Adding the two equations: $\sin^2\theta+\cos^2\theta=1=x^2/a^2+y^2/b^2$. Thus, we have an ellipse.



40. A has coordinates $(a\cos\theta, a\sin\theta)$. Since OA is perpendicular to AB , ΔOAB is a right triangle and B has coordinates $(a\sec\theta, 0)$. It follows that P has coordinates $(a\sec\theta, b\sin\theta)$. Thus, the parametric equations are $x=a\sec\theta$, $y=b\sin\theta$.

41. $C=(2a\cot\theta, 2a)$, so the $x-$ coordinate of P is $x=2a\cot\theta$. Let $B=(0,2a)$. Then $\angle OAB$ is a right angle and $\angle OBA=\theta$, so $|OA|=2a\sin\theta$ and $A=((2a\sin\theta)\cos\theta, (2a\sin\theta)\sin\theta)$. Thus, the $y-$ coordinate of P is $y=2a\sin^2\theta$.

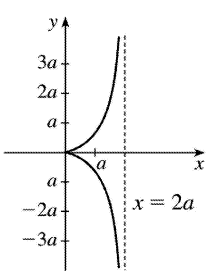


42. Let θ be the angle of inclination of segment OP . Then $|OB|=\dfrac{2a}{\cos\theta}$. Let $C=(2a,0)$. Then by use of right triangle OAC we see that $|OA|=2a\cos\theta$. Now

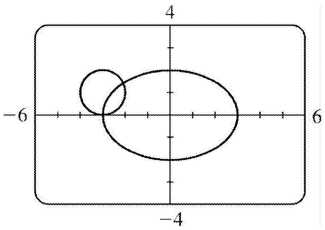
$$|OP| = |AB|=|OB|-|OA|=2a\left(\frac{1}{\cos\theta}-\cos\theta\right)$$

$$= 2a\frac{1-\cos^2\theta}{\cos\theta}=2a\frac{\sin^2\theta}{\cos\theta}=2a\sin\theta\tan\theta$$

So P has coordinates $x=2a\sin\theta\tan\theta\cdot\cos\theta=2a\sin^2\theta$ and $y=2a\sin\theta\tan\theta\cdot\sin\theta=2a\sin^2\theta\tan\theta$.



43. **(a)**



There are 2 points of intersection:
$(-3,0)$ and approximately $(-2.1,1.4)$.

(b) A collision point occurs when $x_1=x_2$ and $y_1=y_2$ for the same t . So solve the equations:

$$\begin{aligned} 3\sin t &= -3+\cos t \quad \textbf{(1)} \\ 2\cos t &= 1+\sin t \quad \textbf{(2)} \end{aligned}$$

From **(2)** , $\sin t=2\cos t-1$. Substituting into **(1)** , we get $3(2\cos t-1)=-3+\cos t\Rightarrow$ $5\cos t=0$ (*) $\Rightarrow \cos t=0\Rightarrow t=\frac{\pi}{2}$ or $\frac{3\pi}{2}$. We check that $t=\frac{3\pi}{2}$ satisfies **(1)** and **(2)** but $t=\frac{\pi}{2}$ does not. So the only collision point occurs when $t=\frac{3\pi}{2}$, and this gives the point $(-3,0)$.

(c) The circle is centered at $(3,1)$ instead of $(-3,1)$. There are still 2 intersection points: $(3,0)$ and $(2.1,1.4)$, but there are no collision points, since (*) in part (b) becomes $5\cos t=6\Rightarrow \cos t=\frac{6}{5}>1$.

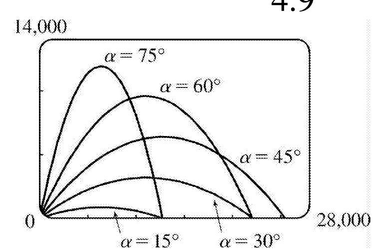
44. **(a)** If $\alpha=30^\circ$ and $v_0=500$ m / s, then the equations become $x=(500\cos 30^\circ)t=250\sqrt{3}t$ and $y=\left(500\sin 30^\circ\right)t-\frac{1}{2}(9.8)t^2=250t-4.9t^2$. $y=0$ when $t=0$ (when the gun is fired) and again when $t=\frac{250}{4.9}\approx 51$ s. Then $x=\left(250\sqrt{3}\right)\left(\frac{250}{4.9}\right)\approx 22,092$ m, so the bullet hits the ground about 22 km from the gun.

The formula for y is quadratic in t . To find the maximum y –value, we will complete the square:

$$y=-4.9\left(t^2-\frac{250}{4.9}t\right)=-4.9\left[t^2-\frac{250}{4.9}t+\left(\frac{125}{4.9}\right)^2\right]+\frac{125^2}{4.9}=-4.9\left(t-\frac{125}{4.9}\right)^2+\frac{125^2}{4.9}\le\frac{125^2}{4.9}$$

with equality when $t=\frac{125}{4.9}$ s, so the maximum height attained is $\frac{125^2}{4.9}\approx 3189$ m.

(b)

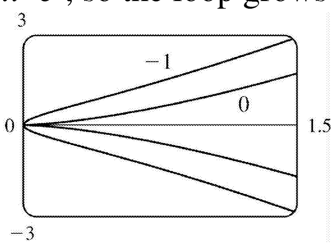


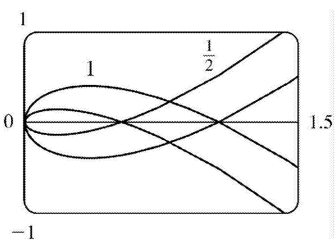
As α ($0^\circ < \alpha < 90^\circ$) increases up to 45° , the projectile attains a greater height and a greater range. As α increases past 45° , the projectile attains a greater height, but its range decreases.

(c) $x=(v_0 \cos\alpha)t \Rightarrow t=\dfrac{x}{v_0\cos\alpha}$.

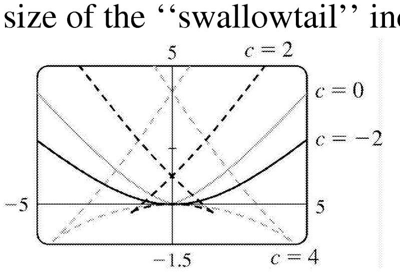
$y=(v_0 \sin\alpha)t-\dfrac{1}{2}gt^2 \Rightarrow y=(v_0\sin\alpha)\dfrac{x}{v_0\cos\alpha}-\dfrac{g}{2}\left(\dfrac{x}{v_0\cos\alpha}\right)^2=(\tan\alpha)x-\left(\dfrac{g}{2v_0^2\cos^2\alpha}\right)x^2$, which is the equation of a parabola (quadratic in x).

45. $x=t^2, y=t^3-ct$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the $x-$ axis. For $c<0$, the graph does not cross itself, but for $c=0$ it has a cusp at $(0,0)$ and for $c>0$ the graph crosses itself at $x=c$, so the loop grows larger as c increases.

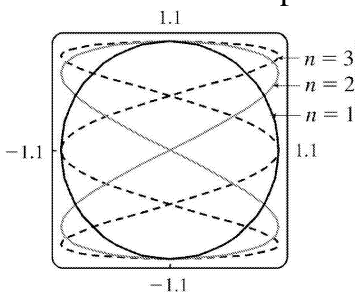




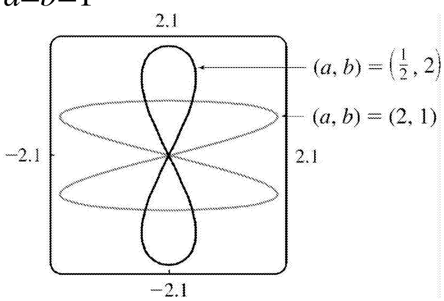
46. $x=2ct-4t^3, y=-ct^2+3t^4$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the $y-$ axis. When $c<0$, the graph resembles that of a polynomial of even degree, but when $c=0$ there is a corner at the origin, and when $c>0$, the graph crosses itself at the origin, and has two cusps below the $x-$ axis. The size of the ''swallowtail'' increases as c increases.



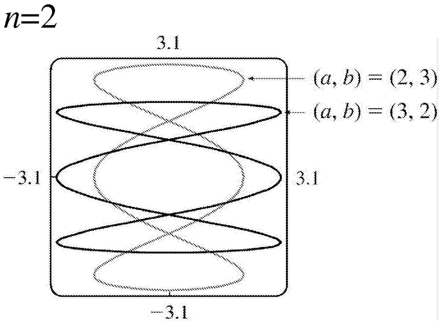
47. Note that all the Lissajous figures are symmetric about the $x-$ axis. The parameters a and b simply stretch the graph in the $x-$ and $y-$ directions respectively. For $a=b=n=1$ the graph is simply a circle with radius 1 . For $n=2$ the graph crosses itself at the origin and there are loops above and below the $x-$ axis. In general, the figures have $n-1$ points of intersection, all of which are on the $y-$ axis, and a total of n closed loops.



$a=b=1$

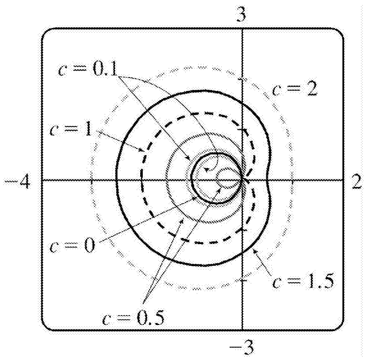


$n=2$



$n=3$

48. We use $-\pi \leq t \leq \pi$ in the viewing rectangle $[-4,2]\times[-3,3]$. We first observe that for $c=0$, we obtain a circle with center $\left(-\frac{1}{2},0\right)$ and radius $\frac{1}{2}$. As the value of c increases, there is a larger outer loop and a smaller inner loop until $c=1$, when we obtain a curve with a dent (called a **cardioid**). As c increases, we get a curve with a dimple (called a **limacon**) until $c=2$. For $c>2$, we have convex limacons. For negative values of c , we obtain the same graphs as for positive c , but with different values of t corresponding to the points on the curve.


3
-4
2
-3
c = 0.1
c = 2
c = 1
c = 0
c = 1.5
c = 0.5

1. $x=t-t^3$, $y=2-5t \Rightarrow \dfrac{dy}{dt}=-5$, $\dfrac{dx}{dt}=1-3t^2$, and $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{-5}{1-3t^2}$ or $\dfrac{5}{3t^2-1}$.

2. $x=te^t$, $y=t+e^t \Rightarrow \dfrac{dy}{dt}=1+e^t$, $\dfrac{dx}{dt}=te^t+e^t$, and $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{1+e^t}{te^t+e^t}$.

3. $x=t^4+1$, $y=t^3+t$; $t=-1$. $\dfrac{dy}{dt}=3t^2+1$, $\dfrac{dx}{dt}=4t^3$, and $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{3t^2+1}{4t^3}$.

When $t=-1$, $(x,y)=(2,-2)$ and $dy/dx=\dfrac{4}{-4}=-1$, so an equation of the tangent to the curve at the point corresponding to $t=-1$ is $y-(-2)=(-1)(x-2)$, or $y=-x$.

4. $x=2t^2+1$, $y=\dfrac{1}{3}t^3-t$; $t=3$. $\dfrac{dy}{dt}=t^2-1$, $\dfrac{dx}{dt}=4t$, and $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{t^2-1}{4t}$. When $t=3$, $(x,y)=(19,6)$ and $dy/dx=\dfrac{8}{12}=\dfrac{2}{3}$, so an equation of the tangent line is $y-6=\dfrac{2}{3}(x-19)$, or $y=\dfrac{2}{3}x-\dfrac{20}{3}$.

5. $x=e^{\sqrt{t}}$, $y=t-\ln t^2$; $t=1$. $\dfrac{dy}{dt}=1-\dfrac{2t}{t^2}=1-\dfrac{2}{t}$, $\dfrac{dx}{dt}=\dfrac{e^{\sqrt{t}}}{2\sqrt{t}}$, and $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{1-2/t}{e^{\sqrt{t}}/(2\sqrt{t})}\cdot\dfrac{2t}{2t}=\dfrac{2t-4}{\sqrt{t}\,e^{\sqrt{t}}}$. When $t=1$, $(x,y)=(e,1)$ and $\dfrac{dy}{dx}=-\dfrac{2}{e}$, so an equation of the tangent line is $y-1=-\dfrac{2}{e}(x-e)$, or $y=-\dfrac{2}{e}x+3$.

6. $x=\cos\theta+\sin 2\theta$, $y=\sin\theta+\cos 2\theta$; $\theta=0$. $\dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}=\dfrac{\cos\theta-2\sin 2\theta}{-\sin\theta+2\cos 2\theta}$. When $\theta=0$, $(x,y)=(1,1)$ and $dy/dx=\dfrac{1}{2}$, so an equation of the tangent to the curve is $y-1=\dfrac{1}{2}(x-1)$, or $y=\dfrac{1}{2}x+\dfrac{1}{2}$.

7. **(a)** $x=e^t$, $y=(t-1)^2$; $(1,1)$. $\dfrac{dy}{dt}=2(t-1)$, $\dfrac{dx}{dt}=e^t$, and $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{2(t-1)}{e^t}$.

At $(1,1)$, $t=0$ and $\dfrac{dy}{dx}=-2$, so an equation of the tangent is $y-1=-2(x-1)$, or $y=-2x+3$.

(b) $x=e^t \Rightarrow t=\ln x$, so $y=(t-1)^2=(\ln x-1)^2$ and $\dfrac{dy}{dx}=2(\ln x-1)\left(\dfrac{1}{x}\right)$. When $x=1$, $\dfrac{dy}{dx}=2(-1)(1)=-2$, so an equation of the tangent is $y=-2x+3$, as in part (a).

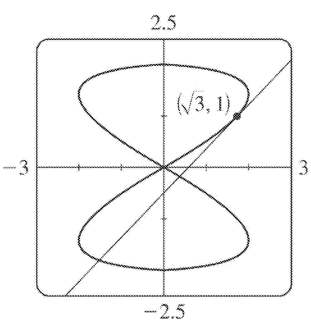
8. **(a)** $x=\tan\theta$, $y=\sec\theta$; $\left(1,\sqrt{2}\right)$. $\dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}=\dfrac{\sec\theta\tan\theta}{\sec^2\theta}=\dfrac{\tan\theta}{\sec\theta}=\sin\theta$.

When (x,y)=$\left(1,\sqrt{2}\right)$, $\theta=\dfrac{\pi}{4}$ (or $\dfrac{\pi}{4}+2\pi n$ for some integer n), so $dy/dx=\sin\dfrac{\pi}{4}=\sqrt{2}/2$. Thus, an equation of the tangent to the curve is $y-\sqrt{2}=\left(\sqrt{2}/2\right)(x-1)$, or $y=\left(\sqrt{2}/2\right)x+\left(\sqrt{2}/2\right)$.

(b) $\tan^2\theta+1=\sec^2\theta\Rightarrow x^2+1=y^2$, so $\dfrac{d}{dx}(x^2+1)=\dfrac{d}{dx}(y^2)\Rightarrow 2x=2y\dfrac{dy}{dx}$.

When (x,y)=$\left(1,\sqrt{2}\right)$, $\dfrac{dy}{dx}=\dfrac{x}{y}=\dfrac{1}{\sqrt{2}}=\dfrac{\sqrt{2}}{2}$, so an equation of the tangent is $y-\sqrt{2}=\left(\sqrt{2}/2\right)(x-1)$, as in part (a).

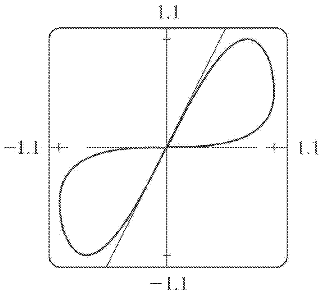
9. $x=2\sin 2t$, $y=2\sin t$; $\left(\sqrt{3},1\right)$. $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{2\cos t}{2\cdot 2\cos 2t}=\dfrac{\cos t}{2\cos 2t}$. The point $\left(\sqrt{3},1\right)$ corresponds to $t=\dfrac{\pi}{6}$, so the slope of the tangent at that point is $\dfrac{\cos\frac{\pi}{6}}{2\cos\frac{\pi}{3}}=\dfrac{\frac{\sqrt{3}}{2}}{2\cdot\frac{1}{2}}=\dfrac{\sqrt{3}}{2}$. An equation of the tangent is therefore $(y-1)=\dfrac{\sqrt{3}}{2}\left(x-\sqrt{3}\right)$, or $y=\dfrac{\sqrt{3}}{2}x-\dfrac{1}{2}$.



10. $x=\sin t$, $y=\sin(t+\sin t)$; $(0,0)$.

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\cos(t+\sin t)(1+\cos t)}{\cos t}=\left(\sec t+1\right)\cos(t+\sin t)$$

Note that there are two tangents at the point $(0,0)$, since both $t=0$ and $t=\pi$ correspond to the origin. The tangent corresponding to $t=0$ has slope $\left(\sec 0+1\right)\cos(0+\sin 0)=2\cos 0=2$, and its equation is $y=2x$. The tangent corresponding to $t=\pi$ has slope $\left(\sec\pi+1\right)\cos(\pi+\sin\pi)=0$, so it is the $x-$ axis.



11. $x=4+t^2$, $y=t^2+t^3 \Rightarrow \dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{2t+3t^2}{2t}=1+\dfrac{3}{2}t \Rightarrow$

$\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)=\dfrac{d(dy/dx)/dt}{dx/dt}=\dfrac{(d/dt)\left(1+\frac{3}{2}t\right)}{2t}=\dfrac{3/2}{2t}=\dfrac{3}{4t}$. The curve is CU when $\dfrac{d^2y}{dx^2}>0$, that is, when $t>0$.

12. $x=t^3-12t$, $y=t^2-1 \Rightarrow \dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{2t}{3t^2-12} \Rightarrow$

$\dfrac{d^2y}{dx^2}=\dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}=\dfrac{\frac{(3t^2-12)\cdot 2-2t(6t)}{(3t^2-12)^2}}{3t^2-12}=\dfrac{-6t^2-24}{(3t^2-12)^3}=\dfrac{-6(t^2+4)}{3^3(t^2-4)^3}=\dfrac{-2(t^2+4)}{9(t^2-4)^3}$. Thus, the curve is CU when $t^2-4<0 \Rightarrow |t|<2 \Rightarrow -2<t<2$.

13. $x=t-e^t$, $y=t+e^{-t} \Rightarrow$

$\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{1-e^{-t}}{1-e^t}=\dfrac{1-\frac{1}{e^t}}{1-e^t}=\dfrac{\frac{e^t-1}{e^t}}{1-e^t}=-e^{-t} \Rightarrow \dfrac{d^2y}{dx^2}=\dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}=\dfrac{\frac{d}{dt}(-e^{-t})}{dx/dt}=\dfrac{e^{-t}}{1-e^t}$. The curve is CU when $e^t<1 \Rightarrow t<0$.

14. $x=t+\ln t$, $y=t-\ln t \Rightarrow \dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{1-1/t}{1+1/t}=\dfrac{t-1}{t+1}=1-\dfrac{2}{t+1} \Rightarrow$

$\dfrac{d^2y}{dx^2}=\dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}=\dfrac{\frac{d}{dt}\left(1-\frac{2}{t+1}\right)}{1+1/t}=\dfrac{2/(t+1)^2}{(t+1)/t}=\dfrac{2t}{(t+1)^3}$, so the curve is CU for all t in its domain, that is, $t>0$.

15. $x=2\sin t$, $y=3\cos t$, $0<t<2\pi$.

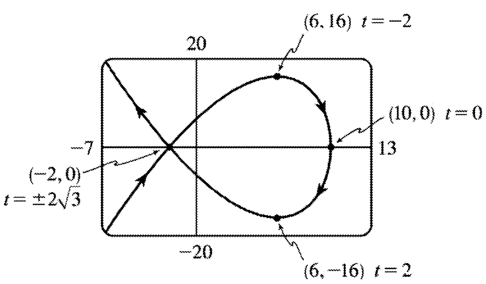
$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-3\sin t}{2\cos t}=-\frac{3}{2}\tan t$, so $\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}=\frac{-\frac{3}{2}\sec^2 t}{2\cos t}=-\frac{3}{4}\sec^3 t$.

The curve is CU when $\sec^3 t<0 \Rightarrow \sec t<0 \Rightarrow \cos t<0 \Rightarrow \frac{\pi}{2}<t<\frac{3\pi}{2}$.

16. $x=\cos 2t$, $y=\cos t$, $0<t<\pi$. $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-\sin t}{-2\sin 2t}=\frac{\sin t}{2\cdot 2\sin t\cos t}=\frac{1}{4\cos t}=\frac{1}{4}\sec t$, so $\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}=\frac{\frac{1}{4}\sec t\tan t}{-4\sin t\cos t}=-\frac{1}{16}\sec^3 t$. The curve is CU when $\sec^3 t<0 \Rightarrow \sec t<0 \Rightarrow \cos t<0 \Rightarrow \frac{\pi}{2}<t<\pi$.

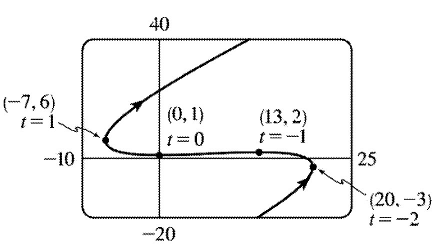
17. $x=10-t^2$, $y=t^3-12t$.

$dy/dt=3t^2-12=3(t+2)(t-2)$, so $dy/dt=0 \Leftrightarrow t=\pm 2 \Leftrightarrow (x,y)=(6,\mp 16)$. $dx/dt=-2t$, so $dx/dt=0 \Leftrightarrow t=0 \Leftrightarrow (x,y)=(10,0)$. The curve has horizontal tangents at $(6,\pm 16)$ and a vertical tangent at $(10,0)$.



18. $x=2t^3+3t^2-12t$, $y=2t^3+3t^2+1$.

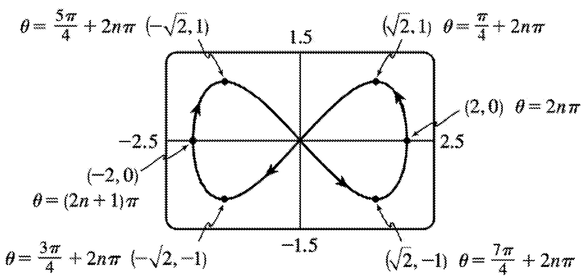
$dy/dt=6t^2+6t=6t(t+1)$, so $dy/dt=0 \Leftrightarrow t=0$ or $-1 \Leftrightarrow (x,y)=(0,1)$ or $(13,2)$. $dx/dt=6t^2+6t-12=6(t+2)(t-1)$, so $dx/dt=0 \Leftrightarrow t=-2$ or $1 \Leftrightarrow$



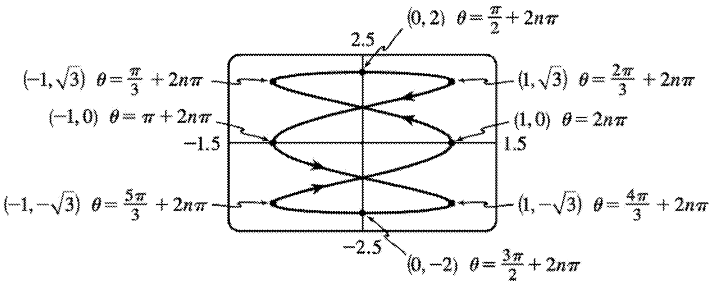
$(x,y)=(20,-3)$ or $(-7,6)$. The curve has horizontal tangents at $(0,1)$ and $(13,2)$, and vertical tangents at $(20,-3)$ and $(-7,6)$.

19. $x = 2\cos\theta$, $y = \sin 2\theta$.

$dy/d\theta = 2\cos 2\theta$, so $dy/d\theta = 0 \Leftrightarrow 2\theta = \frac{\pi}{2} + n\pi$ (n an integer) $\Leftrightarrow \theta = \frac{\pi}{4} + \frac{\pi}{2}n \Leftrightarrow (x,y) = \left(\pm\sqrt{2}, \pm 1\right)$. Also, $dx/d\theta = -2\sin\theta$, so $dx/d\theta = 0 \Leftrightarrow \theta = n\pi \Leftrightarrow (x,y) = (\pm 2, 0)$. The curve has horizontal tangents at $\left(\pm\sqrt{2}, \pm 1\right)$ (four points), and vertical tangents at $(\pm 2, 0)$.

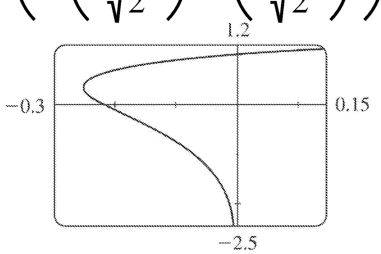


20. $x = \cos 3\theta$, $y = 2\sin\theta$. $dy/d\theta = 2\cos\theta$, so $dy/d\theta = 0 \Leftrightarrow \theta = \frac{\pi}{2} + n\pi$ (n an integer) $\Leftrightarrow (x,y) = (0, \pm 2)$. Also, $dx/d\theta = -3\sin 3\theta$, so $dx/d\theta = 0 \Leftrightarrow 3\theta = n\pi \Leftrightarrow \theta = \frac{\pi}{3}n \Leftrightarrow (x,y) = (\pm 1, 0)$ or $\left(\pm 1, \pm\sqrt{3}\right)$. The curve has horizontal tangents at $(0, \pm 2)$, and vertical tangents at $(\pm 1, 0)$, $\left(\pm 1, -\sqrt{3}\right)$ and $\left(\pm 1, \sqrt{3}\right)$.

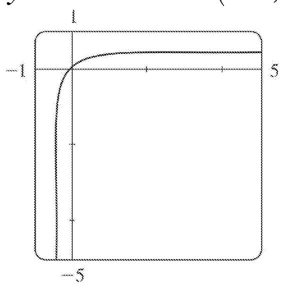


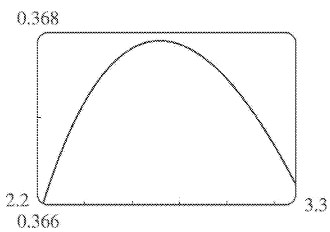
21. From the graph, it appears that the leftmost point on the curve $x = t^4 - t^2$, $y = t + \ln t$ is about $(-0.25, 0.36)$. To find the exact coordinates, we find the value of t for which the graph has a vertical tangent, that is, $0 = dx/dt = 4t^3 - 2t \Leftrightarrow 2t\left(2t^2 - 1\right) = 0 \Leftrightarrow 2t\left(\sqrt{2}\,t + 1\right)\left(\sqrt{2}\,t - 1\right) = 0 \Leftrightarrow t = 0$ or $\pm\frac{1}{\sqrt{2}}$. The negative and 0 roots are inadmissible since $y(t)$ is only defined for $t > 0$, so the leftmost point must be

$$\left(x\left(\frac{1}{\sqrt{2}}\right), y\left(\frac{1}{\sqrt{2}}\right)\right) = \left(\left(\frac{1}{\sqrt{2}}\right)^4 - \left(\frac{1}{\sqrt{2}}\right)^2, \frac{1}{\sqrt{2}} + \ln\frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{4}, \frac{1}{\sqrt{2}} - \frac{1}{2}\ln 2\right)$$



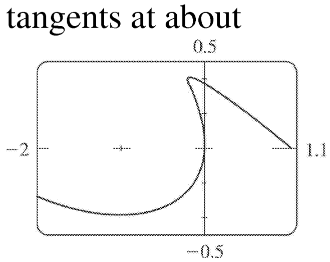
22. The curve is symmetric about the line $y=-x$ since replacing t with $-t$ has the effect of replacing (x,y) with $(-y,-x)$, so if we can find the highest point $\left(x_h,y_h\right)$, then the leftmost point is $\left(x_l,y_l\right)=\left(-y_h,-x_h\right)$. After carefully zooming in, we estimate that the highest point on the curve $x=te^t$, $y=te^{-t}$ is about $(2.7,0.37)$.

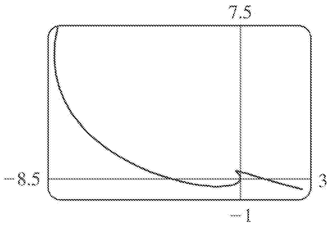




To find the exact coordinates of the highest point, we find the value of t for which the curve has a horizontal tangent, that is, $dy/dt=0 \Leftrightarrow t(-e^{-t})+e^{-t}=0 \Leftrightarrow (1-t)e^{-t}=0 \Leftrightarrow t=1$. This corresponds to the point $(x(1),y(1))=(e,1/e)$. To find the leftmost point, we find the value of t for which $0=dx/dt=te^t+e^t \Leftrightarrow$ $(1+t)e^t=0 \Leftrightarrow t=-1$. This corresponds to the point $(x(-1),y(-1))=(-1/e,-e)$. As $t\to-\infty$, $x(t)=te^t\to0^-$ by l'Hospital's Rule and $y(t)=te^{-t}\to-\infty$, so the $y-$ axis is an asymptote. As $t\to\infty$, $x(t)\to\infty$ and $y(t)\to0^+$, so the $x-$ axis is the other asymptote. The asymptotes can also be determined from the graph, if we use a larger t –interval.

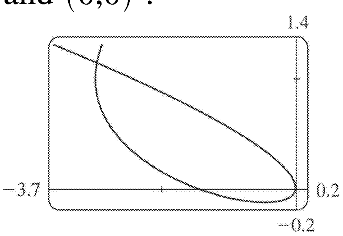
23. We graph the curve $x=t^4-2t^3-2t^2$, $y=t^3-t$ in the viewing rectangle $[-2,1.1]$ by $[-0.5,0.5]$. This rectangle corresponds approximately to $t\in[-1,0.8]$. We estimate that the curve has horizontal tangents at about

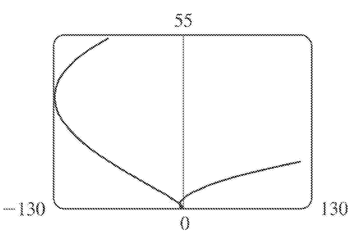




$(-1,-0.4)$ and $(-0.17,0.39)$ and vertical tangents at about $(0,0)$ and $(-0.19,0.37)$. We calculate $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{3t^2-1}{4t^3-6t^2-4t}$. The horizontal tangents occur when $dy/dt=3t^2-1=0\Leftrightarrow t=\pm\frac{1}{\sqrt{3}}$, so both horizontal tangents are shown in our graph. The vertical tangents occur when $dx/dt=2t(2t^2-3t-2)=0\Leftrightarrow 2t(2t+1)(t-2)=0\Leftrightarrow t=0$, $-\frac{1}{2}$ or 2 . It seems that we have missed one vertical tangent, and indeed if we plot the curve on the $t-$ interval $[-1.2,2.2]$ we see that there is another vertical tangent at $(-8,6)$.

24. We graph the curve $x=t^4+4t^3-8t^2$, $y=2t^2-t$ in the viewing rectangle $[-3.7,0.2]$ by $[-0.2,1.4]$. It appears that there is a horizontal tangent at about $(-0.4,-0.1)$, and vertical tangents at about $(-3,1)$ and $(0,0)$.

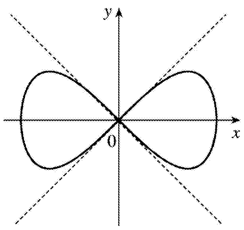




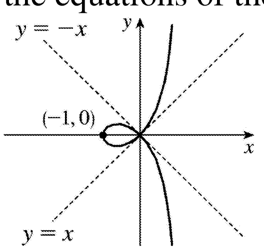
We calculate $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{4t-1}{4t^3+12t^2-16t}$, so there is a horizontal tangent where $dy/dt=4t-1=0\Leftrightarrow t=\frac{1}{4}$. This point (the lowest point) is shown in the first graph. There are vertical tangents where $dx/dt=4t^3+12t^2-16t=0\Leftrightarrow 4t(t^2+3t-4)=0\Leftrightarrow 4t(t+4)(t-1)=0$. We have missed one vertical tangent corresponding to $t=-4$, and if we plot the graph for $t\in[-5,3]$, we see that the curve has another vertical tangent line at approximately $(-128,36)$.

25. $x=\cos t$, $y=\sin t\cos t$. $\frac{dx}{dt}=-\sin t$, $\frac{dy}{dt}=-\sin^2 t+\cos^2 t=\cos 2t$. $(x,y)=(0,0)\Leftrightarrow\cos t=0\Leftrightarrow t$ is an odd multiple of

$\frac{\pi}{2}$. When $t=\frac{\pi}{2}$, $\frac{dx}{dt}=-1$ and $\frac{dy}{dt}=-1$, so $\frac{dy}{dx}=1$. When $t=\frac{3\pi}{2}$, $\frac{dx}{dt}=1$ and $\frac{dy}{dt}=-1$. So $\frac{dy}{dx}=-1$. Thus, $y=x$ and $y=-x$ are both tangent to the curve at $(0,0)$.



26. $x=1-2\cos^2 t=-\cos 2t$, $y=(\tan t)(1-2\cos^2 t)=-(\tan t)\cos 2t$. To find a point where the curve crosses itself, we look for two values of t that give the same point (x,y) . Call these values t_1 and t_2 . Then $\cos^2 t_1=\cos^2 t_2$ (from the equation for x) and either $\tan t_1=\tan t_2$ or $\cos^2 t_1=\cos^2 t_2=\frac{1}{2}$ (from the equation for y). We can satisfy $\cos^2 t_1=\cos^2 t_2$ and $\tan t_1=\tan t_2$ by choosing t_1 arbitrarily and taking $t_2=t_1+\pi$, so evidently the whole curve is retraced every time t traverses an interval of length π . Thus, we can restrict our attention to the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. If $t_2=-t_1$, then $\cos^2 t_2=\cos^2 t_1$, but $\tan t_2=-\tan t_1$. This suggests that we try to satisfy the condition $\cos^2 t_1=\cos^2 t_2=\frac{1}{2}$. Taking $t_1=\frac{\pi}{4}$ and $t_2=-\frac{\pi}{4}$ gives $(x,y)=(0,0)$ for both values of t . $\frac{dx}{dt}=2\sin 2t$,and $\frac{dy}{dt}=2\sin 2t\tan t-\cos 2t\sec^2 t$. When $t=\frac{\pi}{4}$, $\frac{dx}{dt}=2$ and $\frac{dy}{dt}=2$, so $\frac{dy}{dx}=1$. When $t=-\frac{\pi}{4}$, $\frac{dx}{dt}=-2$ and $\frac{dy}{dt}=2$, so $\frac{dy}{dx}=-1$. Thus, the equations of the two tangents at $(0,0)$ are $y=x$ and $y=-x$.



27. **(a)** $x=r\theta-d\sin\theta$, $y=r-d\cos\theta$; $\frac{dx}{d\theta}=r-d\cos\theta$, $\frac{dy}{d\theta}=d\sin\theta$. So $\frac{dy}{dx}=\frac{d\sin\theta}{r-d\cos\theta}$.
(b) If $0<d<r$, then $|d\cos\theta|\le d<r$, so $r-d\cos\theta\ge r-d>0$. This shows that $dx/d\theta$ never vanishes, so the trochoid can have no vertical tangent if $d<r$.

28. $x=a\cos^3\theta$, $y=a\sin^3\theta$.
(a)

$\frac{dx}{d\theta}=-3a\cos^2\theta\sin\theta$, $\frac{dy}{d\theta}=3a\sin^2\theta\cos\theta$, so $\frac{dy}{dx}=-\frac{\sin\theta}{\cos\theta}=-\tan\theta$.

(b) The tangent is horizontal $\Leftrightarrow dy/dx=0 \Leftrightarrow \tan\theta=0 \Leftrightarrow \theta=n\pi \Leftrightarrow (x,y)=(\pm a,0)$. The tangent is vertical $\Leftrightarrow \cos\theta=0 \Leftrightarrow \theta$ is an odd multiple of $\frac{\pi}{2} \Leftrightarrow (x,y)=(0,\pm a)$.

(c) $dy/dx=\pm 1 \Leftrightarrow \tan\theta=\pm 1 \Leftrightarrow \theta$ is an odd multiple of $\frac{\pi}{4} \Leftrightarrow (x,y)=\left(\pm\frac{\sqrt{2}}{4}a, \pm\frac{\sqrt{2}}{4}a\right)$ (All sign choices are valid.)

29. The line with parametric equations $x=-7t$, $y=12t-5$ is $y=12\left(-\frac{1}{7}x\right)-5$, which has slope $-\frac{12}{7}$. The curve $x=t^3+4t$, $y=6t^2$ has slope $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{12t}{3t^2+4}$. This equals $-\frac{12}{7} \Leftrightarrow 3t^2+4=-7t \Leftrightarrow$ $(3t+4)(t+1)=0 \Leftrightarrow t=-1$ or $t=-\frac{4}{3} \Leftrightarrow (x,y)=(-5,6)$ or $\left(-\frac{208}{27},\frac{32}{3}\right)$.

30. $x=3t^2+1$, $y=2t^3+1$, $\frac{dx}{dt}=6t$, $\frac{dy}{dt}=6t^2$, so $\frac{dy}{dx}=\frac{6t^2}{6t}=t$ (even where $t=0$).
So at the point corresponding to parameter value t , an equation of the tangent line is $y-(2t^3+1)=t\left[x-(3t^2+1)\right]$. If this line is to pass through $(4,3)$, we must have $3-(2t^3+1)=t\left[4-(3t^2+1)\right]$ $\Leftrightarrow 2t^3-2=3t^3-3t \Leftrightarrow t^3-3t+2=0 \Leftrightarrow (t-1)^2(t+2)=0 \Leftrightarrow t=1$ or -2 . Hence, the desired equations are $y-3=x-4$, or $y=x-1$, tangent to the curve at $(4,3)$, and $y-(-15)=-2(x-13)$, or $y=-2x+11$, tangent to the curve at $(13,-15)$.

31. By symmetry of the ellipse about the $x-$ and $y-$ axes,

$$A = 4\int_0^a y\,dx=4\int_{\pi/2}^0 b\sin\theta\,(-a\sin\theta)\,d\theta=4ab\int_0^{\pi/2}\sin^2\theta\,d\theta=4ab\int_0^{\pi/2}\frac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$= 2ab\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{\pi/2}=2ab\left(\frac{\pi}{2}\right)=\pi ab$$

32. $t+1/t=2.5 \Leftrightarrow t=\frac{1}{2}$ or 2 , and for $\frac{1}{2}<t<2$, we have $t+1/t<2.5$. $x=-\frac{3}{2}$ when $t=\frac{1}{2}$ and $x=\frac{3}{2}$ when $t=2$. \

$$A = \int_{-3/2}^{3/2}(2.5-y)\,dx=\int_{1/2}^{2}\left(\frac{5}{2}-t-1/t\right)(1+1/t^2)\,dt\ [x=t-1/t,dx=(1+1/t^2)\,dt]$$

$$= \int_{1/2}^{2}\left(-t+\frac{5}{2}-2t^{-1}+\frac{5}{2}t^{-2}-t^{-3}\right)dt=\left[\frac{-t^2}{2}+\frac{5t}{2}-2\ln|t|-\frac{5}{2t}+\frac{1}{2t^2}\right]_{1/2}^{2}$$

$$= \left(-2+5-2\ln 2-\frac{5}{4}+\frac{1}{8} \right)-\left(-\frac{1}{8}+\frac{5}{4}+2\ln 2-5+2 \right)=\frac{15}{4}-4\ln 2$$

33.

$$A=\int_0^1 (y-1)\,dx=\int_{\pi/2}^0 (e^t-1)(-\sin t)\,dt=\int_0^{\pi/2}(e^t\sin t-\sin t)\,dt=\left[\frac{1}{2}e^t(\sin t-\cos t)+\cos t \right]_0^{\pi/2}$$

$$=\frac{1}{2}(e^{\pi/2}-1)$$

34. By symmetry, $A=4\int_0^a y\,dx=4\int_{\pi/2}^0 a\sin^3\theta\,(-3a\cos^2\theta\sin\theta)\,d\theta=12a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta$. Now

$$\int\sin^4\theta\cos^2\theta\,d\theta=\int\sin^2\theta\left(\frac{1}{4}\sin^2 2\theta\right)d\theta=\frac{1}{8}\int(1-\cos 2\theta)\sin^2 2\theta\,d\theta$$

$$=\frac{1}{8}\int\left[\frac{1}{2}(1-\cos 4\theta)-\sin^2 2\theta\cos 2\theta\right]d\theta=\frac{1}{16}\theta-\frac{1}{64}\sin 4\theta-\frac{1}{48}\sin^3 2\theta+C$$

so $\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta=\left[\frac{1}{16}\theta-\frac{1}{64}\sin 4\theta-\frac{1}{48}\sin^3 2\theta\right]_0^{\pi/2}=\frac{\pi}{32}$. Thus, $A=12a^2\left(\frac{\pi}{32}\right)=\frac{3}{8}\pi a^2$.

35.

$$A=\int_0^{2\pi r}y\,dx=\int_0^{2\pi}(r-d\cos\theta)(r-d\cos\theta)\,d\theta=\int_0^{2\pi}(r^2-2dr\cos\theta+d^2\cos^2\theta)\,d\theta$$

$$=\left[r^2\theta-2dr\sin\theta+\frac{1}{2}d^2\left(\theta+\frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi}=2\pi r^2+\pi d^2$$

36. **(a)** By symmetry, the area of is twice the area inside above the $x-$ axis. The top half of the loop is described by $x=t^2$, $y=t^3-3t$, $-\sqrt{3}\le t\le 0$, so, using the Substitution Rule with $y=t^3-3t$ and $dx=2t\,dt$, we find that

$$\text{area}=2\int_0^3 y\,dx=2\int_0^{-\sqrt{3}}(t^3-3t)2t\,dt=2\int_0^{-\sqrt{3}}(2t^4-6t^2)\,dt=2\left[\frac{2}{5}t^5-2t^3\right]_0^{-\sqrt{3}}$$

$$=2\left[\frac{2}{5}(-3^{1/2})^5-2(-3^{1/2})^3\right]=2\left[\frac{2}{5}\left(-9\sqrt{3}\right)-2\left(-3\sqrt{3}\right)\right]=\frac{24}{5}\sqrt{3}$$

(b) Here we use the formula for disks and use the Substitution Rule as in part (a):

$$\text{volume}=\pi\int_0^3 y^2\,dx=\pi\int_0^{-\sqrt{3}}(t^3-3t)^2 2t\,dt=2\pi\int_0^{-\sqrt{3}}(t^6-6t^4+9t^2)t\,dt$$

$$=2\pi\left[\frac{1}{8}t^8-t^6+\frac{9}{4}t^4\right]_0^{-\sqrt{3}}=2\pi\left[\frac{1}{8}(-3^{1/2})^8-(-3^{1/2})^6+\frac{9}{4}(-3^{1/2})^4\right]$$

$$= 2\pi\left[\frac{81}{8} - 27 + \frac{81}{4}\right] = \frac{27}{4}\pi$$

(c) By symmetry, the $y-$ coordinate of the centroid is 0 . To find the $x-$ coordinate, we note that it is the same as the $x-$ coordinate of the centroid of the top half of , the area of which is $\frac{1}{2}\cdot\frac{24}{5}\sqrt{3} = \frac{12}{5}\sqrt{3}$. So, using Formula .3.8 with $A = \frac{12}{5}\sqrt{3}$, we get

$$\bar{x} = \frac{5}{12\sqrt{3}}\int_0^3 xy\,dx = \frac{5}{12\sqrt{3}}\int_0^{-\sqrt{3}} t^2(t^3-3t)2t\,dt = \frac{5}{6\sqrt{3}}\left[\frac{1}{7}t^7 - \frac{3}{5}t^5\right]_0^{-\sqrt{3}}$$

$$= \frac{5}{6\sqrt{3}}\left[\frac{1}{7}(-3^{1/2})^7 - \frac{3}{5}(-3^{1/2})^5\right] = \frac{5}{6\sqrt{3}}\left[-\frac{27}{7}\sqrt{3} + \frac{27}{5}\sqrt{3}\right] = \frac{9}{7}$$

So the coordinates of the centroid of are $(\mathrm{x},\mathrm{y}) = \left(\frac{9}{7}, 0\right)$.

37. $x = t - t^2$, $y = \frac{4}{3}t^{3/2}$, $1 \le t \le 2$. $dx/dt = 1-2t$ and $dy/dt = 2t^{1/2}$, so $(dx/dt)^2 + (dy/dt)^2 = (1-2t)^2 + (2t^{1/2})^2 = 1-4t+4t^2+4t = 1+4t^2$. Thus,
$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_1^2 \sqrt{1+4t^2}\,dt$.

38. $x = 1+e^t$, $y = t^2$, $-3 \le t \le 3$. $dx/dt = e^t$ and $dy/dt = 2t$, so $(dx/dt)^2 + (dy/dt)^2 = e^{2t} + 4t^2$. Thus,
$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_{-3}^3 \sqrt{e^{2t}+4t^2}\,dt$.

39. $x = t + \cos t$, $y = t - \sin t$, $0 \le t \le 2\pi$. $dx/dt = 1 - \sin t$ and $dy/dt = 1 - \cos t$, so

$$\begin{aligned}(dx/dt)^2 + (dy/dt)^2 &= (1-\sin t)^2 + (1-\cos t)^2 = (1-2\sin t + \sin^2 t) + (1 - 2\cos t + \cos^2 t)\\ &= 3 - 2\sin t - 2\cos t\end{aligned}$$

Thus, $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^{2\pi}\sqrt{3 - 2\sin t - 2\cos t}\,dt$.

40. $x = \ln t$, $y = \sqrt{t+1}$, $1 \le t \le 5$. $\frac{dx}{dt} = \frac{1}{t}$ and $\frac{dy}{dt} = \frac{1}{2\sqrt{t+1}}$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{1}{t^2} + \frac{1}{4(t+1)} = \frac{t^2+4t+4}{4t^2(t+1)}$$. Thus,

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_1^5 \sqrt{\frac{t^2+4t+4}{4t^2(t+1)}}\,dt = \int_1^5 \sqrt{\frac{(t+2)^2}{(2t)^2(t+1)}}\,dt = \int_1^5 \frac{t+2}{2t\sqrt{t+1}}\,dt$$.

41. $x=1+3t^2$, $y=4+2t^3$, $0\le t\le 1$. $dx/dt=6t$ and $dy/dt=6t^2$, so $(dx/dt)^2+(dy/dt)^2=36t^2+36t^4$. Thus,

$$L=\int_0^1\sqrt{36t^2+36t^4}\,dt=\int_0^1 6t\sqrt{1+t^2}\,dt=6\int_1^2\sqrt{u}\left(\frac{1}{2}du\right)\ [\,u=1+t^2\ ,\ du=2t\,dt\,]$$

$$=3\left[\frac{2}{3}u^{3/2}\right]_1^2=2(2^{3/2}-1)=2\left(2\sqrt{2}-1\right)$$

42. $x=a(\cos\theta+\theta\sin\theta)$, $y=a(\sin\theta-\theta\cos\theta)$, $0\le\theta\le\pi$.

$$\begin{aligned}(dx/d\theta)^2+(dy/d\theta)^2 &= a^2\left[(-\sin\theta+\theta\cos\theta+\sin\theta)^2+(\cos\theta+\theta\sin\theta-\cos\theta)^2\right]\\ &= a^2\theta^2\left(\cos^2\theta+\sin^2\theta\right)=(a\theta)^2\end{aligned}$$

Thus, $L=\int_0^\pi a\theta\,d\theta=a\left[\frac{1}{2}\theta^2\right]_0^\pi=\frac{1}{2}\pi^2 a$.

43. $x=\frac{t}{1+t}$, $y=\ln(1+t)$, $0\le t\le 2$. $\frac{dx}{dt}=\frac{(1+t)\cdot 1-t\cdot 1}{(1+t)^2}=\frac{1}{(1+t)^2}$ and $\frac{dy}{dt}=\frac{1}{1+t}$, so

$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\frac{1}{(1+t)^4}+\frac{1}{(1+t)^2}=\frac{1}{(1+t)^4}\left[1+(1+t)^2\right]=\frac{t^2+2t+2}{(1+t)^4}$. Thus,

$$L=\int_0^2\frac{\sqrt{t^2+2t+2}}{(1+t)^2}\,dt=\int_1^3\frac{\sqrt{u^2+1}}{u^2}\,du[u=t+1,du=dt]=\left[-\frac{\sqrt{u^2+1}}{u}+\ln\left(u+\sqrt{u^2+1}\right)\right]_1^3$$

$$=-\frac{\sqrt{10}}{3}+\ln\left(3+\sqrt{10}\right)+\sqrt{2}-\ln\left(1+\sqrt{2}\right)$$

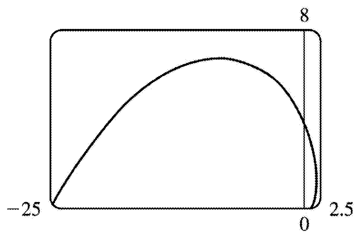
44. $x=e^t+e^{-t}$, $y=5-2t$, $0\le t\le 3$. $dx/dt=e^t-e^{-t}$ and $dy/dt=-2$, so $(dx/dt)^2+(dy/dt)^2=e^{2t}-2+e^{-2t}+4=e^{2t}+2+e^{-2t}=(e^t+e^{-t})^2$. Thus, $L=\int_0^3(e^t+e^{-t})dt=\left[e^t-e^{-t}\right]_0^3=e^3-e^{-3}-(1-1)=e^3-e^{-3}$.

45. $x=e^t\cos t$, $y=e^t\sin t$, $0\le t\le\pi$.

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2 &= \left[e^t(\cos t-\sin t)\right]^2+\left[e^t(\sin t+\cos t)\right]^2\\ &= \left(e^t\right)^2\left(\cos^2 t-2\cos t\sin t+\sin^2 t\right)\end{aligned}$$

$$+\left(e^{t}\right)^{2}\left(\sin^{2}t+2\sin t\cos t+\cos^{2}t\right)$$
$$=\ e^{2t}\left(2\cos^{2}t+2\sin^{2}t\right)=2e^{2t}$$

Thus, $L=\int_0^{\pi}\sqrt{2e^{2t}}\,dt=\int_0^{\pi}\sqrt{2}\,e^{t}\,dt=\sqrt{2}\left[e^{t}\right]_0^{\pi}=\sqrt{2}\left(e^{\pi}-1\right)$.

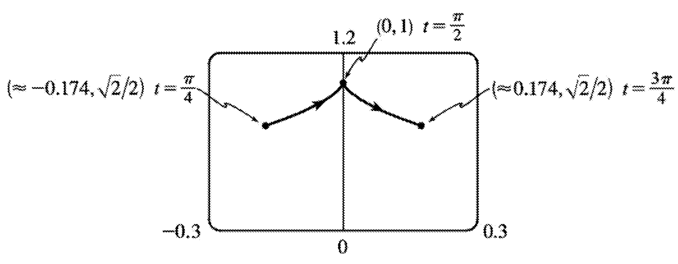


46. $x=\cos t+\ln(\tan\frac{1}{2}t)$, $y=\sin t$, $\pi/4\le t\le 3\pi/4$.

$$\frac{dx}{dt}=-\sin t+\frac{\frac{1}{2}\sec^{2}(t/2)}{\tan(t/2)}=-\sin t+\frac{1}{2\sin(t/2)\cos(t/2)}=-\sin t+\frac{1}{\sin t}$$

and $\frac{dy}{dt}=\cos t$, so $\left(\frac{dx}{dt}\right)^{2}+\left(\frac{dy}{dt}\right)^{2}=\sin^{2}t-2+\frac{1}{\sin^{2}t}+\cos^{2}t=1-2+\csc^{2}t=\cot^{2}t$. Thus,

$$L=\int_{\pi/4}^{3\pi/4}|\cot t|\,dt=2\int_{\pi/4}^{\pi/2}\cot t\,dt=2[\ln|\sin t|]_{\pi/4}^{\pi/2}=2\left(\ln 1-\ln\frac{1}{\sqrt{2}}\right)=2\left(0+\ln\sqrt{2}\right)=2\left(\frac{1}{2}\ln 2\right)=\ln 2.$$



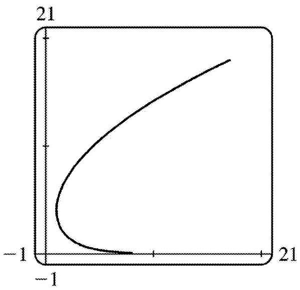
47. $x=e^{t}-t$, $y=4e^{t/2}$, $-8\le t\le 3$.

$$(dx/dt)^{2}+(dy/dt)^{2}=(e^{t}-1)^{2}+\left(2e^{t/2}\right)^{2}=e^{2t}-2e^{t}+1+4e^{t}=e^{2t}+2e^{t}+1=(e^{t}+1)^{2}$$

Thus,

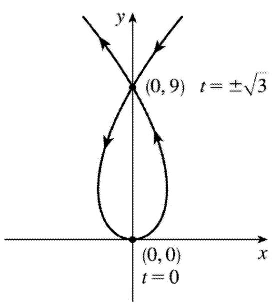
$$L=$$

$$\int_{-8}^{3}\sqrt{(e^{t}+1)^{2}}\,dt=\int_{-8}^{3}(e^{t}+1)\,dt=\left[e^{t}+t\right]_{-8}^{3}=(e^{3}+3)-(e^{-8}-8)=e^{3}-e^{-8}+11.$$



48. $x=3t-t^{3}$, $y=3t^{2}$. $dx/dt=3-3t^{2}$ and $dy/dt=6t$, so $\left(\frac{dx}{dt}\right)^{2}+\left(\frac{dy}{dt}\right)^{2}=(3-3t^{2})^{2}+(6t)^{2}=(3+3t^{2})^{2}$ and the length of the loop is given by

$$L=\int_{-\sqrt{3}}^{\sqrt{3}}(3+3t^{2})\,dt=2\int_{0}^{\sqrt{3}}(3+3t^{2})\,dt=2[3t+t^{3}]_{0}^{\sqrt{3}}$$
$$=2\left(3\sqrt{3}+3\sqrt{3}\right)=12\sqrt{3}.$$



49. $x=t-e^{t}$, $y=t+e^{t}$, $-6\le t\le 6$.
$\left(\frac{dx}{dt}\right)^{2}+\left(\frac{dy}{dt}\right)^{2}=\left(1-e^{t}\right)^{2}+\left(1+e^{t}\right)^{2}=\left(1-2e^{t}+e^{2t}\right)+\left(1+2e^{t}+e^{2t}\right)=2+2e^{2t}$, so $L=\int_{-6}^{6}\sqrt{2+2e^{2t}}\,dt$. Set $f(t)=\sqrt{2+2e^{2t}}$. Then by Simpson's Rule with $n=6$ and $\Delta t=\frac{6-(-6)}{6}=2$, we get

$$L\approx\frac{2}{3}[f(-6)+4f(-4)+2f(-2)+4f(0)+2f(2)+4f(4)+f(6)]\approx 612.3053 .$$

50. $x=2a\cot\theta\Rightarrow dx/dt=-2a\csc^{2}\theta$ and $y=2a\sin^{2}\theta\Rightarrow dy/dt=4a\sin\theta\cos\theta=2a\sin 2\theta$.
So $L=\int_{\pi/4}^{\pi/2}\sqrt{4a^{2}\csc^{4}\theta+4a^{2}\sin^{2}2\theta}\;d\theta=2a\int_{\pi/4}^{\pi/2}\sqrt{\csc^{4}\theta+\sin^{2}2\theta}\;d\theta$. Using Simpson's Rule with $n=4$,

$\Delta\theta=\dfrac{\pi/2-\pi/4}{4}=\dfrac{\pi}{16}$, and $f(\theta)=\sqrt{\csc^4\theta+\sin^2 2\theta}$, we get

$L\approx 2a\cdot S_4=(2a)\dfrac{\pi}{16\cdot 3}\left[f\left(\dfrac{\pi}{4}\right)+4f\left(\dfrac{5\pi}{16}\right)+2f\left(\dfrac{3\pi}{8}\right)+4f\left(\dfrac{7\pi}{16}\right)+f\left(\dfrac{\pi}{2}\right)\right]\approx 2.2605a$.

51. $x=\sin^2 t$, $y=\cos^2 t$, $0\le t\le 3\pi$.

$(dx/dt)^2+(dy/dt)^2=(2\sin t\cos t)^2+(-2\cos t\sin t)^2=8\sin^2 t\cos^2 t=2\sin^2 2t\Rightarrow$

Distance $=\int_0^{3\pi}\sqrt{2}\,|\sin 2t|\,dt=6\sqrt{2}\int_0^{\pi/2}\sin 2t\,dt$ [bysymmetry]$=-3\sqrt{2}\,[\cos 2t]_0^{\pi/2}$

$=-3\sqrt{2}\,(-1-1)=6\sqrt{2}$

The full curve is traversed as t goes from 0 to $\dfrac{\pi}{2}$, because the curve is the segment of $x+y=1$ that lies in the first quadrant (since x , $y\ge 0$), and this segment is completely traversed as t goes from 0 to $\dfrac{\pi}{2}$. Thus, $L=\int_0^{\pi/2}\sin 2t\,dt=\sqrt{2}$, as above.

52. $x=\cos^2 t$, $y=\cos t$, $0\le t\le 4\pi$. $\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2=(-2\cos t\sin t)^2+(-\sin t)^2=\sin^2 t(4\cos^2 t+1)$

Distance $=\int_0^{4\pi}|\sin t|\sqrt{4\cos^2 t+1}\,dt=4\int_0^{\pi}\sin t\sqrt{4\cos^2 t+1}\,dt$

$=-4\int_1^{-1}\sqrt{4u^2+1}\,du[u=\cos t,du=-\sin t\,dt]=4\int_{-1}^{1}\sqrt{4u^2+1}\,du=8\int_0^1\sqrt{4u^2+1}\,du$

$=8\int_0^{\tan^{-1}2}\sec\theta\cdot\dfrac{1}{2}\sec^2\theta\,d\theta\ \ 2u=\tan\theta,2du=\sec^2\theta\,d\theta\,]$

$=4\int_0^{\tan^{-1}2}\sec^3\theta\,d\theta=[2\sec\theta\tan\theta+2\ln|\sec\theta+\tan\theta|]_0^{\tan^{-1}2}=4\sqrt{5}+2\ln\left(\sqrt{5}+2\right)$

Thus, $L=\int_0^{\pi}|\sin t|\sqrt{4\cos^2 t+1}\,dt=\sqrt{5}+\dfrac{1}{2}\ln\left(\sqrt{5}+2\right)$.

53. $x=a\sin\theta$, $y=b\cos\theta$, $0\le\theta\le 2\pi$.

$\left(\dfrac{dx}{d\theta}\right)^2+\left(\dfrac{dy}{d\theta}\right)^2=(a\cos\theta)^2+(-b\sin\theta)^2=a^2\cos^2\theta+b^2\sin^2\theta=a^2\left(1-\sin^2\theta\right)+b^2\sin^2\theta$

$=a^2-\left(a^2-b^2\right)\sin^2\theta=a^2-c^2\sin^2\theta=a^2\left(1-\dfrac{c^2}{a^2}\sin^2\theta\right)=a^2\left(1-e^2\sin^2\theta\right)$

So $L=4\int_0^{\pi/2}\sqrt{a^2\left(1-e^2\sin^2\theta\right)}\,d\theta=4a\int_0^{\pi/2}\sqrt{1-e^2\sin^2\theta}\,d\theta$.

54. $x=a\cos^3\theta$, $y=a\sin^3\theta$.

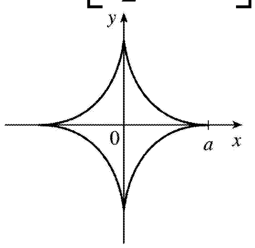
$(dx/d\theta)^2+(dy/d\theta)^2=\left(-3a\cos^2\theta\sin\theta\right)^2+\left(3a\sin^2\theta\cos\theta\right)^2=9a^2\cos^4\theta\sin^2\theta+9a^2\sin^4\theta\cos^2\theta$

$=9a^2\sin^2\theta\cos^2\theta\left(\cos^2\theta+\sin^2\theta\right)=9a^2\sin^2\theta\cos^2\theta$.

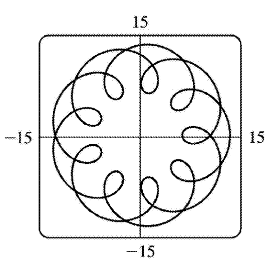
The graph has four–fold symmetry and the curve in the first quadrant corresponds to $0\le\theta\le\pi/2$. Thus,

$L=4\int_0^{\pi/2}3a\sin\theta\cos\theta\,d\theta$ [since $a>0$ and $\sin\theta$ and $\cos\theta$ are positive for $0\le\theta\le\pi/2$]

$=12a\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}=12a\left(\frac{1}{2}-0\right)=6a$.



55. **(a)** Notice that $0\le t\le 2\pi$ does not give the complete curve because $x(0)\ne x(2\pi)$. In fact, we must take $t\in[0,4\pi]$ in order to obtain the complete curve, since the first term in each of the parametric equations has period 2π and the second has period $\frac{2\pi}{11/2}=\frac{4\pi}{11}$, and the least common integer multiple of these two numbers is 4π .

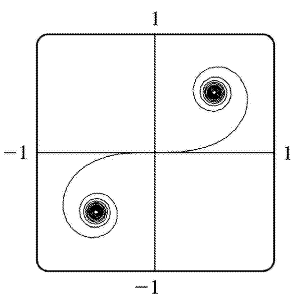


(b) We use the CAS to find the derivatives dx/dt and dy/dt , and then use Formula 1 to find the arc length. Recent versions of Maple express the integral $\int_0^{4\pi}\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt$ as $88E\left(2\sqrt{2}\,i\right)$, where $E(x)$ is the elliptic integral $\int_0^1\frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}}\,dt$ and i is the imaginary number $\sqrt{-1}$. Some earlier versions of Maple (as well as Mathematica) cannot do the integral exactly, so we use the command `evalf(Int(sqrt(diff(x,t)^2+diff(y,t)^2),t=0..4*Pi));` to estimate the length, and find that the arc length is approximately 294.03 . Derive's `Para_arc_length` function in the utility file `Int_apps` simplifies the integral to

$11\int_0^{4\pi}\sqrt{-4\cos t\cos\left(\frac{11t}{2}\right)-4\sin t\sin\left(\frac{11t}{2}\right)+5}\,dt$.

56. **(a)** It appears that as $t\to\infty$, $(x,y)\to\left(\frac{1}{2},\frac{1}{2}\right)$, and as $t\to-\infty$, $(x,y)\to\left(-\frac{1}{2},-\frac{1}{2}\right)$.

(b) By the Fundamental Theorem of Calculus, $dx/dt=\cos\left(\frac{\pi}{2}t^2\right)$ and $dy/dt=\sin\left(\frac{\pi}{2}t^2\right)$, so by Formula 6, the length of the curve from the origin to the point with parameter value t is



$$L=\int_0^t\sqrt{(dx/du)^2+(dy/du)^2}\,du=\int_0^t\sqrt{\cos^2\left(\frac{\pi}{2}u^2\right)+\sin^2\left(\frac{\pi}{2}u^2\right)}\,du$$

$$=\int_0^t 1\,du=t\ [\text{or}-t\ \text{if}\ t<0]$$

We have used u as the dummy variable so as not to confuse it with the upper limit of integration.

57. $x=t-t^2$, $y=\frac{4}{3}t^{3/2}$, $1\le t\le 2$. $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=(1-2t)^2+(2t^{1/2})^2=1-4t+4t^2+4t=1+4t^2$, so $S=\int_1^2 2\pi y\,ds=\int_1^2 2\pi\cdot\frac{4}{3}t^{3/2}\sqrt{1+4t^2}\,dt=\int_1^2\frac{8\pi}{3}t^{3/2}\sqrt{1+4t^2}\,dt$.

58. $x=\sin^2 t$, $y=\sin 3t$, $0\le t\le\frac{\pi}{3}$. $dx/dt=2\sin t\cos t=\sin 2t$ and $dy/dt=3\cos 3t$, so $(dx/dt)^2+(dy/dt)^2=\sin^2 2t+9\cos^2 3t$ and $S=\int 2\pi y\,ds=\int_0^{\pi/3}2\pi\sin 3t\sqrt{\sin^2 2t+9\cos^2 3t}\,dt$.

59. $x=t^3$, $y=t^2$, $0\le t\le 1$. $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(3t^2\right)^2+(2t)^2=9t^4+4t^2$.

$$S=\int_0^1 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_0^1 2\pi t^2\sqrt{9t^4+4t^2}\,dt=2\pi\int_0^1 t^2\sqrt{t^2(9t^2+4)}\,dt$$

$$= 2\pi\int_4^{13}\left(\frac{u-4}{9}\right)\sqrt{u}\left(\frac{1}{18}\,du\right)\left[\begin{array}{cc} u=9t^2+4 & t^2=(u-4)/9 \\ du=18t\,dt & \text{so, } t\,dt=\frac{1}{18}\,du\end{array}\right] = \frac{2\pi}{9\cdot 18}\int_4^{13}(u^{3/2}-4u^{1/2})du$$

$$= \frac{\pi}{81}\left[\frac{2}{5}u^{5/2}-\frac{8}{3}u^{3/2}\right]_4^{13} = \frac{\pi}{81}\cdot\frac{2}{15}[3u^{5/2}-20u^{3/2}]_4^{13}$$

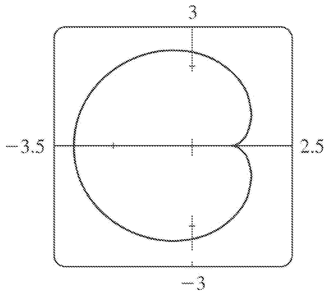
$$= \frac{2\pi}{1215}\left[\left(3\cdot 13^2\sqrt{13}-20\cdot 13\sqrt{13}\right)-(3\cdot 32-20\cdot 8)\right]$$

$$= \frac{2\pi}{1215}\left(247\sqrt{13}+64\right)$$

60. $x=3t-t^3$, $y=3t^2$, $0\le t\le 1$. $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(3-3t^2\right)^2+(6t)^2=9(1+2t^2+t^4)=\left[3(1+t^2)\right]^2$.
$S=\int_0^1 2\pi\cdot 3t^2\cdot 3(1+t^2)dt=18\pi\int_0^1\left(t^2+t^4\right)dt=18\pi\left[\frac{1}{3}t^3+\frac{1}{5}t^5\right]_0^1=\frac{48}{5}\pi$

61. $x=a\cos^3\theta$, $y=a\sin^3\theta$, $0\le\theta\le\frac{\pi}{2}$.
$\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=\left(-3a\cos^2\theta\sin\theta\right)^2+\left(3a\sin^2\theta\cos\theta\right)^2=9a^2\sin^2\theta\cos^2\theta$.
$S=\int_0^{\pi/2}2\pi\cdot a\sin^3\theta\cdot 3a\sin\theta\cos\theta\,d\theta=6\pi a^2\int_0^{\pi/2}\sin^4\theta\cos\theta\,d\theta=\frac{6}{5}\pi a^2\left[\sin^5\theta\right]_0^{\pi/2}=\frac{6}{5}\pi a^2$

62. $\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=(-2\sin\theta+2\sin 2\theta)^2+(2\cos\theta-2\cos 2\theta)^2$
$=4\left[\left(\sin^2\theta-2\sin\theta\sin 2\theta+\sin^2 2\theta\right)+\left(\cos^2\theta-2\cos\theta\cos 2\theta+\cos^2 2\theta\right)\right]$
$=4[1+1-2(\cos 2\theta\cos\theta+\sin 2\theta\sin\theta)]=8[1-\cos(2\theta-\theta)]=8(1-\cos\theta)$
We plot the graph with parameter interval $[0,2\pi]$, and see that we should only integrate between 0 and π . (If the interval $[0,2\pi]$ were taken, the surface of revolution would be generated twice.) Also note that $y=2\sin\theta-\sin 2\theta=2\sin\theta\,(1-\cos\theta)$. So $S=\int_0^{\pi}2\pi\cdot 2\sin\theta\,(1-\cos\theta)2\sqrt{2}\sqrt{1-\cos\theta}\,d\theta$ =
$8\sqrt{2}\pi\int_0^{\pi}(1-\cos\theta)^{3/2}\sin\theta\,d\theta=8\sqrt{2}\pi\int_0^2\sqrt{u^3}\,du$ [where $u=1-\cos\theta$, $du=\sin\theta\,d\theta$] =
$8\sqrt{2}\pi\left[\left(\frac{2}{5}\right)u^{5/2}\right]_0^2=\frac{16}{5}\sqrt{2}\pi(2)^{\frac{5}{2}}=\frac{128}{5}\pi$



63. $x=t+t^3$, $y=t-\frac{1}{t^2}$, $1\le t\le 2$. $\frac{dx}{dt}=1+3t^2$ and $\frac{dy}{dt}=1+\frac{2}{t^3}$, so

$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(1+3t^2\right)^2+\left(1+\frac{2}{t^3}\right)^2$ and

$S=\int 2\pi y\,ds=\int_1^2 2\pi\left(t-\frac{1}{t^2}\right)\sqrt{\left(1+3t^2\right)^2+\left(1+\frac{2}{t^3}\right)^2}\,dt\approx 59.101$.

64. $S=\int_{\pi/4}^{\pi/2}2\pi\cdot 2a\sin^2\theta\sqrt{\csc^4\theta+\sin^2 2\theta}\,dt=4\pi a\int_{\pi/4}^{\pi/2}\sin^2\theta\sqrt{\csc^4\theta+\sin^2 2\theta}\,d\theta$.

Using Simpson's Rule with $n=4$, $\Delta\theta=\frac{\pi/2-\pi/4}{4}=\frac{\pi}{16}$, and $f(\theta)=\sin^2\theta\sqrt{\csc^4\theta+\sin^2 2\theta}$, we get

$S\approx(4\pi a)\frac{\pi}{16\cdot 3}\left[f\left(\frac{\pi}{4}\right)+4f\left(\frac{5\pi}{16}\right)+2f\left(\frac{3\pi}{8}\right)+4f\left(\frac{7\pi}{16}\right)+f\left(\frac{\pi}{2}\right)\right]\approx 11.0893a$.

65. $x=3t^2$, $y=2t^3$, $0\le t\le 5\Rightarrow\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=(6t)^2+(6t^2)^2=36t^2(1+t^2)\Rightarrow$

$$
\begin{aligned}
S &= \int_0^5 2\pi x\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt=\int_0^5 2\pi(3t^2)6t\sqrt{1+t^2}\,dt=18\pi\int_0^5 t^2\sqrt{1+t^2}\,2t\,dt\\
&= 18\pi\int_1^{26}(u-1)\sqrt{u}\,du[\text{where}\,u=1+t^2, du=2t\,dt]=18\pi\int_1^{26}\left(u^{3/2}-u^{1/2}\right)du\\
&= 18\pi\left[\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right]_1^{26}=18\pi\left[\left(\frac{2}{5}\cdot 676\sqrt{26}-\frac{2}{3}\cdot 26\sqrt{26}\right)-\left(\frac{2}{5}-\frac{2}{3}\right)\right]\\
&= \frac{24}{5}\pi\left(949\sqrt{26}+1\right)
\end{aligned}
$$

66. $x=e^t-t$, $y=4e^{t/2}$, $0\le t\le 1$. $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(e^t-1\right)^2+\left(2e^{t/2}\right)^2=e^{2t}+2e^t+1=\left(e^t+1\right)^2$.

$$S=\int_0^1 2\pi\left(e^t-t\right)\sqrt{\left(e^t-1\right)^2+\left(2e^{t/2}\right)^2}\,dt=\int_0^1 2\pi\left(e^t-t\right)\left(e^t+1\right)dt$$

$$= 2\pi \left[\frac{1}{2} e^{2t} + e^{t} - (t-1)\, e^{t} - \frac{1}{2} t^{2} \right]_0^1 = \pi \left(e^2 + 2e - 6 \right)$$

67. If f' is continuous and $f'(t) \neq 0$ for $a \le t \le b$, then either $f'(t) > 0$ for all t in $[a,b]$ or $f'(t) < 0$ for all t in $[a,b]$. Thus, f is monotonic (in fact, strictly increasing or strictly decreasing) on $[a,b]$. It follows that f has an inverse. Set $F = g \circ f^{-1}$, that is, define F by $F(x) = g(f^{-1}(x))$. Then $x = f(t) \Rightarrow f^{-1}(x) = t$, so $y = g(t) = g(f^{-1}(x)) = F(x)$.

68. By Formula .2.5 with $y = F(x)$, $S = \int_a^b 2\pi F(x) \sqrt{1 + [F'(x)]^2}\, dx$. But by Formula .2.2,

$1 + [F'(x)]^2 = 1 + \left(\frac{dy}{dx} \right)^2 = 1 + \left(\frac{dy/dt}{dx/dt} \right)^2 = \frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}$. Using the Substitution Rule with

$x = x(t)$, where $a = x(\alpha)$ and $b = x(\beta)$, we have (since $dx = \frac{dx}{dt}\, dt$)

$$S = \int_\alpha^\beta 2\pi F(x(t)) \sqrt{\frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}}\, \frac{dx}{dt}\, dt = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2}\, dt$$

which is Formula .2.7.

69. **(a)** $\phi = \tan^{-1}\left(\frac{dy}{dx} \right) \Rightarrow \frac{d\phi}{dt} = \frac{d}{dt} \tan^{-1}\left(\frac{dy}{dx} \right) = \frac{1}{1 + (dy/dx)^2} \left[\frac{d}{dt} \left(\frac{dy}{dx} \right) \right]$. But

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}} \Rightarrow \frac{d}{dt} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left(\frac{\dot{y}}{\dot{x}} \right) = \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2} \Rightarrow \frac{d\phi}{dt} = \frac{1}{1 + \left(\dot{y}/\dot{x} \right)^2} \left(\frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2} \right) = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}$$

.

Using the Chain Rule, and the fact that $s = \int_0^t \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2}\, dt \Rightarrow$

$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} = \left(\dot{x}^2 + \dot{y}^2 \right)^{1/2}$, we have that

$$\frac{d\phi}{ds}=\frac{d\phi/dt}{ds/dt}=\left(\frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{\dot{x}^2+\dot{y}^2}\right)\frac{1}{\left(\dot{x}^2+\dot{y}^2\right)^{1/2}}=\frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}$$. So

$$\kappa=\left|\frac{d\phi}{ds}\right|=\left|\frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}\right|=\frac{\left|\dot{x}\ddot{y}-\ddot{x}\dot{y}\right|}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}.$$

(b) $x=x$ and $y=f(x)\Rightarrow \dot{x}=1$, $\ddot{x}=0$ and $\dot{y}=\frac{dy}{dx}$, $\ddot{y}=\frac{d^2y}{dx^2}$. So

$$\kappa=\frac{\left|1\cdot\left(d^2y/dx^2\right)-0\cdot(dy/dx)\right|}{\left[1+(dy/dx)^2\right]^{3/2}}=\frac{\left|d^2y/dx^2\right|}{\left[1+(dy/dx)^2\right]^{3/2}}.$$

70. **(a)** $y=x^2\Rightarrow \frac{dy}{dx}=2x\Rightarrow \frac{d^2y}{dx^2}=2$. So $\kappa=\frac{\left|d^2y/dx^2\right|}{\left[1+(dy/dx)^2\right]^{3/2}}=\frac{2}{\left(1+4x^2\right)^{3/2}}$, and at $(1,1)$,

$$\kappa=\frac{2}{5^{3/2}}=\frac{2}{5\sqrt{5}}.$$

(b) $\kappa'=\frac{d\kappa}{dx}=-3(1+4x^2)^{-5/2}(8x)=0\Leftrightarrow x=0\Rightarrow y=0$. This is a maximum since $\kappa'>0$ for $x<0$ and $\kappa'<0$ for $x>0$. So the parabola $y=x^2$ has maximum curvature at the origin.

71. $x=\theta-\sin\theta\Rightarrow \dot{x}=1-\cos\theta\Rightarrow \ddot{x}=\sin\theta$, and $y=1-\cos\theta\Rightarrow \dot{y}=\sin\theta\Rightarrow \ddot{y}=\cos\theta$. Therefore,

$$\kappa=\frac{\left|\cos\theta-\cos^2\theta-\sin^2\theta\right|}{\left[(1-\cos\theta)^2+\sin^2\theta\right]^{3/2}}=\frac{\left|\cos\theta-\left(\cos^2\theta+\sin^2\theta\right)\right|}{\left(1-2\cos\theta+\cos^2\theta+\sin^2\theta\right)^{3/2}}=\frac{|\cos\theta-1|}{(2-2\cos\theta)^{3/2}}$$. The top of the arch is characterized by a horizontal tangent, and from Example 2(b) in Section .2, the tangent is horizontal when $\theta=(2n-1)\pi$, so take $n=1$ and substitute $\theta=\pi$ into the expression for κ :

$$\kappa=\frac{|\cos\pi-1|}{(2-2\cos\pi)^{3/2}}=\frac{|-1-1|}{[2-2(-1)]^{3/2}}=\frac{1}{4}.$$

72. **(a)** Every straight line has parametrizations of the form $x=a+vt$, $y=b+wt$, where a , b are arbitrary and v , $w\neq 0$. For example, a straight line passing through distinct points (a,b) and (c,d)

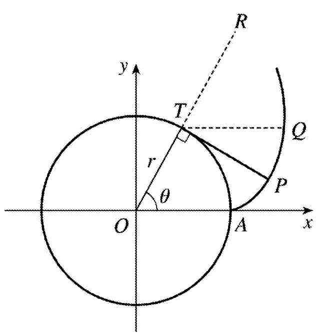
can be described as the parametrized curve $x=a+(c-a)t$, $y=b+(d-b)t$. Starting with $x=a+vt$, $y=b+wt$, we compute $\dot{x}=v$, $\dot{y}=w$, $\ddot{x}=\ddot{y}=0$, and $\kappa=\frac{|v\cdot 0-w\cdot 0|}{\left(v^2+w^2\right)^{3/2}}=0$.

(b) Parametric equations for a circle of radius r are $x=r\cos\theta$ and $y=r\sin\theta$. We can take the center to be the origin. So $\dot{x}=-r\sin\theta\Rightarrow\ddot{x}=-r\cos\theta$ and $\dot{y}=r\cos\theta\Rightarrow\ddot{y}=-r\sin\theta$. Therefore,

$\kappa=\frac{\left|r^2\sin^2\theta+r^2\cos^2\theta\right|}{\left(r^2\sin^2\theta+r^2\cos^2\theta\right)^{3/2}}=\frac{r^2}{r^3}=\frac{1}{r}$. And so for any θ (and thus any point), $\kappa=\frac{1}{r}$.

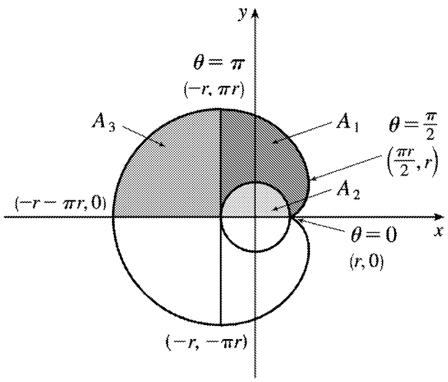
73. The coordinates of T are $(r\cos\theta, r\sin\theta)$. Since TP was unwound from arc TA , TP has length $r\theta$. Also $\angle PTQ=\angle PTR-\angle QTR=\frac{1}{2}\pi-\theta$, so P has coordinates

$x=r\cos\theta+r\theta\cos\left(\frac{1}{2}\pi-\theta\right)=r(\cos\theta+\theta\sin\theta)$, $y=r\sin\theta-r\theta\sin\left(\frac{1}{2}\pi-\theta\right)=r(\sin\theta-\theta\cos\theta)$.



74. If the cow walks with the rope taut, it traces out the portion of the involute in Exercise 73 corresponding to the range $0\le\theta\le\pi$, arriving at the point $(-r,\pi r)$ when $\theta=\pi$. With the rope now fully extended, the cow walks in a semicircle of radius πr , arriving at $(-r,-\pi r)$. Finally, the cow traces out another portion of the involute, namely the reflection about the $x-$ axis of the initial involute path. (This corresponds to the range $-\pi\le\theta\le 0$.) Referring to the figure, we see that the total grazing area is $2(A_1+A_3)$. A_3 is one–quarter of the area of a circle of radius πr , so

$A_3=\frac{1}{4}\pi(\pi r)^2=\frac{1}{4}\pi^3r^2$. We will compute A_1+A_2 and then subtract $A_2=\frac{1}{2}\pi r^2$ to obtain A_1 .



To find A_1+A_2 , first note that the rightmost point of the involute is $\left(\frac{\pi}{2}\text{r,r}\right)$. The leftmost point of the involute is $(-\text{r},\pi r)$. Thus, $A_1+A_2=\int_{\theta=\pi}^{\pi/2} y\,dx-\int_{\theta=0}^{\pi/2} y\,dx=\int_{\theta=\pi}^{0} y\,dx$. Now $y\,dx=r(\sin\theta-\theta\cos\theta)r\theta\cos\theta\,d\theta=r^2(\theta\sin\theta\cos\theta-\theta^2\cos^2\theta)d\theta$. Integrate: $\left(1/r^2\right)\int y\,dx=-\theta\cos^2\theta-\frac{1}{2}\left(\theta^2-1\right)\sin\theta\cos\theta-\frac{1}{6}\theta^3+\frac{1}{2}\theta+C$. This enables us to compute

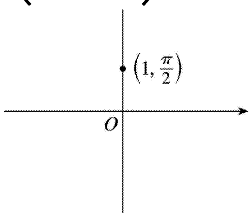
$$A_1+A_2 = r^2\left[-\theta\cos^2\theta-\frac{1}{2}\left(\theta^2-1\right)\sin\theta\cos\theta-\frac{1}{6}\theta^3+\frac{1}{2}\theta\right]_{\pi}^{0}=r^2\left[0-\left(-\pi-\frac{\pi^3}{6}+\frac{\pi}{2}\right)\right]$$

$$= r^2\left(\frac{\pi}{2}+\frac{\pi^3}{6}\right)$$

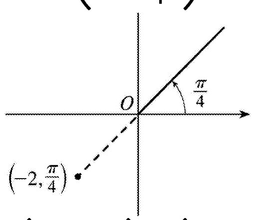
Therefore, $A_1=\left(A_1+A_2\right)-A_2=\frac{1}{6}\pi^3r^2$, so the grazing area is

$$2\left(A_1+A_3\right)=2\left(\frac{1}{6}\pi^3r^2+\frac{1}{4}\pi^3r^2\right)=\frac{5}{6}\pi^3r^2 .$$

1. **(a)** By adding 2π to $\frac{\pi}{2}$, we obtain the point $\left(1, \frac{5\pi}{2}\right)$. The direction opposite $\frac{\pi}{2}$ is $\frac{3\pi}{2}$, so $\left(-1, \frac{3\pi}{2}\right)$ is a point that satisfies the $r<0$ requirement.

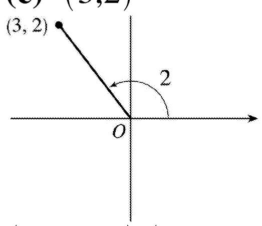


(b) $\left(-2, \frac{\pi}{4}\right)$



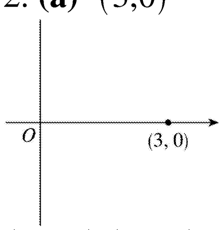
$\left(2, \frac{5\pi}{4}\right), \left(-2, \frac{9\pi}{4}\right)$

(c) $(3,2)$



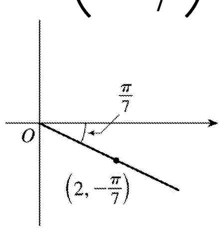
$(3,2+2\pi),(-3,2+\pi)$

2. **(a)** $(3,0)$



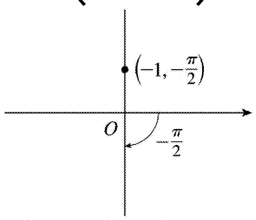
$(3,2\pi),(-3,\pi)$

(b) $\left(2,-\frac{\pi}{7}\right)$

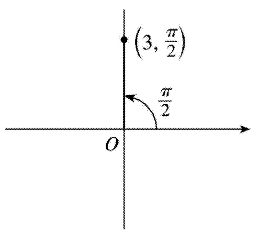


$\left(2,\dfrac{13\pi}{7}\right),\left(-2,\dfrac{6\pi}{7}\right)$

(c) $\left(-1,-\dfrac{\pi}{2}\right)$



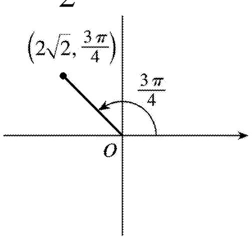
$\left(1,\dfrac{\pi}{2}\right),\left(-1,\dfrac{3\pi}{2}\right)$



3. **(a)**

$x=3\cos\dfrac{\pi}{2}=3(0)=0$ and

$y=3\sin\dfrac{\pi}{2}=3(1)=3$ give us the Cartesian coordinates $(0,3)$.

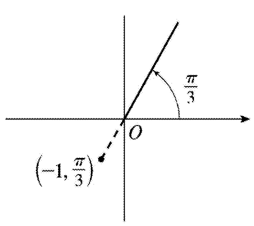


(b)

$x=2\sqrt{2}\cos\dfrac{3\pi}{4}$

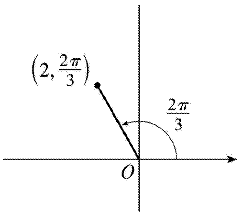
$=2\sqrt{2}\left(-\dfrac{1}{\sqrt{2}}\right)=-2$ and

$y=2\sqrt{2}\sin\dfrac{3\pi}{4}=2\sqrt{2}\left(\dfrac{1}{\sqrt{2}}\right)=2$ give us $(-2,2)$.



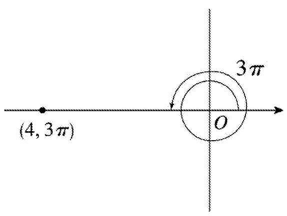
(c)

$x=-1\cos\ \frac{\pi}{3}=-\frac{1}{2}$ and
$y=-1\sin\ \frac{\pi}{3}=-\frac{\sqrt{3}}{2}$ give
us $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$.



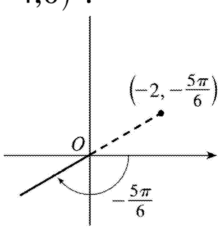
4. **(a)**

$x=2\cos\ \frac{2\pi}{3}=-1$ and $y=2\sin\ \frac{2\pi}{3}=\sqrt{3}$ give us $\left(-1,\sqrt{3}\right)$.



(b)

$x=4\cos\ 3\pi=-4$ and $y=4\sin\ 3\pi=0$ give
us $(-4,0)$.



(c)

$x=-2\cos\left(-\frac{5\pi}{6}\right)=\sqrt{3}$ and $y=-2\sin\left(-\frac{5\pi}{6}\right)=1$ give us $\left(\sqrt{3},1\right)$.

5. **(a)** $x=1$ and $y=1\Rightarrow r=\sqrt{1^2+1^2}=\sqrt{2}$ and $\theta=\tan^{-1}\left(\frac{1}{1}\right)=\frac{\pi}{4}$. Since $(1,1)$ is in the first quadrant, the polar coordinates are (i) $\left(\sqrt{2},\frac{\pi}{4}\right)$ and (ii) $\left(-\sqrt{2},\frac{5\pi}{4}\right)$.

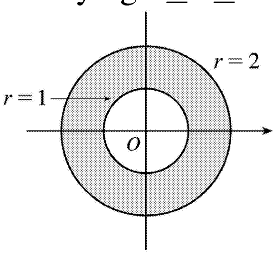
(b) $x=2\sqrt{3}$ and $y=-2\Rightarrow r=\sqrt{\left(2\sqrt{3}\right)^2+(-2)^2}=\sqrt{12+4}=\sqrt{16}=4$ and
$\theta=\tan^{-1}\left(-\frac{2}{2\sqrt{3}}\right)=\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)=-\frac{\pi}{6}$. Since $\left(2\sqrt{3},-2\right)$ is in the fourth quadrant and

$0 \le \theta \le 2\pi$, the polar coordinates are (i) $\left(4, \frac{11\pi}{6}\right)$ and (ii) $\left(-4, \frac{5\pi}{6}\right)$.

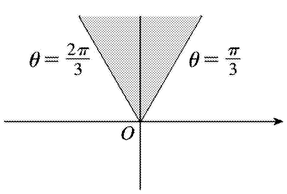
6. **(a)** $(x,y)=(-1,-\sqrt{3})$, $r=\sqrt{1+3}=2$, $\tan\theta = y/x = \sqrt{3}$ and (x,y) is in the third quadrant, so $\theta = \frac{4\pi}{3}$. The polar coordinates are (i) $\left(2, \frac{4\pi}{3}\right)$ and (ii) $\left(-2, \frac{\pi}{3}\right)$.

(b) $(x,y)=(-2,3)$, $r=\sqrt{4+9}=\sqrt{13}$, $\tan\theta = y/x = -\frac{3}{2}$ and (x,y) is in the second quadrant, so $\theta = \tan^{-1}\left(-\frac{3}{2}\right)+\pi$. The polar coordinates are (i) $(\sqrt{13}, \theta)$ and (ii) $(-\sqrt{13}, \theta+\pi)$.

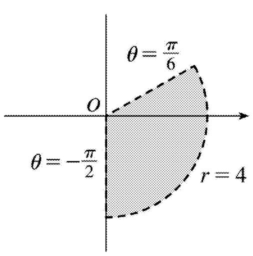
7. The curves $r=1$ and $r=2$ represent circles with center O and radii 1 and 2. The region in the plane satisfying $1 \le r \le 2$ consists of both circles and the shaded region between them in the figure.



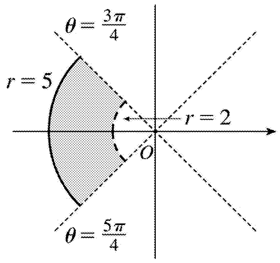
8. $r \ge 0$, $\pi/3 \le \theta \le 2\pi/3$



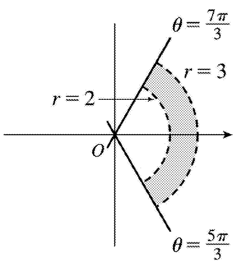
9. The region satisfying $0 \le r < 4$ and $-\pi/2 \le \theta < \pi/6$ does not include the circle $r=4$ nor the line $\theta = \frac{\pi}{6}$.



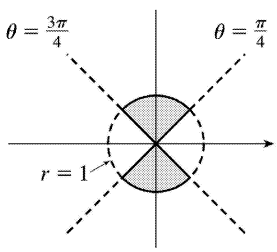
10. $2 < r \le 5$, $3\pi/4 < \theta < 5\pi/4$



11. $2<r<3$, $\frac{5\pi}{3} \le \theta \le \frac{7\pi}{3}$



12. $-1 \le r \le 1$, $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$



13. $(r,\theta)=\left(1, \frac{\pi}{6}\right) \Rightarrow x=1\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$ and $y=1\sin\frac{\pi}{6}=\frac{1}{2}$.

$(r,\theta)=\left(3, \frac{3\pi}{4}\right) \Rightarrow x=3\cos\frac{3\pi}{4}=-\frac{3\sqrt{2}}{2}$ and $y=3\sin\frac{3\pi}{4}=\frac{3\sqrt{2}}{2}$. The distance between them is

$$\sqrt{\left[\frac{\sqrt{3}}{2}-\left(-\frac{3\sqrt{2}}{2}\right)\right]^2+\left(\frac{1}{2}-\frac{3\sqrt{2}}{2}\right)^2}=\sqrt{\frac{1}{4}\left(\sqrt{3}+3\sqrt{2}\right)^2+\frac{1}{4}\left(1-3\sqrt{2}\right)^2}$$

$$=\sqrt{\frac{1}{4}}\sqrt{(3+6\sqrt{6}+18)+(1-6\sqrt{2}+18)}=\frac{1}{2}\sqrt{40+6\sqrt{6}-6\sqrt{2}}$$

14. The points (r_1,θ_1) and (r_2,θ_2) in Cartesian coordinates are $(r_1\cos\theta_1, r_1\sin\theta_1)$ and $(r_2\cos\theta_2, r_2\sin\theta_2)$, respectively.The *square* of the distance between them is

$\left(r_2\cos\theta_2-r_1\cos\theta_1\right)^2+\left(r_2\sin\theta_2-r_1\sin\theta_1\right)^2=$
$\left(r_2^2\cos^2\theta_2-2r_1r_2\cos\theta_1\cos\theta_2+r_1^2\cos^2\theta_1\right)+\left(r_2^2\sin^2\theta_2-2r_1r_2\sin\theta_1\sin\theta_2+r_1^2\sin^2\theta_1\right)=r_1^2\left(\sin^2\theta_1+\cos^2\theta_1\right)+r_2^2$
$=r_1^2-2r_1r_2\cos\left(\theta_1-\theta_2\right)+r_2^2,$

so the distance between them is $\sqrt{r_1^2-2r_1r_2\cos(\theta_1-\theta_2)+r_2^2}$.

15. $r=2\Leftrightarrow\sqrt{x^2+y^2}=2\Leftrightarrow x^2+y^2=4$, a circle of radius 2 centered at the origin.

16. $r\cos\theta=1\Leftrightarrow x=1$, a vertical line.

17. $r=3\sin\theta\Rightarrow r^2=3r\sin\theta\Leftrightarrow x^2+y^2=3y\Leftrightarrow x^2+\left(y-\frac{3}{2}\right)^2=\left(\frac{3}{2}\right)^2$, a circle of radius $\frac{3}{2}$ centered at $\left(0,\frac{3}{2}\right)$. The first two equations are actually equivalent since $r^2=3r\sin\theta\Rightarrow r(r-3\sin\theta)=0\Rightarrow r=0$ or $r=3\sin\theta$. But $r=3\sin\theta$ gives the point $r=0$ (the pole) when $\theta=0$. Thus, the single equation $r=3\sin\theta$ is equivalent to the compound condition ($r=0$ or $r=3\sin\theta$).

18. $r=2\sin\theta+2\cos\theta\Rightarrow r^2=2r\sin\theta+2r\cos\theta\Leftrightarrow x^2+y^2=2y+2x\Leftrightarrow\left(x^2-2x+1\right)+\left(y^2-2y+1\right)=2\Leftrightarrow$ $(x-1)^2+(y-1)^2=2$. The first implication is reversible since $r^2=2r\sin\theta+2r\cos\theta\Rightarrow r=0$ or $r=2\sin\theta+2\cos\theta$, but the curve $r=2\sin\theta+2\cos\theta$ passes through the pole $(r=0)$ when $\theta=-\frac{\pi}{4}$, so $r=2\sin\theta+2\cos\theta$ includes the single point of $r=0$. The curve is a circle of radius $\sqrt{2}$, centered at $(1,1)$.

19. $r=\csc\theta\Leftrightarrow r=\frac{1}{\sin\theta}\Leftrightarrow r\sin\theta=1\Leftrightarrow y=1$, a horizontal line 1 unit above the x –axis.

20. $r=\tan\theta\sec\theta=\frac{\sin\theta}{\cos^2\theta}\Rightarrow r\cos^2\theta=\sin\theta\Leftrightarrow(r\cos\theta)^2=r\sin\theta\Leftrightarrow x^2=y$, a parabola with vertex at the origin opening upward. The first implication is reversible since $\cos\theta=0$ would imply $\sin\theta=r\cos^2\theta=0$, contradicting the fact that $\cos^2\theta+\sin^2\theta=1$.

21. $x=3\Leftrightarrow r\cos\theta=3\Leftrightarrow r=3/\cos\theta\Leftrightarrow r=3\sec\theta$.

22. $x^2+y^2=9 \Leftrightarrow r^2=9 \Leftrightarrow r=3$. [$r=-3$ gives the same curve.]

23. $x=-y^2 \Leftrightarrow r\cos\theta=-r^2\sin^2\theta \Leftrightarrow \cos\theta=-r\sin^2\theta \Leftrightarrow r=-\dfrac{\cos\theta}{\sin^2\theta}=-\cot\theta\csc\theta$.

24. $x+y=9 \Leftrightarrow r\cos\theta+r\sin\theta=9 \Leftrightarrow r=9/(\cos\theta+\sin\theta)$.

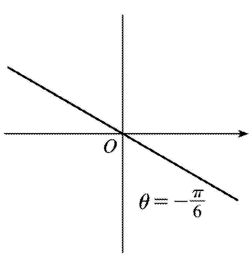
25. $x^2+y^2=2cx \Leftrightarrow r^2=2cr\cos\theta \Leftrightarrow r^2-2cr\cos\theta=0 \Leftrightarrow r(r-2c\cos\theta)=0 \Leftrightarrow r=0$ or $r=2c\cos\theta$. $r=0$ is included in $r=2c\cos\theta$ when $\theta=\dfrac{\pi}{2}+n\pi$, so the curve is represented by the single equation $r=2c\cos\theta$.

26. $x^2-y^2=1 \Leftrightarrow (r\cos\theta)^2-(r\sin\theta)^2=1 \Leftrightarrow r^2\left(\cos^2\theta-\sin^2\theta\right)=1 \Leftrightarrow r^2\cos 2\theta=1 \Rightarrow r^2=\sec 2\theta$

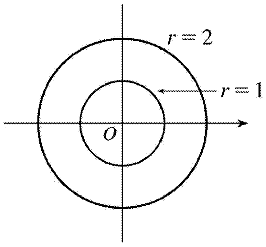
27. **(a)** The description leads immediately to the polar equation $\theta=\dfrac{\pi}{6}$, and the Cartesian equation $\tan\theta=y/x \Rightarrow y=\left(\tan\dfrac{\pi}{6}\right)x=\dfrac{1}{\sqrt{3}}x$ is slightly more difficult to derive.
(b) The easier description here is the Cartesian equation $x=3$.

28. **(a)** Because its center is not at the origin, it is more easily described by its Cartesian equation, $(x-2)^2+(y-3)^2=5^2$.
(b) This circle is more easily given in polar coordinates: $r=4$. The Cartesian equation is also simple: $x^2+y^2=16$.

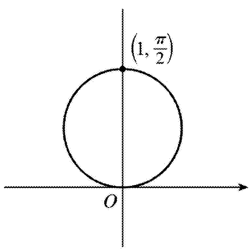
29. $\theta=-\pi/6$



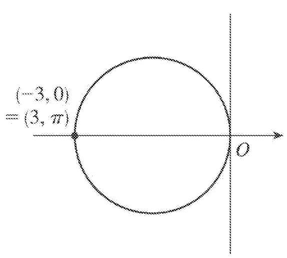
30. $r^2-3r+2=0 \Leftrightarrow (r-1)(r-2)=0 \Leftrightarrow r=1$ or $r=2$



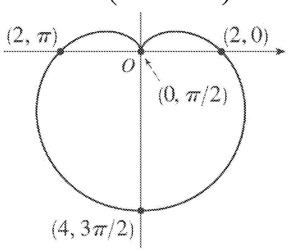
31. $r=\sin\theta \Leftrightarrow r^2=r\sin\theta \Leftrightarrow x^2+y^2=y \Leftrightarrow x^2+\left(y-\frac{1}{2}\right)^2=\left(\frac{1}{2}\right)^2$. The reasoning here is the same as in Exercise 17. This is a circle of radius $\frac{1}{2}$ centered at $\left(0,\frac{1}{2}\right)$.



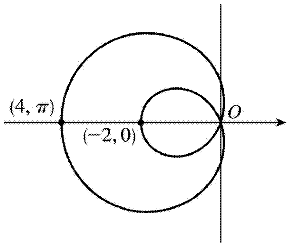
32. $r=-3\cos\theta \Leftrightarrow r^2=-3r\cos\theta \Leftrightarrow x^2+y^2=-3x \Leftrightarrow \left(x+\frac{3}{2}\right)^2+y^2=\left(\frac{3}{2}\right)^2$. This curve is a circle of radius $\frac{3}{2}$ centered at $\left(-\frac{3}{2},0\right)$.



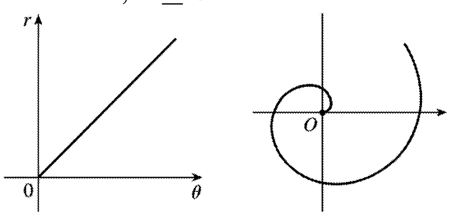
33. $r=2(1-\sin\theta)$. This curve is a cardioid.



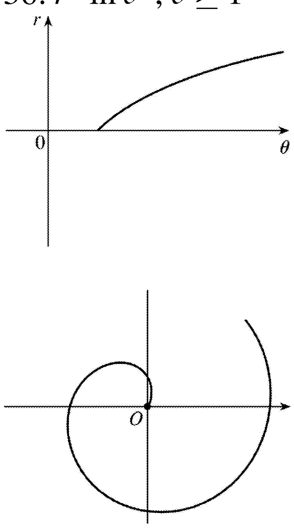
34. $r=1-3\cos\theta$. This is a limacon.



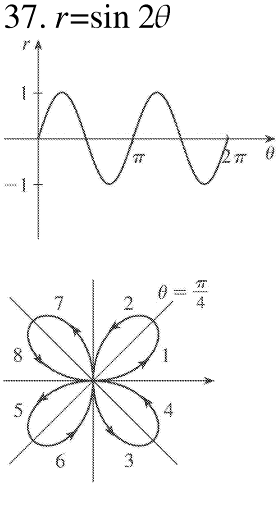
35. $r = \theta \ , \theta \geq 0$



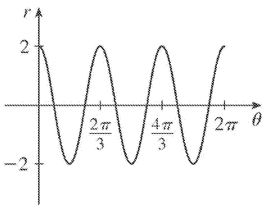
36. $r = \ln \theta \ , \theta \geq 1$

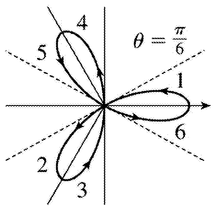


37. $r = \sin 2\theta$

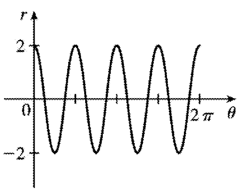


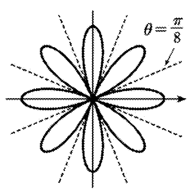
38. r=2cos 3θ



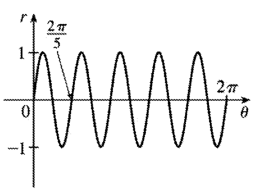


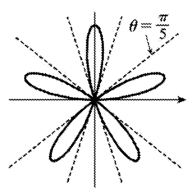
39. r=2cos 4θ



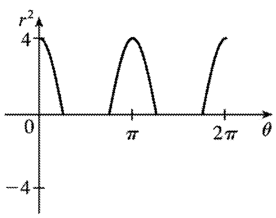


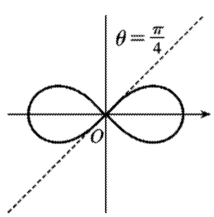
40. r=sin 5θ



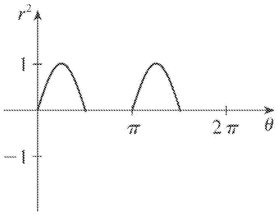


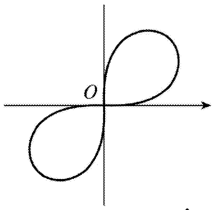
41. r^2=4cos 2θ



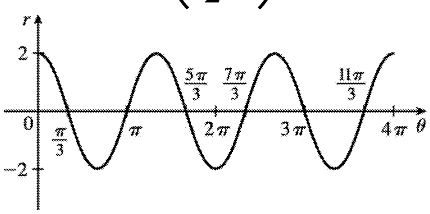


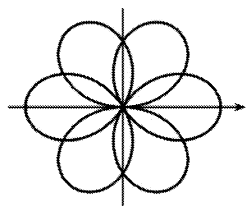
42. r^2=sin 2θ



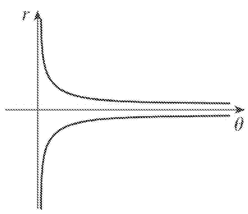


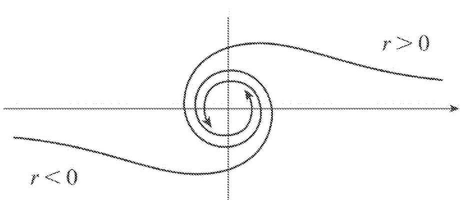
43. $r = 2\cos\left(\frac{3}{2}\theta\right)$



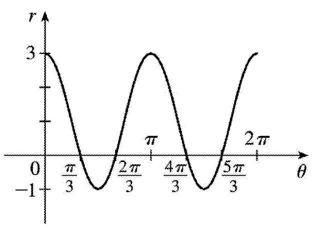


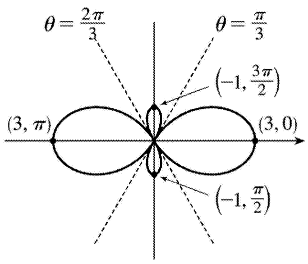
44. $r^2\theta = 1 \Leftrightarrow r = \pm 1/\sqrt{\theta}$ for $\theta > 0$



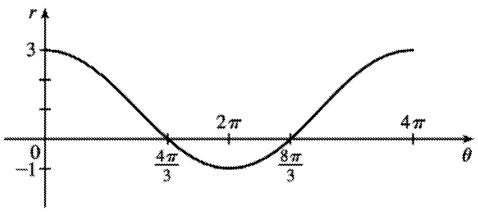


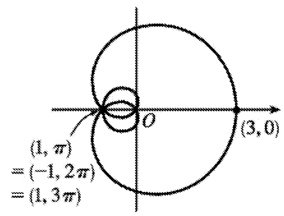
45. $r = 1 + 2\cos 2\theta$



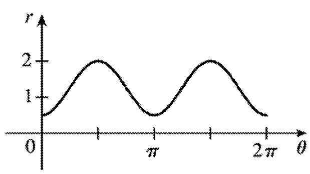


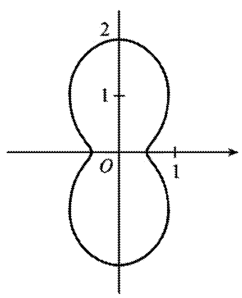
46. $r=1+2\cos(\theta/2)$



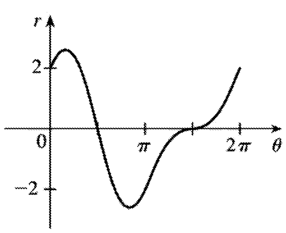


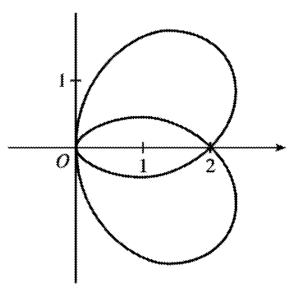
47. For $\theta=0$, π , and 2π , r has its minimum value of about 0.5 . For $\theta=\dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$, r attains its maximum value of 2 . We see that the graph has a similar shape for $0\leq\theta\leq\pi$ and $\pi\leq\theta\leq 2\pi$.





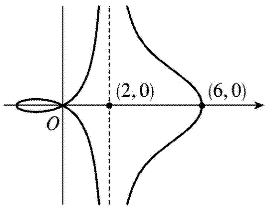
48.





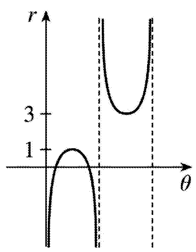
49. $x=r\cos\theta=(4+2\sec\theta)\cos\theta=4\cos\theta+2$. Now, $r\to\infty\Rightarrow(4+2\sec\theta)\to\infty\Rightarrow\theta\to\left(\dfrac{\pi}{2}\right)^{-}$ or $\theta\to\left(\dfrac{3\pi}{2}\right)^{+}$ (since we need only consider $0\leq\theta<2\pi$), so $\lim\limits_{r\to\infty} x=\lim\limits_{\theta\to\pi/2^{-}}(4\cos\theta+2)=2$. Also, $r\to-\infty\Rightarrow(4+2\sec\theta)\to-\infty\Rightarrow\theta\to\left(\dfrac{\pi}{2}\right)^{+}$ or $\theta\to\left(\dfrac{3\pi}{2}\right)^{-}$, so $\lim\limits_{r\to-\infty} x=\lim\limits_{\theta\to\pi/2^{+}}(4\cos\theta+2)=2$.

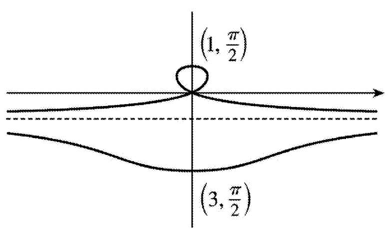
Therefore, $\lim_{r \to \pm\infty} x = 2 \Rightarrow x = 2$ is a vertical asymptote.



50. $y = r\sin\theta = 2\sin\theta - \csc\theta\sin\theta = 2\sin\theta - 1$. $r \to \infty \Rightarrow (2 - \csc\theta) \to \infty \Rightarrow \csc\theta \to -\infty \Rightarrow \theta \to \pi^+$ (since we need only consider $0 \le \theta \le 2\pi$) and so $\lim_{r \to \infty} y = \lim_{\theta \to \pi^+} 2\sin\theta - 1 = -1$. Also $r \to -\infty \Rightarrow (2 - \csc\theta) \to -\infty \Rightarrow$

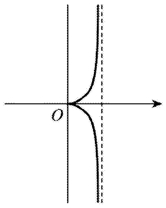
$\csc\theta \to \infty \Rightarrow$





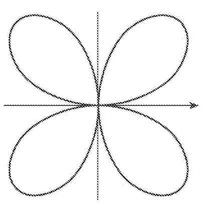
$\theta \to \pi^-$ and so $\lim_{r \to -\infty} x = \lim_{\theta \to \pi^-} 2\sin\theta - 1 = -1$. Therefore $\lim_{r \to \pm\infty} y = -1 \Rightarrow y = -1$ is a horizontal asymptote.

51. To show that $x = 1$ is an asymptote we must prove $\lim_{r \to \pm\infty} x = 1$. $x = r\cos\theta = (\sin\theta\tan\theta)\cos\theta = \sin^2\theta$. Now, $r \to \infty \Rightarrow \sin\theta\tan\theta \to \infty \Rightarrow \theta \to \left(\frac{\pi}{2}\right)^-$, so $\lim_{r \to \infty} x = \lim_{\theta \to \pi/2^-} \sin^2\theta = 1$. Also, $r \to -\infty \Rightarrow$ $\sin\theta\tan\theta \to -\infty \Rightarrow \theta \to \left(\frac{\pi}{2}\right)^+$, so $\lim_{r \to -\infty} x = \lim_{\theta \to \pi/2^+} \sin^2\theta = 1$.



Therefore, $\lim_{r\to\pm\infty} x=1 \Rightarrow x=1$ is a vertical asymptote. Also notice that $x=\sin^2\theta \geq 0$ for all θ, and $x=\sin^2\theta \leq 1$ for all θ. And $x\neq 1$, since the curve is not defined at odd multiples of $\frac{\pi}{2}$. Therefore, the curve lies entirely within the vertical strip $0\leq x<1$.

52. The equation is $\left(x^2+y^2\right)^3=4x^2y^2$, but using polar coordinates we know that $x^2+y^2=r^2$ and $x=r\cos\theta$ and $y=r\sin\theta$. Substituting into the given equation: $r^6=4r^2\cos^2\theta\, r^2\sin^2\theta \Rightarrow$ $r^2=4\cos^2\theta\sin^2\theta \Rightarrow r=\pm 2\cos\theta\sin\theta=\pm\sin 2\theta$. $r=\pm\sin 2\theta$ is sketched at right.



53. **(a)** We see that the curve crosses itself at the origin, where $r=0$ (in fact the inner loop corresponds to negative $r-$ values), so we solve the equation of the limacon for $r=0 \Leftrightarrow c\sin\theta=-1 \Leftrightarrow \sin\theta=-1/c$. Now if $|c|<1$, then this equation has no solution and hence there is no inner loop. But if $c<-1$, then on the interval $(0,2\pi)$ the equation has the two solutions $\theta=\sin^{-1}(-1/c)$ and $\theta=\pi-\sin^{-1}(-1/c)$, and if $c>1$, the solutions are $\theta=\pi+\sin^{-1}(1/c)$ and $\theta=2\pi-\sin^{-1}(1/c)$. In each case, $r<0$ for θ between the two solutions, indicating a loop.
(b) For $0<c<1$, the dimple (if it exists) is characterized by the fact that y has a local maximum at $\theta=\frac{3\pi}{2}$. So we determine for what $c-$ values $\frac{d^2y}{d\theta^2}$ is negative at $\theta=\frac{3\pi}{2}$, since by the Second Derivative Test this indicates a maximum: $y=r\sin\theta=\sin\theta+c\sin^2\theta \Rightarrow$ $\frac{dy}{d\theta}=\cos\theta+2c\sin\theta\cos\theta=\cos\theta+c\sin 2\theta \Rightarrow \frac{d^2y}{d\theta^2}=-\sin\theta+2c\cos 2\theta$. At $\theta=\frac{3\pi}{2}$, this is equal to $-(-1)+2c(-1)=1-2c$, which is negative only for $c>\frac{1}{2}$. A similar argument shows that for $-1<c<0$, y only has a local minimum at

$\theta = \frac{\pi}{2}$ (indicating a dimple) for $c < -\frac{1}{2}$.

54. **(a)** $r = \sin\ (\theta/2)$. This equation must correspond to one of II, III or VI, since these are the only graphs which are bounded. In fact it must be VI, since this is the only graph which is completed after a rotation of exactly 4π .
(b) $r = \sin\ (\theta/2)$. This equation must correspond to one of II, III or VI, since these are the only graphs which are bounded. In fact it must be VI, since this is the only graph which is completed after a rotation of exactly 4π .
(c) $r = \sin\ (\theta/4)$. This equation must correspond to III, since this is the only graph which is completed after a rotation of exactly 8π .
(d) $r = \sin\ (\theta/4)$. This equation must correspond to III, since this is the only graph which is completed after a rotation of exactly 8π .
(e) $r = \sec\ (3\theta)$. This must correspond to IV, since the graph is unbounded at $\theta = \frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{2\pi}{3}$, and so on.
(f) $r = \sec\ (3\theta)$. This must correspond to IV, since the graph is unbounded at $\theta = \frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{2\pi}{3}$, and so on.
(g) $r = \theta \sin \theta$. This must correspond to V. Note that $r = 0$ whenever θ is a multiple of π . This graph is unbounded, and each time θ moves through an interval of 2π , the same basic shape is repeated (because of the periodic $\sin \theta$ factor) but it gets larger each time (since θ increases each time we go around.)
(h) $r = \theta \sin \theta$. This must correspond to V. Note that $r = 0$ whenever θ is a multiple of π . This graph is unbounded, and each time θ moves through an interval of 2π , the same basic shape is repeated (because of the periodic $\sin \theta$ factor) but it gets larger each time (since θ increases each time we go around.)
(i) $r = 1 + 4\cos 5\theta$. This corresponds to II, since it is bounded, has fivefold rotational symmetry, and takes only one takes only one rotation through 2π to be complete.
(j) $r = 1 + 4\cos 5\theta$. This corresponds to II, since it is bounded, has fivefold rotational symmetry, and takes only one takes only one rotation through 2π to be complete.
(k) $r = 1/\sqrt{\theta}$. This corresponds to I, since it is unbounded at $\theta = 0$, and r decreases as θ increases; in fact $r \to 0$ as $\theta \to \infty$.
(l) $r = 1/\sqrt{\theta}$. This corresponds to I, since it is unbounded at $\theta = 0$, and r decreases as θ increases; in fact $r \to 0$ as $\theta \to \infty$.

55. $r = 2\sin \theta \Rightarrow x = r\cos \theta = 2\sin \theta \cos \theta = \sin 2\theta$, $y = r\sin \theta = 2\sin^2 \theta \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2 \cdot 2\sin \theta \cos \theta}{\cos 2\theta \cdot 2} = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$$

When $\theta = \frac{\pi}{6}$, $\frac{dy}{dx} = \tan\left(2\cdot\frac{\pi}{6}\right) = \tan\frac{\pi}{3} = \sqrt{3}$.

56. $r = 2 - \sin\theta \Rightarrow x = r\cos\theta = (2-\sin\theta)\cos\theta$, $y = r\sin\theta = (2-\sin\theta)\sin\theta \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(2-\sin\theta)\cos\theta + \sin\theta(-\cos\theta)}{(2-\sin\theta)(-\sin\theta) + \cos\theta(-\cos\theta)} = \frac{2\cos\theta - 2\sin\theta\cos\theta}{-2\sin\theta + \sin^2\theta - \cos^2\theta} = \frac{2\cos\theta - \sin 2\theta}{-2\sin\theta - \cos 2\theta}$$

When $\theta = \frac{\pi}{3}$, $\frac{dy}{dx} = \frac{2(1/2) - \left(\sqrt{3}/2\right)}{-2\left(\sqrt{3}/2\right) - (-1/2)} = \frac{1 - \sqrt{3}/2}{-\sqrt{3} + 1/2}\cdot\frac{2}{2} = \frac{2-\sqrt{3}}{1-2\sqrt{3}}$.

57. $r = 1/\theta \Rightarrow x = r\cos\theta = (\cos\theta)/\theta$, $y = r\sin\theta = (\sin\theta)/\theta \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta(-1/\theta^2) + (1/\theta)\cos\theta}{\cos\theta(-1/\theta^2) - (1/\theta)\sin\theta}\cdot\frac{\theta^2}{\theta^2} = \frac{-\sin\theta + \theta\cos\theta}{-\cos\theta - \theta\sin\theta}$$

When $\theta = \pi$, $\frac{dy}{dx} = \frac{-0 + \pi(-1)}{-(-1) - \pi(0)} = \frac{-\pi}{1} = -\pi$.

58. $r = \ln\theta \Rightarrow x = r\cos\theta = \ln\theta\cos\theta$, $y = r\sin\theta = \ln\theta\sin\theta \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta\,(1/\theta) + \ln\theta\cos\theta}{\cos\theta\,(1/\theta) - \ln\theta\sin\theta}\cdot\frac{\theta}{\theta} = \frac{\sin\theta + \theta\ln\theta\cos\theta}{\cos\theta - \theta\ln\theta\sin\theta}$$

When $\theta = e$, $\frac{dy}{dx} = \frac{\sin e + e\ln e\cos e}{\cos e - e\ln e\sin e} = \frac{\sin e + e\cos e}{\cos e - e\sin e}$.

59. $r = 1 + \cos\theta \Rightarrow x = r\cos\theta = \cos\theta + \cos^2\theta$, $y = r\sin\theta = \sin\theta + \sin\theta\cos\theta \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - 2\cos\theta\sin\theta} = \frac{\cos\theta + \cos 2\theta}{-\sin\theta - \sin 2\theta}$$

When $\theta = \frac{\pi}{6}$, $\frac{dy}{dx} = \frac{\frac{\sqrt{3}}{2} + \frac{1}{2}}{-\frac{1}{2} - \frac{\sqrt{3}}{2}} = \frac{\frac{\sqrt{3}}{2} + \frac{1}{2}}{-\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)} = -1$.

60. $r = \sin 3\theta \Rightarrow x = r\cos\theta = \sin 3\theta\cos\theta$, $y = r\sin\theta = \sin 3\theta\sin\theta \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos 3\theta\sin\theta + \sin 3\theta\cos\theta}{3\cos 3\theta\cos\theta - \sin 3\theta\sin\theta}$$

When $\theta = \frac{\pi}{6}$, $\frac{dy}{dx} = \frac{3(0)(1/2) + 1\left(\sqrt{3}/2\right)}{3(0)\left(\sqrt{3}/2\right) - 1(1/2)} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$.

61. $r=3\cos\theta \Rightarrow x=r\cos\theta=3\cos\theta\cos\theta$, $y=r\sin\theta=3\cos\theta\sin\theta \Rightarrow dy/d\theta=-3\sin^2\theta+3\cos^2\theta=3\cos 2\theta=0$ $\Rightarrow 2\theta=\frac{\pi}{2}$ or $\frac{3\pi}{2} \Leftrightarrow \theta=\frac{\pi}{4}$ or $\frac{3\pi}{4}$. So the tangent is horizontal at $\left(\frac{3}{\sqrt{2}},\frac{\pi}{4}\right)$ and $\left(-\frac{3}{\sqrt{2}},\frac{3\pi}{4}\right)$ $\left[\text{sameas}\left(\frac{3}{\sqrt{2}},-\frac{\pi}{4}\right)\right]$. $dx/d\theta=-6\sin\theta\cos\theta=-3\sin 2\theta=0 \Rightarrow 2\theta=0$ or $\pi \Leftrightarrow \theta=0$ or $\frac{\pi}{2}$. So the tangent is vertical at $(3,0)$ and $\left(0,\frac{\pi}{2}\right)$.

62. $y=r\sin\theta=\cos\theta\sin\theta+\sin^2\theta=\frac{1}{2}\sin 2\theta+\sin^2\theta \Rightarrow dy/d\theta=\cos 2\theta+\sin 2\theta=0 \Rightarrow \tan 2\theta=-1 \Rightarrow 2\theta=\frac{3\pi}{4}$ or $\frac{7\pi}{4} \Leftrightarrow \theta=\frac{3\pi}{8}$ or $\frac{7\pi}{8} \Rightarrow$ horizontal tangents at $\left(\cos\frac{3\pi}{8}+\sin\frac{3\pi}{8},\frac{3\pi}{8}\right)$ and $\left(\cos\frac{7\pi}{8}+\sin\frac{7\pi}{8},\frac{7\pi}{8}\right)$. $x=r\cos\theta=\cos^2\theta+\cos\theta\sin\theta \Rightarrow dx/d\theta=-\sin 2\theta+\cos 2\theta=0 \Rightarrow \tan 2\theta=1 \Rightarrow$ $2\theta=\frac{\pi}{4}$ or $\frac{5\pi}{4} \Leftrightarrow \theta=\frac{\pi}{8}$ or $\frac{5\pi}{8} \Rightarrow$ vertical tangents at $\left(\cos\frac{\pi}{8}+\sin\frac{\pi}{8},\frac{\pi}{8}\right)$ and $\left(\cos\frac{5\pi}{8}+\sin\frac{5\pi}{8},\frac{5\pi}{8}\right)$.

Note: These expressions can be simplified using trigonometric identities. For example, $\cos\frac{\pi}{8}+\sin\frac{\pi}{8}=\frac{1}{2}\sqrt{4+2\sqrt{2}}$.

63. $r=1+\cos\theta \Rightarrow x=r\cos\theta=\cos\theta(1+\cos\theta)$, $y=r\sin\theta=\sin\theta(1+\cos\theta) \Rightarrow$ $dy/d\theta=(1+\cos\theta)\cos\theta-\sin^2\theta=2\cos^2\theta+\cos\theta-1=(2\cos\theta-1)(\cos\theta+1)=0 \Rightarrow \cos\theta=\frac{1}{2}$ or $-1 \Rightarrow \theta=\frac{\pi}{3}$, π , or $\frac{5\pi}{3} \Rightarrow$ horizontal tangent at $\left(\frac{3}{2},\frac{\pi}{3}\right)$, $(0,\pi)$[the pole] , and $\left(\frac{3}{2},\frac{5\pi}{3}\right)$. $dx/d\theta=-(1+\cos\theta)\sin\theta-\cos\theta\sin\theta=-\sin\theta(1+2\cos\theta)=0 \Rightarrow \sin\theta=0$ or $\cos\theta=-\frac{1}{2} \Rightarrow \theta=0$, π , $\frac{2\pi}{3}$, or $\frac{4\pi}{3} \Rightarrow$ vertical tangent at $(2,0)$, $\left(\frac{1}{2},\frac{2\pi}{3}\right)$, and $\left(\frac{1}{2},\frac{4\pi}{3}\right)$. Note that the tangent is horizontal, not vertical when $\theta=\pi$, since $\lim_{\theta\to\pi}\frac{dy/d\theta}{dx/d\theta}=0$.

64. $\frac{dy}{d\theta}=e^{\theta}\sin\theta+e^{\theta}\cos\theta=e^{\theta}(\sin\theta+\cos\theta)=0 \Rightarrow \sin\theta=-\cos\theta \Rightarrow \tan\theta=-1 \Rightarrow \theta=-\frac{1}{4}\pi+n\pi$ (n any integer) $\Rightarrow$ horizontal tangents at $\left(e^{\pi(n-1/4)},\pi\left(n-\frac{1}{4}\right)\right)$.
$\frac{dx}{d\theta}=e^{\theta}\cos\theta-e^{\theta}\sin\theta=e^{\theta}(\cos\theta-\sin\theta)=0 \Rightarrow \sin\theta=\cos\theta \Rightarrow \tan\theta=1 \Rightarrow \theta=\frac{1}{4}\pi+n\pi$ (n any integer) $\Rightarrow$

vertical tangents at $\left(e^{\pi(n+1/4)},\pi\left(n+\frac{1}{4}\right)\right)$.

65. $r=\cos 2\theta \Rightarrow x=r\cos\theta=\cos 2\theta\cos\theta$, $y=r\sin\theta=\cos 2\theta\sin\theta \Rightarrow$

$$dy/d\theta=-2\sin 2\theta\sin\theta+\cos 2\theta\cos\theta=-4\sin^2\theta\cos\theta+\left(\cos^3\theta-\sin^2\theta\cos\theta\right)=$$
$$\cos\theta\left(\cos^2\theta-5\sin^2\theta\right)=\cos\theta\left(1-6\sin^2\theta\right)=0\Rightarrow$$

$\cos\theta=0$ or $\sin\theta=\pm\frac{1}{\sqrt{6}}\Rightarrow\theta=\frac{\pi}{2}$, $\frac{3\pi}{2}$, α , $\pi-\alpha$, $\pi+\alpha$, or $2\pi-\alpha$ (where $\alpha=\sin^{-1}\frac{1}{\sqrt{6}}$).
So the tangent is horizontal at $\left(-1,\frac{\pi}{2}\right)$, $\left(-1,\frac{3\pi}{2}\right)$, $\left(\frac{2}{3},\alpha\right)$, $\left(\frac{2}{3},\pi-\alpha\right)$, $\left(\frac{2}{3},\pi+\alpha\right)$, and $\left(\frac{2}{3},2\pi-\alpha\right)$.

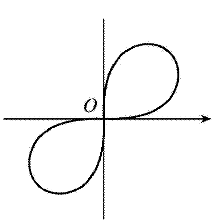
$$dx/d\theta=-2\sin 2\theta\cos\theta-\cos 2\theta\sin\theta=-4\sin\theta\cos^2\theta-(2\cos^2\theta-1)\sin\theta=\sin\theta(1-6\cos^2\theta)=0\Rightarrow$$

$\sin\theta=0$ or $\cos\theta=\pm\frac{1}{\sqrt{6}}\Rightarrow\theta=0$, π , β , $\pi-\beta$, $\pi+\beta$, or $2\pi-\beta$ (where $\beta=\cos^{-1}\frac{1}{\sqrt{6}}$).
So the tangent is vertical at $(1,0)$, $(1,\pi)$, $\left(-\frac{2}{3},\beta\right)$, $\left(-\frac{2}{3},\pi-\beta\right)$, $\left(-\frac{2}{3},\pi+\beta\right)$, and $\left(-\frac{2}{3},2\pi-\beta\right)$.

66.

By differentiating implicitly, $r^2=\sin 2\theta \Rightarrow$
$2r(dr/d\theta)=2\cos 2\theta \Rightarrow dr/d\theta=(1/r)\cos 2\theta$, so $\frac{dy}{d\theta}$ $=\frac{1}{r}\cos 2\theta\sin\theta+r\cos\theta=\frac{1}{r}\left(\cos 2\theta\sin\theta+r^2\cos\theta\right)$

$=\frac{1}{r}(\cos 2\theta\sin\theta+\sin 2\theta\cos\theta)=\frac{1}{r}\sin 3\theta$



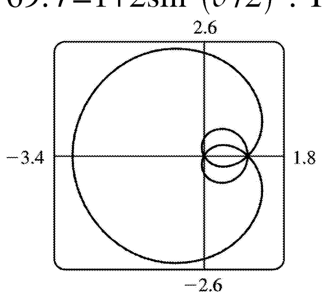
This is 0 when $\sin 3\theta=0\Rightarrow\theta=0$,

$\frac{\pi}{3}$ or $\frac{4\pi}{3}$ (restricting θ to the domain of the lemniscate), so there are horizontal tangents at $\left(\sqrt[4]{\frac{3}{4}}, \frac{\pi}{3}\right)$, $\left(\sqrt[4]{\frac{3}{4}}, \frac{4\pi}{3}\right)$ and $(0,0)$. Similarly, $dx/d\theta = (1/r)\cos 3\theta = 0$ when $\theta = \frac{\pi}{6}$ or $\frac{7\pi}{6}$, so there are vertical tangents at $\left(\sqrt[4]{\frac{3}{4}}, \frac{\pi}{6}\right)$ and $\left(\sqrt[4]{\frac{3}{4}}, \frac{7\pi}{6}\right)$.

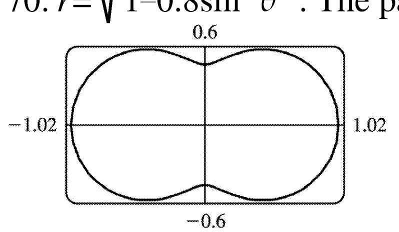
67. $r = a\sin\theta + b\cos\theta \Rightarrow r^2 = ar\sin\theta + br\cos\theta \Rightarrow x^2 + y^2 = ay + bx \Rightarrow$
$x^2 - bx + \left(\frac{1}{2}b\right)^2 + y^2 - ay + \left(\frac{1}{2}a\right)^2 = \left(\frac{1}{2}b\right)^2 + \left(\frac{1}{2}a\right)^2 \Rightarrow \left(x - \frac{1}{2}b\right)^2 + \left(y - \frac{1}{2}a\right)^2 = \frac{1}{4}\left(a^2 + b^2\right)$, and this is a circle with center $\left(\frac{1}{2}b, \frac{1}{2}a\right)$ and radius $\frac{1}{2}\sqrt{a^2 + b^2}$.

68. These curves are circles which intersect at the origin and at $\left(\frac{1}{\sqrt{2}}a, \frac{\pi}{4}\right)$. At the origin, the first circle has a horizontal tangent and the second a vertical one, so the tangents are perpendicular here. For the first circle ($r = a\sin\theta$), $dy/d\theta = a\cos\theta\sin\theta + a\sin\theta\cos\theta = a\sin 2\theta = a$ at $\theta = \frac{\pi}{4}$ and $dx/d\theta = a\cos^2\theta - a\sin^2\theta = a\cos 2\theta = 0$ at $\theta = \frac{\pi}{4}$, so the tangent here is vertical. Similarly, for the second circle ($r = a\cos\theta$), $dy/d\theta = a\cos 2\theta = 0$ and $dx/d\theta = -a\sin 2\theta = -a$ at $\theta = \frac{\pi}{4}$, so the tangent is horizontal, and again the tangents are perpendicular.

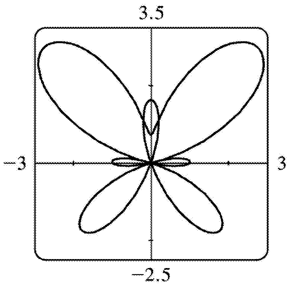
69. $r = 1 + 2\sin(\theta/2)$. The parameter interval is $[0, 4\pi]$.



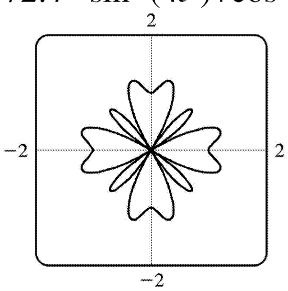
70. $r = \sqrt{1 - 0.8\sin^2\theta}$. The parameter interval is $[0, 2\pi]$.



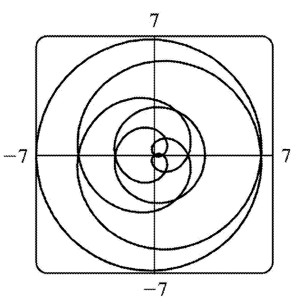
71. $r=e^{\sin\theta}-2\cos(4\theta)$. The parameter interval is $[0,2\pi]$.



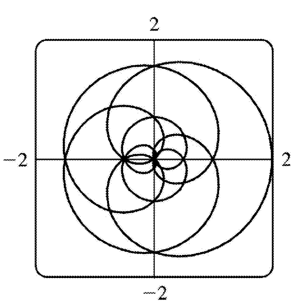
72. $r=\sin^2(4\theta)+\cos(4\theta)$. The parameter interval is $[0,2\pi]$.



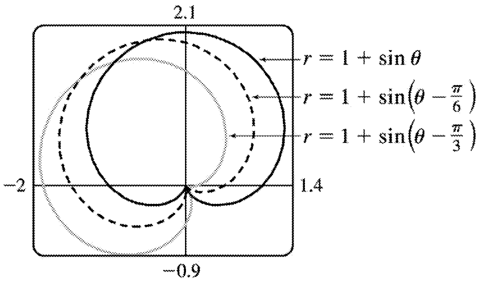
73. $r=2-5\sin(\theta/6)$. The parameter interval is $[-6\pi,6\pi]$.



74. $r=\cos(\theta/2)+\cos(\theta/3)$. The parameter interval is $[-6\pi,6\pi]$.

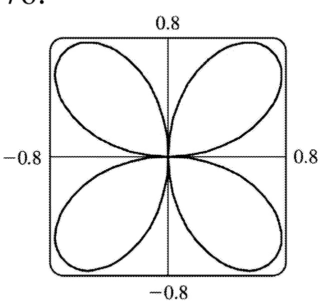


75.



It appears that the graph of $r=1+\sin\left(\theta-\frac{\pi}{6}\right)$ is the same shape as the graph of $r=1+\sin\theta$, but rotated counterclockwise about the origin by $\frac{\pi}{6}$. Similarly, the graph of $r=1+\sin\left(\theta-\frac{\pi}{3}\right)$ is rotated by $\frac{\pi}{3}$. In general, the graph of $r=f(\theta-\alpha)$ is the same shape as that of $r=f(\theta)$, but rotated counterclockwise through α about the origin. That is, for any point $\left(r_0,\theta_0\right)$ on the curve $r=f(\theta)$, the point $\left(r_0,\theta_0+\alpha\right)$ is on the curve $r=f(\theta-\alpha)$, since $r_0=f\left(\theta_0\right)=f\left(\left(\theta_0+\alpha\right)-\alpha\right)$.

76.



From the graph, the highest points seem to have $y\approx 0.77$. To find the exact value, we solve $dy/d\theta=0$.
$y=r\sin\theta=\sin\theta\sin 2\theta \Rightarrow$

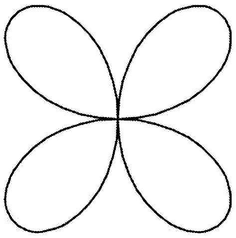
$$\begin{aligned} dy/d\theta &= 2\sin\theta\cos 2\theta+\cos\theta\sin 2\theta \\ &= 2\sin\theta\left(2\cos^2\theta-1\right)+\cos\theta\,(2\sin\theta\cos\theta) \\ &= 2\sin\theta\left(3\cos^2\theta-1\right) \end{aligned}$$

In the first quadrant, this is 0 when $\cos\theta=\frac{1}{\sqrt{3}} \Leftrightarrow \sin\theta=\sqrt{\frac{2}{3}} \Leftrightarrow$

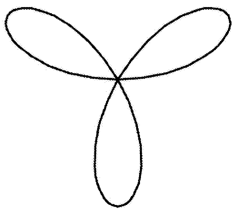
$y=2\sin^2\theta\cos\theta=2\cdot\frac{2}{3}\cdot\frac{1}{\sqrt{3}}=\frac{4\sqrt{3}}{9}\approx 0.77$.

77. **(a)** $r=\sin n\theta$. From the graphs, it seems that when n is even, the number of loops in the curve (called a rose) is $2n$, and when n is odd, the number of loops is simply n.
This is because in the case of n odd, every point on the graph is traversed twice, due to the fact that

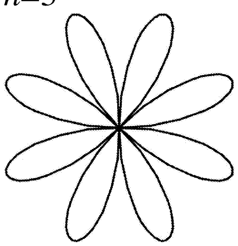
$$r(\theta+\pi)=\sin\left[n(\theta+\pi)\right]=\sin n\theta\cos n\pi+\cos n\theta\sin n\pi=\begin{cases}\sin n\theta & \text{if } n \text{ is even}\\ -\sin n\theta & \text{if } n \text{ is odd}\end{cases}$$



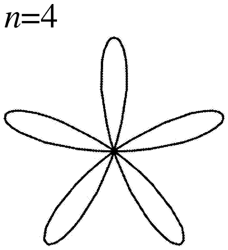
n=2



n=3

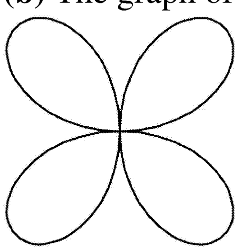


n=4

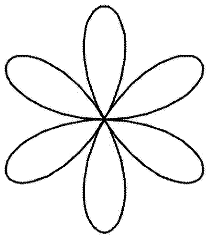


n=5

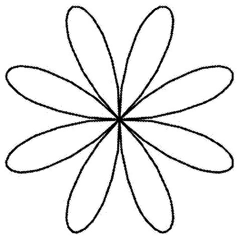
(b) The graph of $r=|\sin n\theta|$ has $2n$ loops whether n is odd or even, since $r(\theta+\pi)=r(\theta)$.



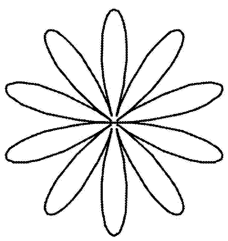
n=2



$n=3$



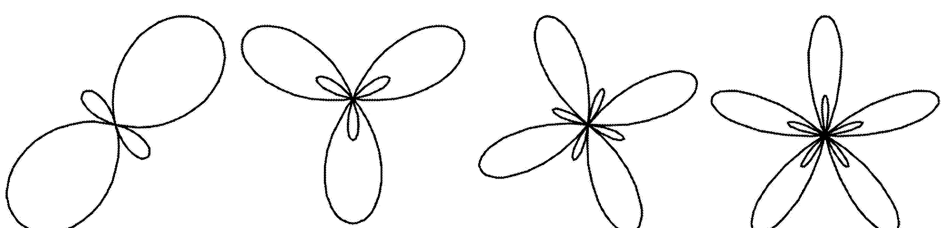
$n=4$



$n=5$

78. $r=1+c\sin n\theta$. We vary n while keeping c constant at 2 . As n changes, the curves change in the same way as those in Exercise 77: the number of loops increases. Note that if n is even, the smaller loops are outside the larger ones; if n is odd, they are inside.

$c=2$

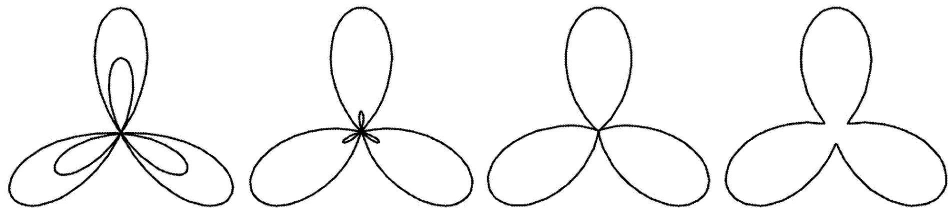


$n=2$ $n=3$ $n=4$ $n=5$

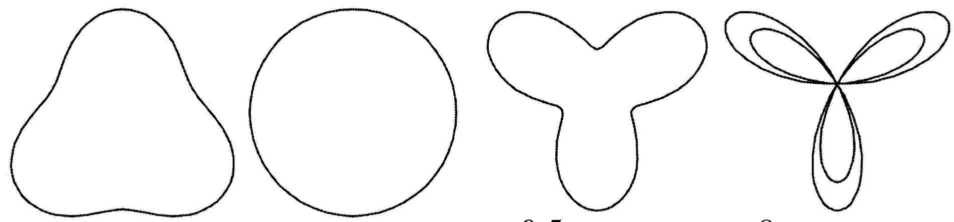
Now we vary c while keeping $n=3$. As c increases toward 0 , the entire graph gets smaller (the graphs below are not to scale) and the smaller loops shrink in relation to the large ones. At $c=-1$, the small loops disappear entirely, and for $-1<c<1$, the graph is a simple, closed curve (at $c=0$ it is a circle). As c continues to increase, the same changes are seen, but in reverse order, since $1+(-c)\sin n\theta=1+c\sin n(\theta+\pi)$, so the graph for $c=c_0$ is the same as that for $c=-c_0$, with a rotation through π . As $c\to\infty$, the smaller loops get relatively closer in size to the large ones. Note that the distance between the outermost points of corresponding inner and outer loops is always 2 .

Maple's `animate` command (or Mathematica's `Animate`) is very useful for seeing the changes that occur as c varies.

$n=3$

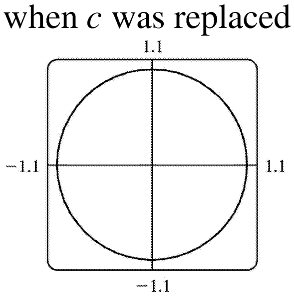


$c=-4$ $c=-1.4$ $c=-1$ $c=-0.8$

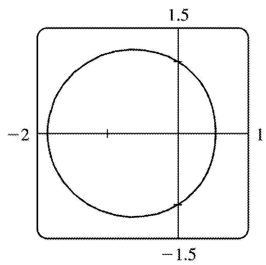


$c=-0.2$ $c=0$ $c=0.5$ $c=8$

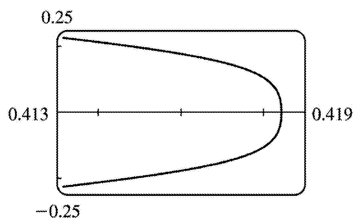
79. $r=\dfrac{1-a\cos\theta}{1+a\cos\theta}$. We start with $a=0$, since in this case the curve is simply the circle $r=1$. As a increases, the graph moves to the left, and its right side becomes flattened. As a increases through about 0.4 , the right side seems to grow a dimple, which upon closer investigation (with narrower $\theta-$ ranges) seems to appear at $a\approx 0.42$ (the actual value is $\sqrt{2}-1$). As $a\to 1$, this dimple becomes more pronounced, and the curve begins to stretch out horizontally, until at $a=1$ the denominator vanishes at $\theta=\pi$, and the dimple becomes an actual cusp. For $a>1$ we must choose our parameter interval carefully, since $r\to\infty$ as $1+a\cos\theta\to 0\Leftrightarrow\theta\to\pm\cos^{-1}(-1/a)$. As a increases from 1 , the curve splits into two parts. The left part has a loop, which grows larger as a increases, and the right part grows broader vertically, and its left tip develops a dimple when $a\approx 2.42$ (actually, $\sqrt{2}+1$). As a increases, the dimple grows more and more pronounced. If $a<0$, we get the same graph as we do for the corresponding positive $a-$ value, but with a rotation through π about the pole, as happened when c was replaced with $-c$ in Exercise 78.



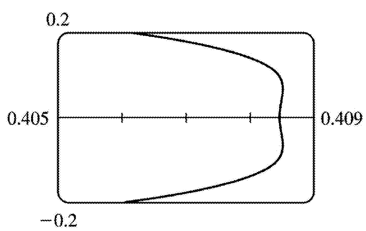
$a=0$



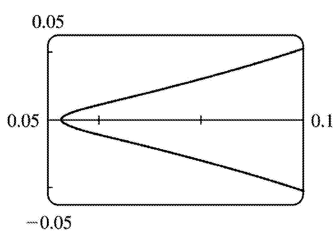
a=0.3



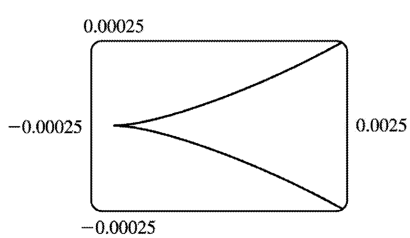
a=0.41,$|\theta| \leq 0.5$



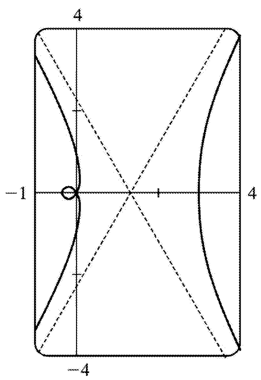
a=0.42,$|\theta| \leq 0.5$



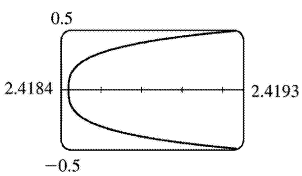
a=0.9,$|\theta| \leq 0.5$



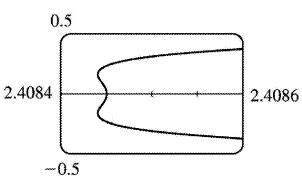
a=1,$|\theta| \leq 0.1$



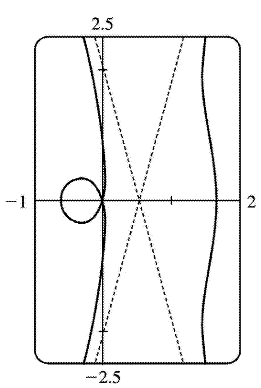
a=2



a=2.41,$|\theta - \pi| \leq 0.2$



a=2.42,$|\theta - \pi| \leq 0.2$



a=4

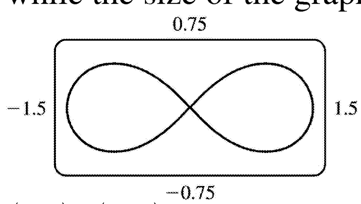
80. Most graphing devices cannot plot implicit polar equations, so we must first find an explicit expression (or expressions) for r in terms of θ , a , and c . We note that the given equation is a quadratic in r^2 , so we use the quadratic formula and find that

$$r^2=\frac{2c^2\cos 2\theta \pm\sqrt{4c^4\cos^2 2\theta-4\left(c^4-a^4\right)}}{2}=c^2\cos 2\theta\pm\sqrt{a^4-c^4\sin^2 2\theta}\quad\text{so}$$

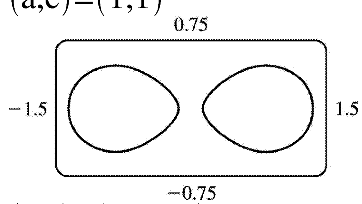
$$r=\pm\sqrt{c^2\cos 2\theta\pm\sqrt{a^4-c^4\sin^2 2\theta}}\ .$$

So for each graph, we must plot four curves to be sure of plotting all the points which satisfy the given equation. Note that all four functions have period π .

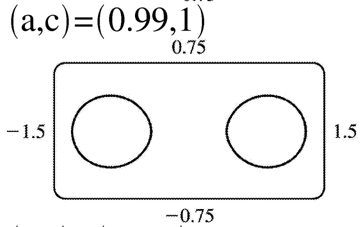
We start with the case $a=c=1$, and the resulting curve resembles the symbol for infinity. If we let a decrease, the curve splits into two symmetric parts, and as a decreases further, the parts become smaller, further apart, and rounder. If instead we let a increase from 1 , the two lobes of the curve join together, and as a increases further they continue to merge, until at $a\approx 1.4$, the graph no longer has dimples, and has an oval shape. As $a\to\infty$, the oval becomes larger and rounder, since the c^2 and c^4 terms lose their significance. Note that the shape of the graph seems to depend only on the ratio c/a , while the size of the graph varies as c and a jointly increase.



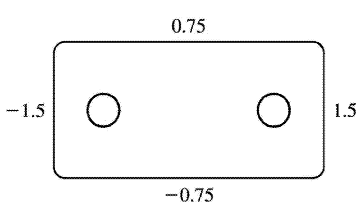
(a,c)=(1,1)



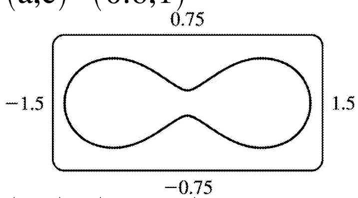
(a,c)=(0.99,1)



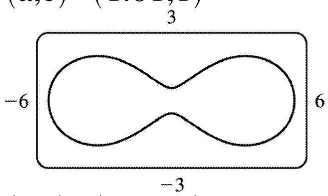
(a,c)=(0.9,1)



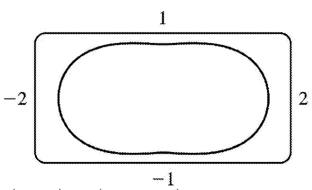
(a,c)=(0.6,1)



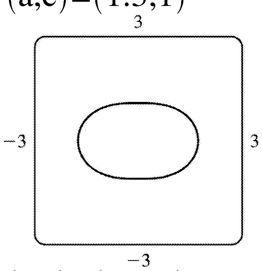
(a,c)=(1.01,1)



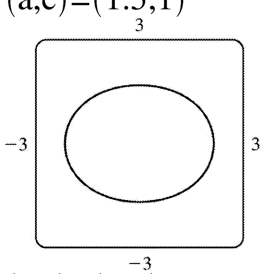
(a,c)=(4.04,4)



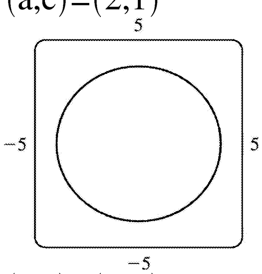
(a,c)=(1.3,1)



(a,c)=(1.5,1)



(a,c)=(2,1)



(a,c)=(4,1)

81.

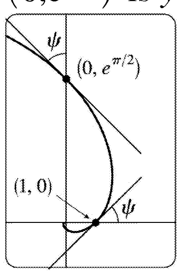
$$\tan\psi = \tan(\phi-\theta) = \frac{\tan\phi-\tan\theta}{1+\tan\phi\,\tan\theta} = \frac{\frac{dy}{dx}-\tan\theta}{1+\frac{dy}{dx}\tan\theta} = \frac{\frac{dy/d\theta}{dx/d\theta}-\tan\theta}{1+\frac{dy/d\theta}{dx/d\theta}\tan\theta}$$

$$= \frac{\frac{dy}{d\theta}-\frac{dx}{d\theta}\tan\theta}{\frac{dx}{d\theta}+\frac{dy}{d\theta}\tan\theta} = \frac{\left(\frac{dr}{d\theta}\sin\theta+r\cos\theta\right)-\tan\theta\left(\frac{dr}{d\theta}\cos\theta-r\sin\theta\right)}{\left(\frac{dr}{d\theta}\cos\theta-r\sin\theta\right)+\tan\theta\left(\frac{dr}{d\theta}\sin\theta+r\cos\theta\right)}$$

$$= \frac{r\cos\theta+r\cdot\frac{\sin^2\theta}{\cos\theta}}{\frac{dr}{d\theta}\cos\theta+\frac{dr}{d\theta}\cdot\frac{\sin^2\theta}{\cos\theta}} = \frac{r\cos^2\theta+r\sin^2\theta}{\frac{dr}{d\theta}\cos^2\theta+\frac{dr}{d\theta}\sin^2\theta} = \frac{r}{dr/d\theta}$$

82. **(a)** $r=e^{\theta} \Rightarrow dr/d\theta=e^{\theta}$, so by Exercise 81, $\tan\psi=r/e^{\theta}=1\Rightarrow\psi=\arctan 1=\frac{\pi}{4}$.

(b) The Cartesian equation of the tangent line at $(1,0)$ is $y=x-1$, and that of the tangent line at $\left(0,e^{\pi/2}\right)$ is $y=e^{\pi/2}-x$.



(c) Let a be the tangent of the angle between the tangent and radial lines, that is, $a=\tan\psi$. Then, by Exercise 81 , $a=\frac{r}{dr/d\theta} \Rightarrow \frac{dr}{d\theta}=\frac{1}{a}r\Rightarrow r=Ce^{\theta/a}$ (by Theorem 10.4.2).

1. $r=\sqrt{\theta}\ ,\ 0\le\theta\le\dfrac{\pi}{4}\ .\ A=\int_0^{\pi/4}\dfrac{1}{2}r^2d\theta=\int_0^{\pi/4}\dfrac{1}{2}\left(\sqrt{\theta}\right)^2d\theta=\int_0^{\pi/4}\dfrac{1}{2}\theta\,d\theta=\left[\dfrac{1}{4}\theta^2\right]_0^{\pi/4}=\dfrac{1}{64}\pi^2$

2. $r=e^{\theta/2}\ ,\ \pi\le\theta\le2\pi\ .\ A=\int_\pi^{2\pi}\dfrac{1}{2}\left(e^{\theta/2}\right)^2d\theta=\int_\pi^{2\pi}\dfrac{1}{2}e^\theta\,d\theta=\dfrac{1}{2}\left[e^\theta\right]_\pi^{2\pi}=\dfrac{1}{2}\left(e^{2\pi}-e^\pi\right)$

3. $r=\sin\theta\ ,\ \dfrac{\pi}{3}\le\theta\le\dfrac{2\pi}{3}\ .$

$$A=\int_{\pi/3}^{2\pi/3}\frac{1}{2}\sin^2\theta\,d\theta=\frac{1}{4}\int_{\pi/3}^{2\pi/3}(1-\cos2\theta)d\theta=\frac{1}{4}\left[\theta-\frac{1}{2}\sin2\theta\right]_{\pi/3}^{2\pi/3}$$

$$=\frac{1}{4}\left[\frac{2\pi}{3}-\frac{1}{2}\sin\frac{4\pi}{3}-\frac{\pi}{3}+\frac{1}{2}\sin\frac{2\pi}{3}\right]=\frac{1}{4}\left[\frac{2\pi}{3}-\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)-\frac{\pi}{3}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right]=\frac{1}{4}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)$$

4. $r=\sqrt{\sin\theta}\ ,\ 0\le\theta\le\pi\ .\ A=\int_0^\pi\dfrac{1}{2}\left(\sqrt{\sin\theta}\right)^2d\theta=\int_0^\pi\dfrac{1}{2}\sin\theta\,d\theta=\left[-\dfrac{1}{2}\cos\theta\right]_0^\pi=\dfrac{1}{2}+\dfrac{1}{2}=1$

5. $r=\theta\ ,\ 0\le\theta\le\pi\ .\ A=\int_0^\pi\dfrac{1}{2}\theta^2d\theta=\left[\dfrac{1}{6}\theta^3\right]_0^\pi=\dfrac{1}{6}\pi^3$

6. $r=1+\sin\theta\ ,\ \dfrac{\pi}{2}\le\theta\le\pi\ .$

$$A=\int_{\pi/2}^\pi\frac{1}{2}(1+\sin\theta)^2d\theta=\frac{1}{2}\int_{\pi/2}^\pi\left(1+2\sin\theta+\sin^2\theta\right)d\theta=\frac{1}{2}\int_{\pi/2}^\pi\left[1+2\sin\theta+\frac{1}{2}(1-\cos2\theta)\right]d\theta$$

$$=\frac{1}{2}\left[\theta-2\cos\theta+\frac{1}{2}\theta-\frac{1}{4}\sin2\theta\right]_{\pi/2}^\pi=\frac{1}{2}\left[\pi+2+\frac{\pi}{2}-0-\left(\frac{\pi}{2}-0+\frac{\pi}{4}-0\right)\right]=\frac{1}{2}\left(\frac{3\pi}{4}+2\right)=\frac{3\pi}{8}+1$$

7. $r=4+3\sin\theta\ ,\ -\dfrac{\pi}{2}\le\theta\le\dfrac{\pi}{2}\ .$

$$A=\int_{-\pi/2}^{\pi/2}\frac{1}{2}(4+3\sin\theta)^2d\theta=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\left(16+24\sin\theta+9\sin^2\theta\right)d\theta$$

$$=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\left(16+9\sin^2\theta\right)d\theta\ \text{[byTheorem5.5.(b)]}$$

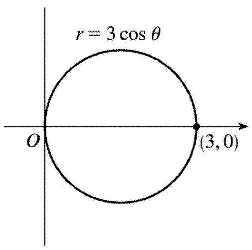
$$=\frac{1}{2}\cdot2\int_0^{\pi/2}\left[16+9\cdot\frac{1}{2}(1-\cos2\theta)\right]d\theta\ \text{[byTheorem5.5.(a)]}$$

$$=\int_0^{\pi/2}\left(\frac{41}{2}-\frac{9}{2}\cos2\theta\right)d\theta=\left[\frac{41}{2}\theta-\frac{9}{4}\sin2\theta\right]_0^{\pi/2}=\left(\frac{41\pi}{4}-0\right)-(0-0)=\frac{41\pi}{4}$$

8. $r=\sin4\theta\ ,\ 0\le\theta\le\dfrac{\pi}{4}\ .\ A=\int_0^{\pi/4}\dfrac{1}{2}\sin^24\theta\,d\theta=\int_0^{\pi/4}\dfrac{1}{4}(1-\cos8\theta)d\theta=\left[\dfrac{1}{4}\theta-\dfrac{1}{32}\sin8\theta\right]_0^{\pi/4}=\dfrac{\pi}{16}$

9. The area above the polar axis is bounded by $r=3\cos\theta$ for $\theta=0$ to $\theta=\pi/2$ (*not* π) . By symmetry,

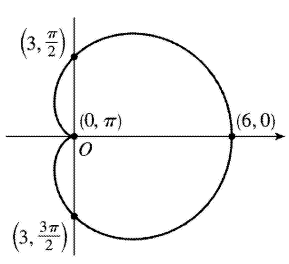
$$A = 2\int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \int_0^{\pi/2} (3\cos\theta)^2 d\theta$$
$$= 3^2 \int_0^{\pi/2} \cos^2\theta \, d\theta = 9\int_0^{\pi/2} \frac{1}{2} (1+\cos 2\theta) d\theta$$
$$= \frac{9}{2} \left[\theta + \frac{1}{2}\sin 2\theta \right]_0^{\pi/2} = \frac{9}{2} \left[\left(\frac{\pi}{2} + 0 \right) - (0+0) \right] = \frac{9\pi}{4} .$$



Also, note that this is a circle with radius $\frac{3}{2}$, so its area is $\pi \left(\frac{3}{2} \right)^2 = \frac{9\pi}{4}$.

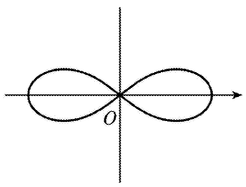
10.

$$A = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} [3(1+\cos\theta)]^2 d\theta$$
$$= \frac{9}{2} \int_0^{2\pi} \left(1 + 2\cos\theta + \cos^2\theta \right) d\theta$$
$$= \frac{9}{2} \int_0^{2\pi} \left[1 + 2\cos\theta + \frac{1}{2} (1+\cos 2\theta) \right] d\theta$$
$$= \frac{9}{2} \left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta \right]_0^{2\pi} = \frac{27}{2}\pi$$



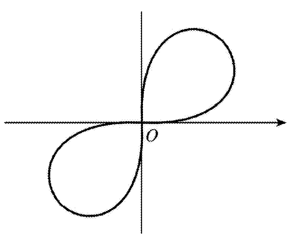
11. The curve $r^2 = 4\cos 2\theta$ goes through the pole when $\theta = \pi/4$, so we'll find the area for $0 \le \theta \le \pi/4$ and multiply it by 4 .

$$A = 4\int_0^{\pi/4} \frac{1}{2} r^2 d\theta = 2\int_0^{\pi/4} (4\cos 2\theta)\, d\theta$$
$$= 8\int_0^{\pi/4} \cos 2\theta\, d\theta = 4[\sin 2\theta]_0^{\pi/4} = 4(1-0) = 4$$



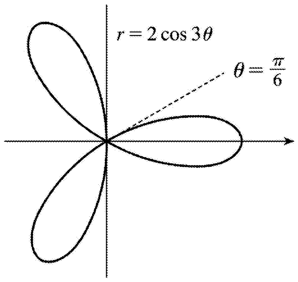
12. The curve $r^2 = \sin 2\theta$ goes through the pole when $\theta = \pi/2$, so we'll find the area for $0 \le \theta \le \pi/2$ and multiply it by 2 .

$$A = 2\int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \int_0^{\pi/2} \sin 2\theta\, d\theta = -\frac{1}{2}[\cos 2\theta]_0^{\pi/2}$$
$$= -\frac{1}{2}(-1-1) = -\frac{1}{2}(-2) = 1$$



13. One–sixth of the area lies above the polar axis and is bounded by the curve $r = 2\cos 3\theta$ for $\theta = 0$ to $\theta = \pi/6$.

$$A = 6\int_0^{\pi/6} \frac{1}{2} (2\cos 3\theta)^2 d\theta = 12\int_0^{\pi/6} \cos^2 3\theta\, d\theta$$
$$= \frac{12}{2}\int_0^{\pi/6} (1+\cos 6\theta) d\theta$$
$$= 6\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = 6\left(\frac{\pi}{6}\right) = \pi$$



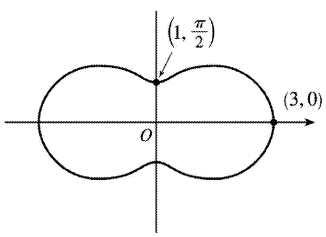
14.

$$A = \int_0^{2\pi} \frac{1}{2}(2+\cos 2\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi}(4+4\cos 2\theta + \cos^2 2\theta)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(4+4\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta\right)d\theta$$

$$= \frac{1}{2}\left[\frac{9}{2}\theta + 2\sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}(9\pi) = \frac{9\pi}{2}$$



15.

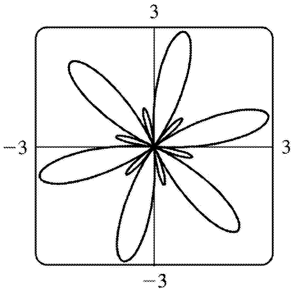
$$A = \int_0^{2\pi} \frac{1}{2}(1+2\sin 6\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi}(1+4\sin 6\theta + 4\sin^2 6\theta)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[1+4\sin 6\theta + 4\cdot\frac{1}{2}(1-\cos 12\theta)\right]d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}(3+4\sin 6\theta - 2\cos 12\theta)d\theta$$

$$= \frac{1}{2}\left[3\theta - \frac{2}{3}\cos 6\theta - \frac{1}{6}\sin 12\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[(6\pi - \frac{2}{3} - 0) - \left(0 - \frac{2}{3} - 0\right)\right] = 3\pi \;.$$



16.

$$A = \int_0^{\pi} \frac{1}{2}(2\sin\theta + 3\sin 9\theta)^2 d\theta = 2\int_0^{\pi/2} \frac{1}{2}(2\sin\theta + 3\sin 9\theta)^2 d\theta$$

$$= \int_0^{\pi/2} (4\sin^2\theta + 12\sin\theta\sin 9\theta + 9\sin^2 9\theta)\, d\theta$$

$$= \int_0^{\pi/2} \left[2(1-\cos 2\theta) + 12\cdot\frac{1}{2}(\cos(\theta - 9\theta) - \cos(\theta + 9\theta)) + \frac{9}{2}(1-\cos 18\theta) \right] d\theta$$

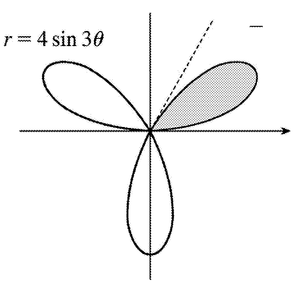
$$= \int_0^{\pi/2} (2 - 2\cos 2\theta + 6\cos 8\theta - 6\cos 10\theta + \frac{9}{2} - \frac{9}{2}\cos 18\theta)\, d\theta$$

$$= \left[\frac{13}{2}\theta - \sin 2\theta + \frac{3}{4}\sin 8\theta - \frac{3}{5}\sin 10\theta - \frac{1}{4}\sin 18\theta \right]_0^{\pi/2} = \frac{13}{4}\pi$$

17. The shaded loop is traced out from $\theta = 0$ to $\theta = \pi/2$. $A = \int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^2 2\theta\, d\theta$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{1}{2}(1-\cos 4\theta)\, d\theta = \frac{1}{4}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{1}{4}\left(\frac{\pi}{2}\right) = \frac{\pi}{8}$$

18. $A = \int_0^{\pi/3} \frac{1}{2}(4\sin 3\theta)^2 d\theta = 8\int_0^{\pi/3} \sin^2 3\theta\, d\theta = 4\int_0^{\pi/3} (1-\cos 6\theta)\, d\theta = 4\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \frac{4\pi}{3}$

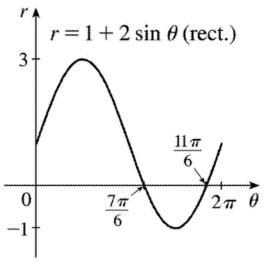


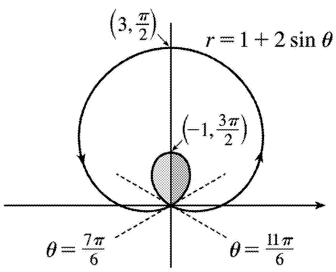
19. $r = 0 \Rightarrow 3\cos 5\theta = 0 \Rightarrow 5\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{10}$.

$$A = \int_{-\pi/10}^{\pi/10} \frac{1}{2}(3\cos 5\theta)^2 d\theta = \int_0^{\pi/10} 9\cos^2 5\theta\, d\theta = \frac{9}{2}\int_0^{\pi/10} (1+\cos 10\theta)\, d\theta = \frac{9}{2}\left[\theta + \frac{1}{10}\sin 10\theta\right]_0^{\pi/10} = \frac{9\pi}{20}$$

20. $A=2\int_0^{\pi/8}\frac{1}{2}(2\cos 4\theta)^2 d\theta=2\int_0^{\pi/8}(1+\cos 8\theta)d\theta=2\left[\theta+\frac{1}{8}\sin 8\theta\right]_0^{\pi/8}=\frac{\pi}{4}$

21.





This is a limacon, with inner loop traced out between $\theta=\frac{7\pi}{6}$ and $\frac{11\pi}{6}$
.

$$A=2\int_{7\pi/6}^{3\pi/2}\frac{1}{2}(1+2\sin\theta)^2 d\theta=\int_{7\pi/6}^{3\pi/2}\left(1+4\sin\theta+4\sin^2\theta\right)d\theta$$

$$=\int_{7\pi/6}^{3\pi/2}\left[1+4\sin\theta+4\cdot\frac{1}{2}(1-\cos 2\theta)\right]d\theta=[\theta-4\cos\theta+2\theta-\sin 2\theta]_{7\pi/6}^{3\pi/2}$$

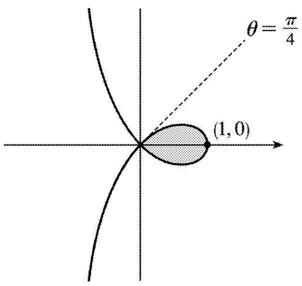
$$=\left(\frac{9\pi}{2}\right)-\left(\frac{7\pi}{2}+2\sqrt{3}-\frac{\sqrt{3}}{2}\right)=\pi-\frac{3\sqrt{3}}{2}$$

22. To determine when the strophoid $r=2\cos\theta-\sec\theta$ passes through the pole, we solve $r=0\Rightarrow 2\cos\theta-\frac{1}{\cos\theta}=0\Rightarrow$ $2\cos^2\theta-1=0\Rightarrow\cos^2\theta=\frac{1}{2}\Rightarrow\cos\theta=\pm\frac{1}{\sqrt{2}}\Rightarrow$ $\theta=\frac{\pi}{4}$ or $\theta=\frac{3\pi}{4}$ for $0\le\theta\le\pi$ with $\theta\ne\frac{\pi}{2}$.

$$A=2\int_0^{\pi/4}\frac{1}{2}(2\cos\theta-\sec\theta)^2 d\theta=\int_0^{\pi/4}(4\cos^2\theta-4+\sec^2\theta)d\theta$$

$$=\int_0^{\pi/4}\left[4\cdot\frac{1}{2}(1+\cos 2\theta)-4+\sec^2\theta\right]d\theta=\int_0^{\pi/4}(-2+2\cos 2\theta+\sec^2\theta)d\theta$$

$$= \left[-2\theta+\sin 2\theta+\tan\theta\right]_0^{\pi/4}=\left(-\frac{\pi}{2}+1+1\right)-0=2-\frac{\pi}{2}$$



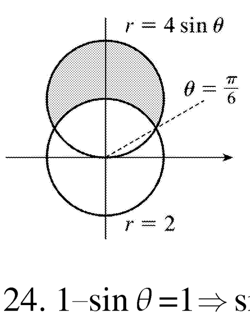
23. $4\sin\theta=2\Leftrightarrow\sin\theta=\frac{1}{2}\Leftrightarrow\theta=\frac{\pi}{6}$ or $\frac{5\pi}{6}$ (for $0\le\theta\le2\pi$). We'll subtract the unshaded area from the shaded area for $\pi/6\le\theta\le\pi/2$ and double that value.

$$A = 2\int_{\pi/6}^{\pi/2}\frac{1}{2}(4\sin\theta)^2\,d\theta-2\int_{\pi/6}^{\pi/2}\frac{1}{2}(2)^2\,d\theta=2\int_{\pi/6}^{\pi/2}\frac{1}{2}\left[(4\sin\theta)^2-2^2\right]d\theta$$

$$= \int_{\pi/6}^{\pi/2}\left(16\sin^2\theta-4\right)d\theta=\int_{\pi/6}^{\pi/2}[8(1-\cos 2\theta)-4]\,d\theta$$

$$= \int_{\pi/6}^{\pi/2}(4-8\cos 2\theta)\,d\theta=\left[4\theta-4\sin 2\theta\right]_{\pi/6}^{\pi/2}$$

$$= (2\pi-0)-\left(\frac{2\pi}{3}-4\cdot\frac{\sqrt{3}}{2}\right)=\frac{4}{3}\pi+2\sqrt{3}$$

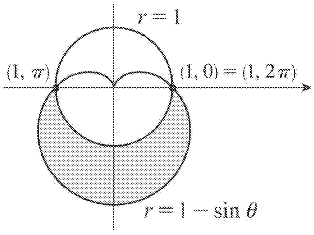


24. $1-\sin\theta=1\Rightarrow\sin\theta=0\Rightarrow\theta=0$ or $\pi\Rightarrow$

$$A = \int_{\pi}^{2\pi}\frac{1}{2}\left[(1-\sin\theta)^2-1\right]d\theta=\frac{1}{2}\int_{\pi}^{2\pi}\left(\sin^2\theta-2\sin\theta\right)d\theta$$

$$= \frac{1}{4}\int_{\pi}^{2\pi}(1-\cos 2\theta-4\sin\theta)\,d\theta=\frac{1}{4}\left[\theta-\frac{1}{2}\sin 2\theta+4\cos\theta\right]_{\pi}^{2\pi}$$

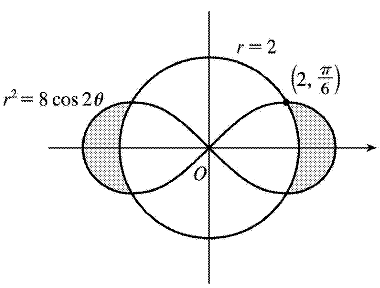
$$= \frac{1}{4}\pi+2$$



25. To find the area inside the leminiscate $r^2=8\cos 2\theta$ and outside the circle $r=2$, we first note that the two curves intersect when $r^2=8\cos 2\theta$ and $r=2$; i.e., when $\cos 2\theta=\frac{1}{2}$. For $-\pi<\theta\leq\pi$, $\cos 2\theta=\frac{1}{2}$ $\Leftrightarrow 2\theta=\pm\pi/3$ or $\pm 5\pi/3 \Leftrightarrow \theta=\pm\pi/6$ or $\pm 5\pi/6$. The figure shows that the desired area is 4 times the area between the curves from 0 to $\pi/6$. Thus,

$$A = 4\int_0^{\pi/6}\left[\frac{1}{2}(8\cos 2\theta)-\frac{1}{2}(2)^2\right]d\theta=8\int_0^{\pi/6}(2\cos 2\theta-1)\,d\theta$$

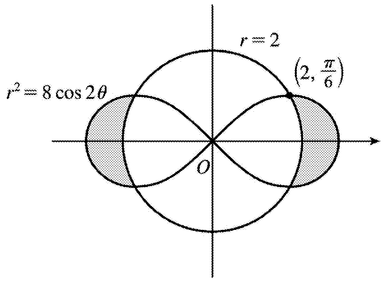
$$= 8\left[\sin 2\theta-\theta\right]_0^{\pi/6}=8\left(\sqrt{3}/2-\pi/6\right)=4\sqrt{3}-4\pi/3$$



26. To find the area inside the leminiscate $r^2=8\cos 2\theta$ and outside the circle $r=2$, we first note that the two curves intersect when $r^2=8\cos 2\theta$ and $r=2$; i.e., when $\cos 2\theta=\frac{1}{2}$. For $-\pi<\theta\leq\pi$, $\cos 2\theta=\frac{1}{2}$ $\Leftrightarrow 2\theta=\pm\pi/3$ or $\pm 5\pi/3 \Leftrightarrow \theta=\pm\pi/6$ or $\pm 5\pi/6$. The figure shows that the desired area is 4 times the area between the curves from 0 to $\pi/6$. Thus,

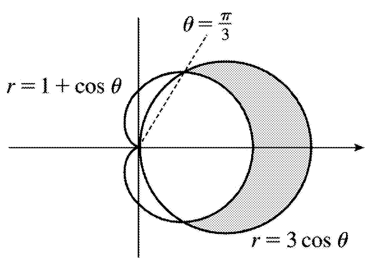
$$A = 4\int_0^{\pi/6}\left[\frac{1}{2}(8\cos 2\theta)-\frac{1}{2}(2)^2\right]d\theta=8\int_0^{\pi/6}(2\cos 2\theta-1)\,d\theta$$

$$= 8\left[\sin 2\theta-\theta\right]_0^{\pi/6}=8\left(\sqrt{3}/2-\pi/6\right)=4\sqrt{3}-4\pi/3$$



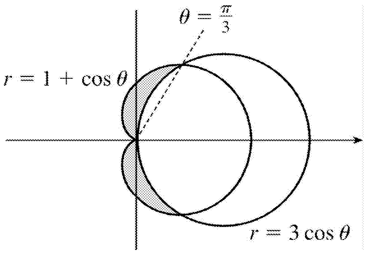
27. $3\cos\theta = 1+\cos\theta \Leftrightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ or $-\frac{\pi}{3}$.

$$A = 2\int_0^{\pi/3} \frac{1}{2}\left[(3\cos\theta)^2 - (1+\cos\theta)^2\right]d\theta$$
$$= \int_0^{\pi/3}\left(8\cos^2\theta - 2\cos\theta - 1\right)d\theta$$
$$= \int_0^{\pi/3}[4(1+\cos 2\theta) - 2\cos\theta - 1]d\theta$$
$$= \int_0^{\pi/3}(3+4\cos 2\theta - 2\cos\theta)d\theta$$
$$= [3\theta + 2\sin 2\theta - 2\sin\theta]_0^{\pi/3}$$
$$= \pi + \sqrt{3} - \sqrt{3} = \pi$$



28. Note that $r=1+\cos\theta$ goes through the pole when $\theta=\pi$, but $r=3\cos\theta$ goes through the pole when $\theta=\pi/2$.

$$A = 2\int_{\pi/3}^{\pi}\frac{1}{2}(1+\cos\theta)^2 d\theta - 2\int_{\pi/3}^{\pi/2}\frac{1}{2}(3\cos\theta)^2 d\theta$$
$$= \int_{\pi/3}^{\pi}\left[1+2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right]d\theta - \frac{9}{2}\int_{\pi/3}^{\pi/2}(1+\cos 2\theta)d\theta$$
$$= \left[\theta + 2\sin\theta + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_{\pi/3}^{\pi} - \frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/3}^{\pi/2}$$
$$= \left(\pi - \frac{9}{8}\sqrt{3}\right) - \frac{9}{2}\left(\frac{\pi}{6} - \frac{1}{4}\sqrt{3}\right) = \frac{\pi}{4}$$

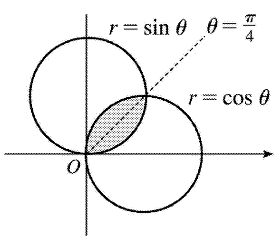


29.

$$A = 2\int_0^{\pi/4} \frac{1}{2}\sin^2\theta \, d\theta = \int_0^{\pi/4} \frac{1}{2}(1 - \cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \frac{1}{2}\left[\left(\frac{\pi}{4} - \frac{1}{2}\cdot 1\right) - (0 - 0)\right]$$

$$= \frac{1}{8}\pi - \frac{1}{4}$$

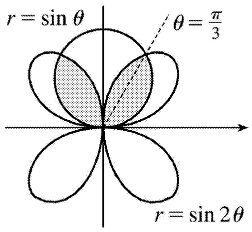


30. $r = \sin 2\theta$ takes on both positive and negative values. $\sin\theta = □\, 2\theta = \pm 2\sin\theta\cos\theta \Rightarrow$ $\sin\theta(1 \pm 2\cos\theta) = 0$. From the figure we can see that the intersections occur where $\cos\theta = \pm\frac{1}{2}$, or $\theta = \frac{\pi}{3}$ and $\frac{2\pi}{3}$.

$$A = 2\left[\int_0^{\pi/3} \frac{1}{2}\sin^2\theta \, d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}\sin^2 2\theta \, d\theta\right]$$

$$= \int_0^{\pi/3} \frac{1}{2}(1 - \cos 2\theta)\, d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(1 - \cos 4\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/3} + \frac{1}{2}\left[\theta - \frac{1}{4}\sin 4\theta\right]_{\pi/3}^{\pi/2} = \frac{4\pi - 3\sqrt{3}}{16}$$

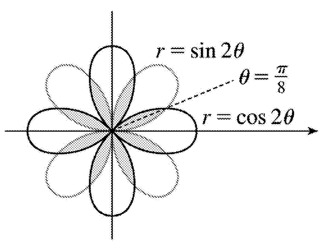


31. $\sin 2\theta = \cos 2\theta \Rightarrow \dfrac{\sin 2\theta}{\cos 2\theta} = 1 \Rightarrow \tan 2\theta = 1 \Rightarrow$

$2\theta = \dfrac{\pi}{4} \Rightarrow \theta = \dfrac{\pi}{8} \Rightarrow$

$$A = 8\cdot 2\int_0^{\pi/8} \frac{1}{2}\sin^2 2\theta\, d\theta = 8\int_0^{\pi/8} \frac{1}{2}(1-\cos 4\theta)\, d\theta$$

$$= 4\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/8} = 4\left(\frac{\pi}{8} - \frac{1}{4}\cdot 1\right) = \frac{1}{2}\pi - 1$$



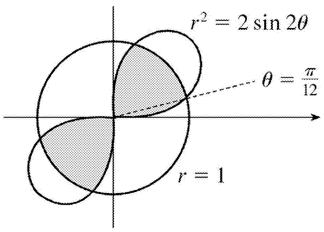
32. $2\sin 2\theta = 1^2 \Rightarrow \sin 2\theta = \dfrac{1}{2} \Rightarrow 2\theta = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6} \Rightarrow \theta = \dfrac{\pi}{12}$ or $\dfrac{5\pi}{12}$.

$$A = 4\left[\int_0^{\pi/12} \frac{1}{2}\cdot 2\sin 2\theta\, d\theta + \int_{\pi/12}^{\pi/4} \frac{1}{2}\left(1^2\right) d\theta\right]$$

$$= [-2\cos 2\theta]_0^{\pi/12} + [2\theta]_{\pi/12}^{\pi/4}$$

$$= -2\left(\frac{\sqrt{3}}{2} - 1\right) + 2\left(\frac{1}{4}\pi - \frac{1}{12}\pi\right)$$

$$= 2 - \sqrt{3} + \frac{\pi}{3}$$

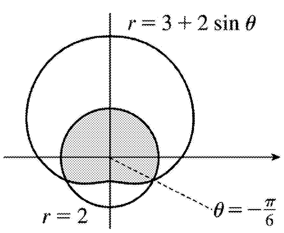


33.

$$A = 2\left[\int_{-\pi/2}^{-\pi/6} \frac{1}{2}\,(3+2\sin\theta)^2\,d\theta + \int_{-\pi/6}^{\pi/2} \frac{1}{2}\,2^2\,d\theta\right]$$

$$= \int_{-\pi/2}^{-\pi/6}\left(9+12\sin\theta+4\sin^2\theta\right)d\theta + \left[4\theta\right]_{-\pi/6}^{\pi/2}$$

$$= \left[9\theta - 12\cos\theta + 2\theta - \sin 2\theta\right]_{-\pi/2}^{-\pi/6} + \frac{8\pi}{3} = \frac{19\pi}{3} - \frac{11\sqrt{3}}{2}$$



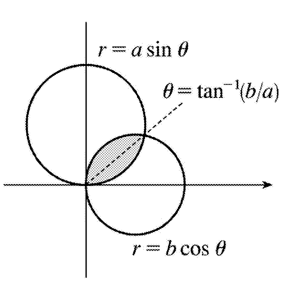
34. Let $\alpha = \tan^{-1}(b/a)$. Then

$$A = \int_0^{\alpha} \frac{1}{2}\,(a\sin\theta)^2\,d\theta + \int_{\alpha}^{\pi/2} \frac{1}{2}\,(b\cos\theta)^2\,d\theta$$

$$= \frac{1}{4}\,a^2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\alpha} + \frac{1}{4}\,b^2\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\alpha}^{\pi/2}$$

$$= \frac{1}{4}\,\alpha\left(a^2 - b^2\right) + \frac{1}{8}\,\pi b^2 - \frac{1}{4}\left(a^2 + b^2\right)(\sin\alpha\cos\alpha)$$

$$= \frac{1}{4}\left(a^2 - b^2\right)\tan^{-1}(b/a) + \frac{1}{8}\,\pi b^2 - \frac{1}{4}\,ab$$



35. The darker shaded region (from $\theta=0$ to $\theta=2\pi/3$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta=2\pi/3$
to $\theta=\pi$), and then double that difference to obtain the desired area.

$$A=2\left[\int_0^{2\pi/3}\frac{1}{2}\left(\frac{1}{2}+\cos\theta\right)^2 d\theta-\int_{2\pi/3}^{\pi}\frac{1}{2}\left(\frac{1}{2}+\cos\theta\right)^2 d\theta\right]$$

$$=\int_0^{2\pi/3}\left(\frac{1}{4}+\cos\theta+\cos^2\theta\right)d\theta-\int_{2\pi/3}^{\pi}\left(\frac{1}{4}+\cos\theta+\cos^2\theta\right)d\theta$$

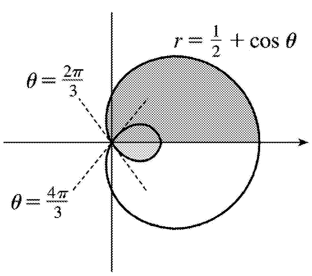
$$=\int_0^{2\pi/3}\left[\frac{1}{4}+\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right]d\theta$$

$$-\int_{2\pi/3}^{\pi}\left[\frac{1}{4}+\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right]d\theta$$

$$=\left[\frac{\theta}{4}+\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^{2\pi/3}-\left[\frac{\theta}{4}+\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_{2\pi/3}^{\pi}$$

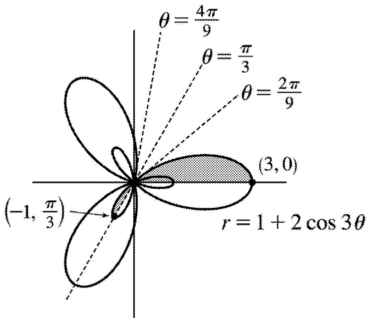
$$=\left(\frac{\pi}{6}+\frac{\sqrt{3}}{2}+\frac{\pi}{3}-\frac{\sqrt{3}}{8}\right)-\left(\frac{\pi}{4}+\frac{\pi}{2}\right)+\left(\frac{\pi}{6}+\frac{\sqrt{3}}{2}+\frac{\pi}{3}-\frac{\sqrt{3}}{8}\right)$$

$$=\frac{\pi}{4}+\frac{3}{4}\sqrt{3}=\frac{1}{4}\left(\pi+3\sqrt{3}\right)$$



36. $r=0 \Rightarrow 1+2\cos 3\theta=0 \Rightarrow \cos 3\theta=-\frac{1}{2} \Rightarrow 3\theta=\frac{2\pi}{3}, \frac{4\pi}{3}$ (for $0\le 3\theta\le 2\pi$) $\Rightarrow \theta=\frac{2\pi}{9}, \frac{4\pi}{9}$. The darker shaded region (from $\theta=0$ to $\theta=2\pi/9$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta=2\pi/9$ to $\theta=\pi/3$), and then double that difference to obtain the desired area.

$$A=2\left[\int_0^{2\pi/9}\frac{1}{2}(1+2\cos 3\theta)^2 d\theta-\int_{2\pi/9}^{\pi/3}\frac{1}{2}(1+2\cos 3\theta)^2 d\theta\right]$$



Now

$$r^2 = (1+2\cos 3\theta)^2 = 1+4\cos 3\theta + 4\cos^2 3\theta = 1+4\cos 3\theta + 4\cdot\frac{1}{2}(1+\cos 6\theta)$$

$$= 1+4\cos 3\theta + 2 + 2\cos 6\theta = 3+4\cos 3\theta + 2\cos 6\theta$$

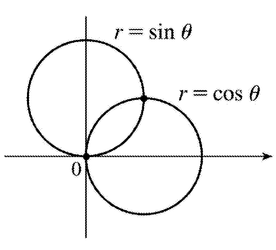
and $\int r^2 d\theta = 3\theta + \frac{4}{3}\sin 3\theta + \frac{1}{3}\sin 6\theta + C$, so

$$A = \left[3\theta + \frac{4}{3}\sin 3\theta + \frac{1}{3}\sin 6\theta\right]_0^{2\pi/9} - \left[3\theta + \frac{4}{3}\sin 3\theta + \frac{1}{3}\sin 6\theta\right]_{2\pi/9}^{\pi/3}$$

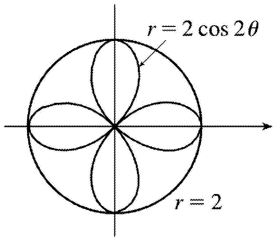
$$= \left[\left(\frac{2\pi}{3} + \frac{4}{3}\cdot\frac{\sqrt{3}}{2} + \frac{1}{3}\cdot\frac{-\sqrt{3}}{2}\right) - 0\right] - \left[(\pi+0+0) - \left(\frac{2\pi}{3} + \frac{4}{3}\cdot\frac{\sqrt{3}}{2} + \frac{1}{3}\cdot\frac{-\sqrt{3}}{2}\right)\right]$$

$$= \frac{4\pi}{3} + \frac{4}{3}\sqrt{3} - \frac{1}{3}\sqrt{3} - \pi = \frac{\pi}{3} + \sqrt{3}$$

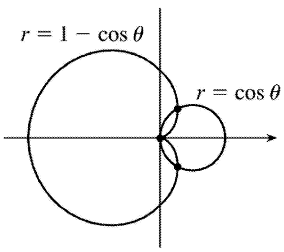
37. The two circles intersect at the pole since $(0,0)$ satisfies the first equation and $\left(0, \frac{\pi}{2}\right)$ the second. The other intersection point $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$ occurs where $\sin\theta = \cos\theta$.



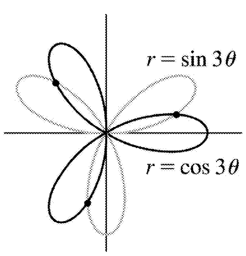
38. $2\cos 2\theta = \pm 2 \Rightarrow \cos 2\theta = \pm 1 \Rightarrow \theta = 0$, $\frac{\pi}{2}$, π , or $\frac{3\pi}{2}$, so the points are $(2,0)$, $\left(2, \frac{\pi}{2}\right)$, $(2,\pi)$, and $\left(2, \frac{3\pi}{2}\right)$.



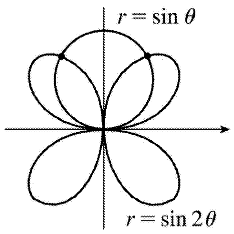
39. The curves intersect at the pole since $\left(0, \frac{\pi}{2}\right)$ satisfies $r = \cos\theta$ and $(0,0)$ satisfies $r = 1 - \cos\theta$. Now $\cos\theta = 1 - \cos\theta \Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ $\Rightarrow$ the other intersection points are $\left(\frac{1}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{1}{2}, \frac{5\pi}{3}\right)$.



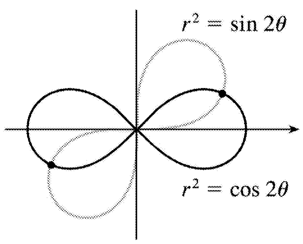
40. Clearly the pole lies on both curves. $\sin 3\theta = \cos 3\theta \Rightarrow \tan 3\theta = 1 \Rightarrow 3\theta = \frac{\pi}{4} + n\pi$ (n any integer) $\Rightarrow$ $\theta = \frac{\pi}{12} + \frac{\pi}{3} n \Rightarrow \theta = \frac{\pi}{12}$, $\frac{5\pi}{12}$, or $\frac{3\pi}{4}$, so the three remaining intersection points are $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{12}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{5\pi}{12}\right)$, and $\left(\frac{1}{\sqrt{2}}, \frac{3\pi}{4}\right)$.



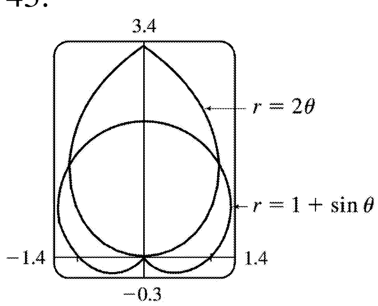
41. The pole is a point of intersection. $\sin\theta = \sin 2\theta = 2\sin\theta\cos\theta \Leftrightarrow \sin\theta\,(1 - 2\cos\theta) = 0 \Leftrightarrow \sin\theta = 0$ or $\cos\theta = \frac{1}{2} \Rightarrow \theta = 0$, π , $\frac{\pi}{3}$, $-\frac{\pi}{3} \Rightarrow \left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$ (by symmetry) are the other intersection points.

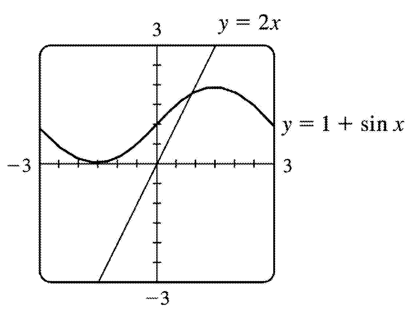


42. Clearly the pole is a point of intersection. $\sin 2\theta = \cos 2\theta \Rightarrow \tan 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{4} + 2n\pi$ (since $\sin 2\theta$ and $\cos 2\theta$ must be positive in the equations) $\Rightarrow \theta = \frac{\pi}{8} + n\pi \Rightarrow \theta = \frac{\pi}{8}$ or $\frac{9\pi}{8}$. So the curves also intersect at
$\left(\frac{1}{\sqrt[4]{2}}, \frac{\pi}{8} \right)$ and $\left(\frac{1}{\sqrt[4]{2}}, \frac{9\pi}{8} \right)$.



43.





From the first graph, we see that the pole is one point of intersection. By zooming in or using the cursor, we find the $\theta-$ values of the intersection points to be $\alpha \approx 0.88786 \approx 0.89$ and $\pi - \alpha \approx 2.25$. (The first of these values may be more easily estimated by plotting $y=1+\sin x$ and $y=2x$ in rectangular coordinates; see the second graph.)

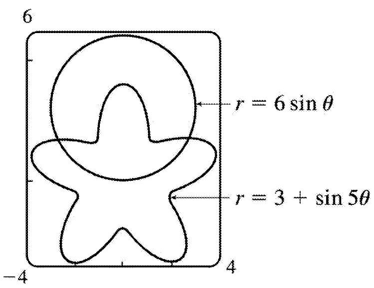
By symmetry, the total area contained is twice the area contained in the first quadrant, that is,

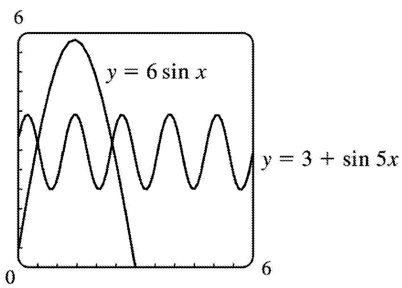
$$A = 2\int_0^{\alpha} \frac{1}{2}\,(2\theta)^2 d\theta + 2\int_{\alpha}^{\pi/2} \frac{1}{2}\,(1+\sin\theta)^2 d\theta = \int_0^{\alpha} 4\theta^2 d\theta + \int_{\alpha}^{\pi/2}\left[1+2\sin\theta+\frac{1}{2}(1-\cos 2\theta)\right]d\theta$$

$$= \left[\frac{4}{3}\theta^3\right]_0^{\alpha} + \left[\theta - 2\cos\theta + \left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\right]_{\alpha}^{\pi/2}$$

$$= \frac{4}{3}\alpha^3 + \left[\left(\frac{\pi}{2}+\frac{\pi}{4}\right) - \left(\alpha - 2\cos\alpha + \frac{1}{2}\alpha - \frac{1}{4}\sin 2\alpha\right)\right] \approx 3.4645$$

44.





From the first graph, it appears that the $\theta-$ values of the points of intersection are $\alpha \approx 0.57504 \approx 0.58$ and $\pi - \alpha \approx 2.57$. (These values may be more easily estimated by plotting $y=3+\sin 5x$ and $y=6\sin x$ in rectangular coordinates; see the second graph.) By symmetry, the total area enclosed in both curves is

$$A = 2\int_0^{\alpha} \frac{1}{2}\,(6\sin\theta)^2 d\theta + 2\int_{\alpha}^{\pi/2} \frac{1}{2}\,(3+\sin 5\theta)^2 d\theta = \int_0^{\alpha} 36\sin^2\theta\, d\theta + \int_{\alpha}^{\pi/2}\left(9+6\sin 5\theta + \sin^2 5\theta\right)d\theta$$

$$= \int_0^{\alpha} 36\cdot\frac{1}{2}(1-\cos 2\theta)\,d\theta + \int_{\alpha}^{\pi/2}\left[9+6\sin 5\theta + \frac{1}{2}(1-\cos 10\theta)\right]d\theta$$

$$= \left[36\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\right]_0^{\alpha} + \left[9\theta - \frac{6}{5}\cos 5\theta + \left(\frac{1}{2}\theta - \frac{1}{20}\sin 10\theta\right)\right]_{\alpha}^{\pi/2} \approx 10.41$$

45.

$$L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{\pi/3}\sqrt{(3\sin\theta)^2 + (3\cos\theta)^2}\, d\theta = \int_0^{\pi/3}\sqrt{9(\sin^2\theta + \cos^2\theta)}\, d\theta$$

$$= 3\int_0^{\pi/3} d\theta = 3\left[\theta\right]_0^{\pi/3} = 3\left(\frac{\pi}{3}\right) = \pi\ .$$

As a check, note that the circumference of a circle with radius $\frac{3}{2}$ is $2\pi\left(\frac{3}{2}\right)=3\pi$, and since $\theta=0$ to $\pi=\frac{\pi}{3}$ traces out $\frac{1}{3}$ of the circle (from $\theta=0$ to $\theta=\pi$), $\frac{1}{3}(3\pi)=\pi$.

46.

$$L=\int_a^b\sqrt{r^2+(dr/d\theta)^2}\,d\theta=\int_0^{2\pi}\sqrt{\left(e^{2\theta}\right)^2+\left(2e^{2\theta}\right)^2}\,d\theta=\int_0^{2\pi}\sqrt{e^{4\theta}+4e^{4\theta}}\,d\theta=\int_0^{2\pi}\sqrt{5e^{4\theta}}\,d\theta$$

$$=\sqrt{5}\int_0^{2\pi}e^{2\theta}\,d\theta=\frac{\sqrt{5}}{2}\left[e^{2\theta}\right]_0^{2\pi}=\frac{\sqrt{5}}{2}\left(e^{4\pi}-1\right)$$

47.

$$L=\int_a^b\sqrt{r^2+(dr/d\theta)^2}\,d\theta=\int_0^{2\pi}\sqrt{\left(\theta^2\right)^2+(2\theta)^2}\,d\theta=\int_0^{2\pi}\sqrt{\theta^4+4\theta^2}\,d\theta$$

$$=\int_0^{2\pi}\sqrt{\theta^2\left(\theta^2+4\right)}\,d\theta=\int_0^{2\pi}\theta\sqrt{\theta^2+4}\,d\theta$$

Now let $u=\theta^2+4$, so that $du=2\theta\,d\theta$ $\left[\theta\,d\theta=\frac{1}{2}\,du\right]$ and

$$\int_0^{2\pi}\theta\sqrt{\theta^2+4}\,\mathrm{d}\theta=\int_4^{4\pi^2+4}\frac{1}{2}\sqrt{u}\,\mathrm{du}=\frac{1}{2}\cdot\frac{2}{3}\left[u^{3/2}\right]_4^{4\left(\pi^2+1\right)}=\frac{1}{3}\left[4^{3/2}\left(\pi^2+1\right)^{3/2}-4^{3/2}\right]$$

$$=\frac{8}{3}\left[\left(\pi^2+1\right)^{3/2}-1\right]$$

48.

$$L=\int_a^b\sqrt{r^2+(dr/d\theta)^2}\,d\theta=\int_0^{2\pi}\sqrt{\theta^2+1}\,d\theta=\left[\frac{\theta}{2}\sqrt{\theta^2+1}+\frac{1}{2}\ln\left(\theta+\sqrt{\theta^2+1}\right)\right]_0^{2\pi}$$

$$=\pi\sqrt{4\pi^2+1}+\frac{1}{2}\ln\left(2\pi+\sqrt{4\pi^2+1}\right)$$

49. $L=\int_a^b\sqrt{r^2+(dr/d\theta)^2}\,d\theta=\int_0^{2\pi}\sqrt{\left(\theta^2\right)^2+(2\theta)^2}\,d\theta=\int_0^{2\pi}\sqrt{\theta^4+4\theta^2}\,d\theta$

$=\int_0^{2\pi}\sqrt{\theta^2\left(\theta^2+4\right)}\,d\theta=\int_0^{2\pi}\theta\sqrt{\theta^2+4}\,d\theta$

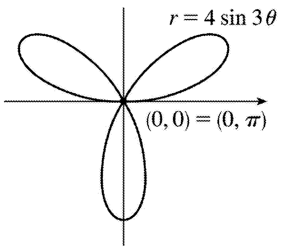
Now let $u=\theta^2+4$, so that $du=2\theta\,d\theta$ $\left[\theta\,d\theta=\frac{1}{2}\,du\right]$ and

$$\int_0^{2\pi} \theta \sqrt{\theta^2+4}\, d\theta = \int_4^{4\pi^2+4} \frac{1}{2} \sqrt{u}\, du = \frac{1}{2} \cdot \frac{2}{3} \left[u^{3/2} \right]_4^{4\left(\pi^2+1\right)} = \frac{1}{3} \left[4^{3/2} \left(\pi^2+1\right)^{3/2} - 4^{3/2} \right] = \frac{8}{3} \left[\left(\pi^2+1\right)^{3/2} - 1 \right]$$

50. The curve $r=4\sin 3\theta$ is completely traced with $0 \le \theta \le \pi$.

$$r^2 + \left(\frac{dr}{d\theta} \right)^2 = (4\sin 3\theta)^2 + (12\cos 3\theta)^2 \Rightarrow$$

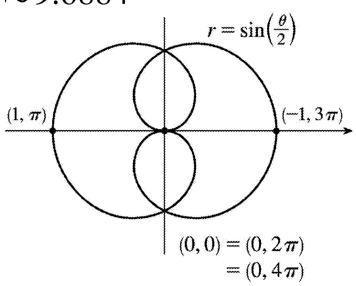
$$L = \int_0^{\pi} \sqrt{16\sin^2 3\theta + 144\cos^2 3\theta}\, d\theta \approx 26.7298$$



51. The curve $r=\sin\left(\frac{\theta}{2}\right)$ is completely traced with $0 \le \theta \le 4\pi$.

$$r^2 + \left(\frac{dr}{d\theta} \right)^2 = \sin^2\left(\frac{\theta}{2} \right) + \left[\frac{1}{2} \cos\left(\frac{\theta}{2} \right) \right]^2 \Rightarrow L = \int_0^{4\pi} \sqrt{\sin^2\left(\frac{\theta}{2} \right) + \frac{1}{4} \cos^2\left(\frac{\theta}{2} \right)}\, d\theta$$

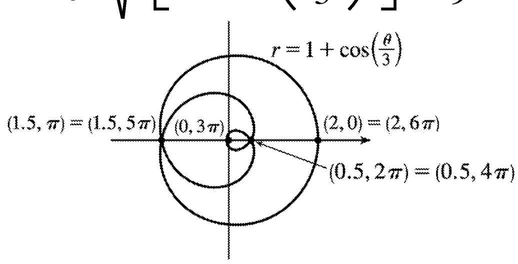
≈ 9.6884



52. The curve $r=1+\cos\left(\frac{\theta}{3}\right)$ is completely traced with $0 \le \theta \le 6\pi$.

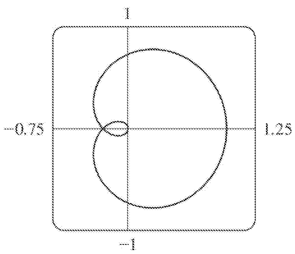
$$r^2 + \left(\frac{dr}{d\theta} \right)^2 = \left[1+\cos\left(\frac{\theta}{3} \right) \right]^2 + \left[-\frac{1}{3} \sin\left(\frac{\theta}{3} \right) \right]^2 \Rightarrow$$

$$L = \int_0^{6\pi} \sqrt{\left[1+\cos\left(\frac{\theta}{3} \right) \right]^2 + \frac{1}{9} \sin^2\left(\frac{\theta}{3} \right)}\, d\theta \approx 19.6676$$



53. The curve $r=\cos^4(\theta/4)$ is completely traced with $0\le\theta\le 4\pi$.

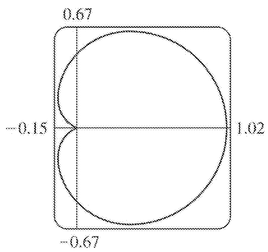
$$r^2+(dr/d\theta)^2=[\cos^4(\theta/4)]^2+\left[4\cos^3(\theta/4)\cdot(-\sin(\theta/4))\cdot\frac{1}{4}\right]^2$$
$$=\cos^8(\theta/4)+\cos^6(\theta/4)\sin^2(\theta/4)$$
$$=\cos^6(\theta/4)[\cos^2(\theta/4)+\sin^2(\theta/4)]$$
$$=\cos^6(\theta/4)$$



$$L=\int_0^{4\pi}\sqrt{\cos^6(\theta/4)}\,d\theta=\int_0^{4\pi}\left|\cos^3(\theta/4)\right|d\theta$$
$$=2\int_0^{2\pi}\cos^3(\theta/4)d\theta\ \ [\text{ since }\cos^3(\theta/4)\ge 0\text{ for }0\le\theta\le 2\pi\ ]=8\int_0^{\pi/2}\cos^3u\,du\left[u=\frac{1}{4}\theta\right]$$
$$=8\left[\frac{1}{3}(2+\cos^2u)\sin u\right]_0^{\pi/2}=\frac{8}{3}[(2\cdot 1)-(3\cdot 0)]=\frac{16}{3}$$

54. The curve $r=\cos^2(\theta/2)$ is completely traced with $0\le\theta\le 2\pi$.

$$r^2+(dr/d\theta)^2=[\cos^2(\theta/2)]^2+\left[2\cos(\theta/2)\cdot(-\sin(\theta/2))\cdot\frac{1}{2}\right]^2$$
$$=\cos^4(\theta/2)+\cos^2(\theta/2)\sin^2(\theta/2)$$
$$=\cos^2(\theta/2)[\cos^2(\theta/2)+\sin^2(\theta/2)]$$
$$=\cos^2(\theta/2)$$



$$L=\int_0^{2\pi}\sqrt{\cos^2(\theta/2)}\,d\theta=\int_0^{2\pi}|\cos(\theta/2)|\,d\theta=2\int_0^{\pi}\cos(\theta/2)\,d\theta\ \left[\text{ since }\cos(\theta/2)\geq 0\text{ for }0\leq\theta\leq\pi\ \right]$$

$$=4\int_0^{\pi/2}\cos u\,du\left[u=\frac{1}{2}\theta\right]=4\left[\sin u\right]_0^{\pi/2}=4(1-0)=4$$

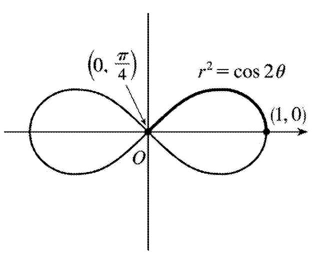
55. **(a)** From (.2.7),

$$S=\int_a^b 2\pi y\sqrt{(dx/d\theta)^2+(dy/d\theta)^2}\,d\theta$$

$$=\int_a^b 2\pi y\sqrt{r^2+(dr/d\theta)^2}\,d\theta\ \text{[fromthederivationofEquation.4.5}$$

$$=\int_a^b 2\pi r\sin\theta\sqrt{r^2+(dr/d\theta)^2}\,d\theta$$

(b) The curve $r^2=\cos 2\theta$ goes through the pole when $\cos 2\theta=0\Rightarrow 2\theta=\frac{\pi}{2}\Rightarrow\theta=\frac{\pi}{4}$. We'll rotate the curve from $\theta=0$ to $\theta=\frac{\pi}{4}$ and double this value to obtain the total surface area generated. $r^2=\cos 2\theta$ $\Rightarrow 2r\frac{dr}{d\theta}=-2\sin 2\theta\Rightarrow\left(\frac{dr}{d\theta}\right)^2=\frac{\sin^2 2\theta}{r^2}=\frac{\sin^2 2\theta}{\cos 2\theta}$.



$$S=2\int_0^{\pi/4}2\pi\sqrt{\cos 2\theta}\sin\theta\sqrt{\cos 2\theta+\left(\sin^2 2\theta\right)/\cos 2\theta}\,d\theta$$

$$=4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\sin\theta\sqrt{\frac{\cos^2 2\theta+\sin^2 2\theta}{\cos 2\theta}}\,d\theta=4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\sin\theta\frac{1}{\sqrt{\cos 2\theta}}\,d\theta$$

$$= 4\pi\int_0^{\pi/4}\sin\theta\, d\theta = 4\pi\left[-\cos\theta\right]_0^{\pi/4} = -4\pi\left(\frac{\sqrt{2}}{2}-1\right) = 2\pi\left(2-\sqrt{2}\right)$$

56. **(a)** Rotation around $\theta=\frac{\pi}{2}$ is the same as rotation around the $y-$ axis, that is, $S=\int_a^b 2\pi x\,ds$ where $ds=\sqrt{(dx/dt)^2+(dy/dt)^2}\;dt$ for a parametric equation, and for the special case of a polar equation, $x=r\cos\theta$ and $ds=\sqrt{(dx/d\theta)^2+(dy/d\theta)^2}\;d\theta=\sqrt{r^2+(dr/d\theta)^2}\;d\theta$. Therefore, for a polar equation rotated around $\theta=\frac{\pi}{2}$, $S=\int_a^b 2\pi r\cos\theta\sqrt{r^2+(dr/d\theta)^2}\;d\theta$.

(b) As in the solution for Exercise 55(b), we can double the surface area generated by rotating the curve from $\theta=0$ to $\theta=\frac{\pi}{4}$ to obtain the total surface area.

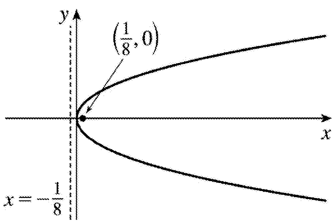
$$S = 2\int_0^{\pi/4}2\pi\sqrt{\cos 2\theta}\,\cos\theta\sqrt{\cos 2\theta+(\sin^2 2\theta)/\cos 2\theta}\;d\theta$$

$$= 4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\,\cos\theta\sqrt{\frac{\cos^2 2\theta+\sin^2 2\theta}{\cos 2\theta}}\;d\theta$$

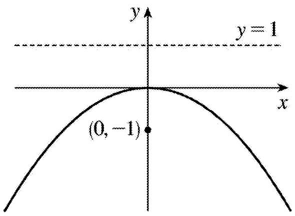
$$= 4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\,\cos\theta\,\frac{1}{\sqrt{\cos 2\theta}}\;d\theta$$

$$= 4\pi\int_0^{\pi/4}\cos\theta\, d\theta = 4\pi\left[\sin\theta\right]_0^{\pi/4} = 4\pi\left(\frac{\sqrt{2}}{2}-0\right) = 2\sqrt{2}\,\pi$$

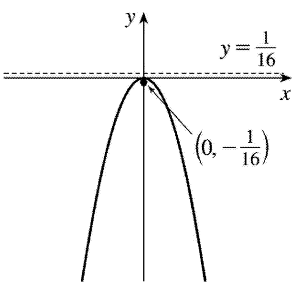
1. $x=2y^2 \Rightarrow y^2=\frac{1}{2}x$. $4p=\frac{1}{2}$, so $p=\frac{1}{8}$. The vertex is $(0,0)$, the focus is $\left(\frac{1}{8},0\right)$, and the directrix is $x=-\frac{1}{8}$.



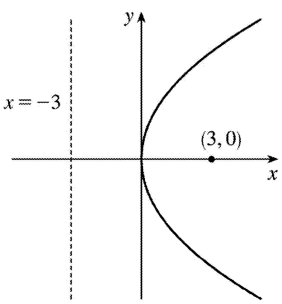
2. $4y+x^2=0 \Rightarrow x^2=-4y$. $4p=-4$, so $p=-1$. The vertex is $(0,0)$, the focus is $(0,-1)$, and the directrix is $y=1$.



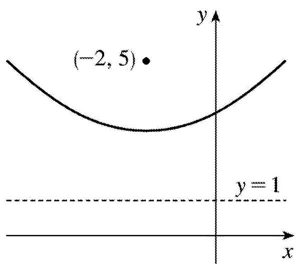
3. $4x^2=-y \Rightarrow x^2=-\frac{1}{4}y$. $4p=-\frac{1}{4}$, so $p=-\frac{1}{16}$. The vertex is $(0,0)$, the focus is $\left(0,-\frac{1}{16}\right)$, and the directrix is $y=\frac{1}{16}$.



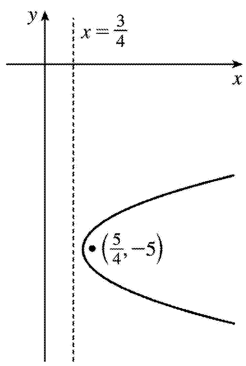
4. $y^2=12x$. $4p=12$, so $p=3$. The vertex is $(0,0)$, the focus is $(3,0)$, and the directrix is $x=-3$.



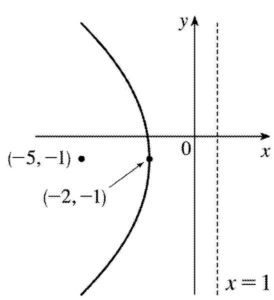
5. $(x+2)^2=8(y-3)$. $4p=8$, so $p=2$. The vertex is $(-2,3)$, the focus is $(-2,5)$, and the directrix is $y=1$.



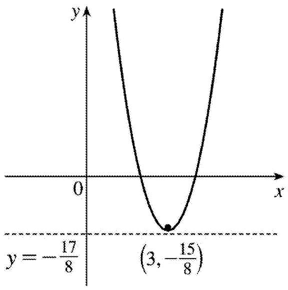
6. $x-1=(y+5)^2$. $4p=1$, so $p=\frac{1}{4}$. The vertex is $(1,-5)$, the focus is $\left(\frac{5}{4},-5\right)$, and the directrix is $x=\frac{3}{4}$.



7. $y^2+2y+12x+25=0 \Rightarrow y^2+2y+1=-12x-24 \Rightarrow (y+1)^2=-12(x+2)$. $4p=-12$, so $p=-3$. The vertex is $(-2,-1)$, the focus is $(-5,-1)$, and the directrix is $x=1$.



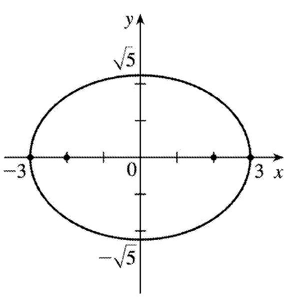
8. $y+12x-2x^2=16 \Rightarrow 2x^2-12x=y-16 \Rightarrow 2(x^2-6x+9)=y-16+18 \Rightarrow 2(x-3)^2=y+2 \Rightarrow (x-3)^2=\frac{1}{2}(y+2)$. $4p=\frac{1}{2}$, so $p=\frac{1}{8}$. The vertex is $(3,-2)$, the focus is $\left(3,-\frac{15}{8}\right)$, and the directrix is $y=-\frac{17}{8}$.



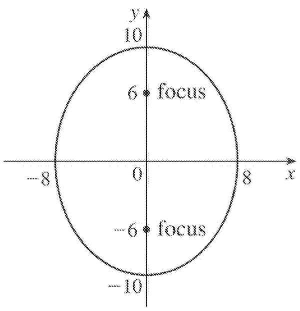
9. The equation has the form $y^2=4px$, where $p<0$. Since the parabola passes through $(-1,1)$, we have $1^2=4p(-1)$, so $4p=-1$ and an equation is $y^2=-x$ or $x=-y^2$. $4p=-1$, so $p=-\frac{1}{4}$ and the focus is $\left(-\frac{1}{4},0\right)$ while the directrix is $x=\frac{1}{4}$.

10. The vertex is $(2,-2)$, so the equation is of the form $(x-2)^2=4p(y+2)$, where $p>0$. The point $(0,0)$ is on the parabola, so $4=4p(2)$ and $4p=2$. Thus, an equation is $(x-2)^2=2(y+2)$. $4p=2$, so $p=\frac{1}{2}$ and the focus is $\left(2,-\frac{3}{2}\right)$ while the directrix is $y=-\frac{5}{2}$.

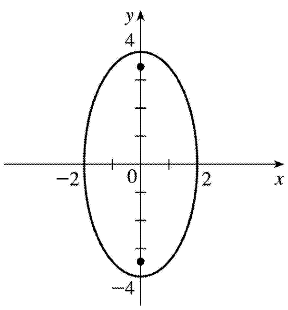
11. $\frac{x^2}{9}+\frac{y^2}{5}=1\Rightarrow a=\sqrt{9}=3$, $b=\sqrt{5}$, $c=\sqrt{a^2-b^2}=\sqrt{9-5}=2$. The ellipse is centered at $(0,0)$, with vertices at $(\pm 3,0)$. The foci are $(\pm 2,0)$.



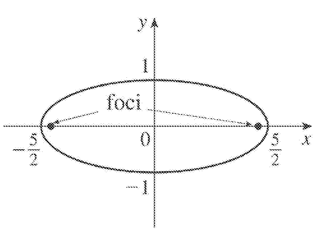
12. $\frac{x^2}{64}+\frac{y^2}{100}=1\Rightarrow a=\sqrt{100}=10$, $b=\sqrt{64}=8$, $c=\sqrt{a^2-b^2}=\sqrt{100-64}=6$. The ellipse is centered at $(0,0)$, with vertices at $(0,\pm 10)$. The foci are $(0,\pm 6)$.



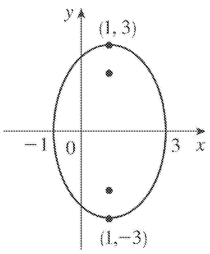
13. $4x^2+y^2=16 \Rightarrow \frac{x^2}{4}+\frac{y^2}{16}=1 \Rightarrow a=\sqrt{16}=4$, $b=\sqrt{4}=2$, $c=\sqrt{a^2-b^2}=\sqrt{16-4}=2\sqrt{3}$. The ellipse is centered at $(0,0)$, with vertices at $(0,\pm 4)$. The foci are $\left(0,\pm 2\sqrt{3}\right)$.



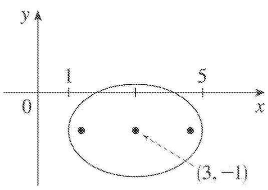
14. $4x^2+25y^2=25 \Rightarrow \frac{x^2}{25/4}+\frac{y^2}{1}=1 \Rightarrow a=\sqrt{\frac{25}{4}}=\frac{5}{2}$, $b=\sqrt{1}=1$, $c=\sqrt{a^2-b^2}=\sqrt{\frac{25}{4}-1}=\sqrt{\frac{21}{4}}=\frac{\sqrt{21}}{2}$. The ellipse is centered at $(0,0)$, with vertices at $\left(\pm\frac{5}{2},0\right)$. The foci are $\left(\pm\frac{\sqrt{21}}{2},0\right)$.



15. $9x^2-18x+4y^2=27 \Leftrightarrow 9(x^2-2x+1)+4y^2=27+9 \Leftrightarrow 9(x-1)^2+4y^2=36 \Leftrightarrow \frac{(x-1)^2}{4}+\frac{y^2}{9}=1 \Rightarrow a=3$, $b=2$, $c=\sqrt{5} \Rightarrow$ center $(1,0)$, vertices $(1,\pm 3)$, foci $\left(1,\pm\sqrt{5}\right)$



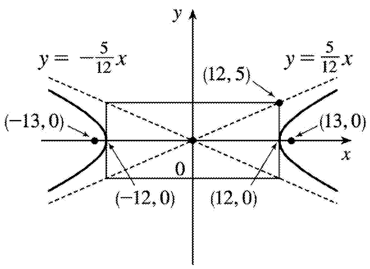
16. $x^2-6x+2y^2+4y=-7 \Leftrightarrow$
$x^2-6x+9+2(y^2+2y+1)=-7+9+2 \Leftrightarrow (x-3)^2+2(y+1)^2=4 \Leftrightarrow$
$\frac{(x-3)^2}{4}+\frac{(y+1)^2}{2}=1 \Rightarrow a=2$, $b=\sqrt{2}=c \Rightarrow$ center $(3,-1)$, vertices $(1,-1)$ and $(5,-1)$, foci $\left(3\pm\sqrt{2},-1\right)$



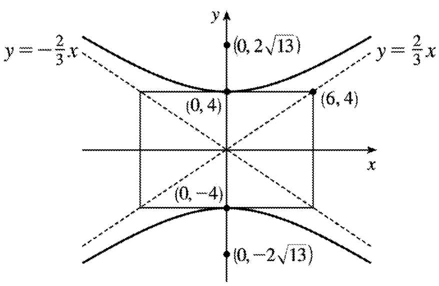
17. The center is $(0,0)$, $a=3$, and $b=2$, so an equation is $\frac{x^2}{4}+\frac{y^2}{9}=1$. $c=\sqrt{a^2-b^2}=\sqrt{5}$, so the foci are $\left(0,\pm\sqrt{5}\right)$.

18. The ellipse is centered at $(2,1)$, with $a=3$ and $b=2$. An equation is $\frac{(x-2)^2}{9}+\frac{(y-1)^2}{4}=1$.
$c=\sqrt{a^2-b^2}=\sqrt{5}$, so the foci are $\left(2\pm\sqrt{5},1\right)$.

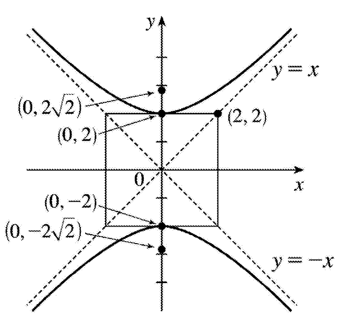
19. $\frac{x^2}{144}-\frac{y^2}{25}=1 \Rightarrow a=12$, $b=5$, $c=\sqrt{144+25}=13 \Rightarrow$ center $(0,0)$, vertices $(\pm 12,0)$, foci $(\pm 13,0)$, asymptotes $y=\pm\frac{5}{12}x$.
Note: It is helpful to draw a $2a$ –by– $2b$ rectangle whose center is the center of the hyperbola. The asymptotes are the extended diagonals of the rectangle.



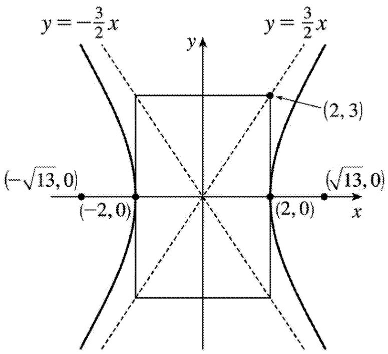
20. $\frac{y^2}{16} - \frac{x^2}{36} = 1 \Rightarrow a = 4$, $b = 6$, $c = \sqrt{a^2 + b^2} = \sqrt{16+36} = \sqrt{52} = 2\sqrt{13}$. The center is $(0,0)$, the vertices are $(0,\pm 4)$, the foci are $\left(0,\pm 2\sqrt{13}\right)$, and the asymptotes are the lines $y = \pm \frac{a}{b} x = \pm \frac{2}{3} x$.



21. $y^2 - x^2 = 4 \Leftrightarrow \frac{y^2}{4} - \frac{x^2}{4} = 1 \Rightarrow a = \sqrt{4} = 2 = b$, $c = \sqrt{4+4} = 2\sqrt{2} \Rightarrow$ center $(0,0)$, vertices $(0,\pm 2)$, foci $\left(0,\pm 2\sqrt{2}\right)$, asymptotes $y = \pm x$

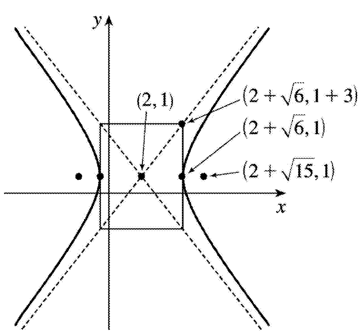


22. $9x^2 - 4y^2 = 36 \Leftrightarrow \frac{x^2}{4} - \frac{y^2}{9} = 1 \Rightarrow a = \sqrt{4} = 2$, $b = \sqrt{9} = 3$, $c = \sqrt{4+9} = \sqrt{13} \Rightarrow$ center $(0,0)$, vertices $(\pm 2,0)$, foci $\left(\pm\sqrt{13},0\right)$, asymptotes $y = \pm \frac{3}{2} x$



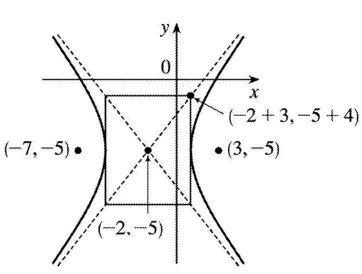
23. $2y^2-4y-3x^2+12x=-8 \Leftrightarrow$

$2(y^2-2y+1)-3(x^2-4x+4)=-8+2-12 \Leftrightarrow 2(y-1)^2-3(x-2)^2=-18 \Leftrightarrow \frac{(x-2)^2}{6}-\frac{(y-1)^2}{9}=1 \Rightarrow a=\sqrt{6}$, $b=3$, $c=\sqrt{15} \Rightarrow$ center $(2,1)$, vertices $\left(2\pm\sqrt{6},1\right)$, foci $\left(2\pm\sqrt{15},1\right)$, asymptotes $y-1=\pm\frac{3}{\sqrt{6}}(x-2)$ or $y-1=\pm\frac{\sqrt{6}}{2}(x-2)$



24. $16x^2+64x-9y^2-90y=305 \Leftrightarrow$

$16(x^2+4x+4)-9(y^2+10y+25)=305+64-225 \Leftrightarrow 16(x+2)^2-9(y+5)^2=144 \Leftrightarrow \frac{(x+2)^2}{9}-\frac{(y+5)^2}{16}=1 \Rightarrow a=3$, $b=4$, $c=5 \Rightarrow$ center $(-2,-5)$, vertices $(-5,-5)$ and $(1,-5)$, foci $(-7,-5)$ and $(3,-5)$, asymptotes $y+5=\pm\frac{4}{3}(x+2)$



25. $x^2=y+1 \Leftrightarrow x^2=1(y+1)$. This is an equation of a *parabola* with $4p=1$, so $p=\frac{1}{4}$. The vertex is $(0,-1)$ and the focus is $\left(0,-\frac{3}{4}\right)$.

26. $x^2=y^2+1 \Leftrightarrow x^2-y^2=1$. This is an equation of a *hyperbola* with vertices $(\pm 1,0)$. The foci are at $(\pm\sqrt{1+1},0)=(\pm\sqrt{2},0)$.

27. $x^2=4y-2y^2 \Leftrightarrow x^2+2y^2-4y=0 \Leftrightarrow x^2+2(y^2-2y+1)=2 \Leftrightarrow x^2+2(y-1)^2=2 \Leftrightarrow \frac{x^2}{2}+\frac{(y-1)^2}{1}=1$. This is an equation of an *ellipse* with vertices at $(\pm\sqrt{2},1)$. The foci are at $(\pm\sqrt{2-1},1)=(\pm 1,1)$.

28. $y^2-8y=6x-16 \Leftrightarrow y^2-8y+16=6x \Leftrightarrow (y-4)^2=6x$. This is an equation of a *parabola* with $4p=6$, so $p=\frac{3}{2}$. The vertex is $(0,4)$ and the focus is $\left(\frac{3}{2},4\right)$.

29. $y^2+2y=4x^2+3 \Leftrightarrow y^2+2y+1=4x^2+4 \Leftrightarrow (y+1)^2-4x^2=4 \Leftrightarrow \frac{(y+1)^2}{4}-x^2=1$. This is an equation of a *hyperbola* with vertices $(0,-1\pm 2)=(0,1)$ and $(0,-3)$. The foci are at $(0,-1\pm\sqrt{4+1})=(0,-1\pm\sqrt{5})$.

30. $4x^2+4x+y^2=0 \Leftrightarrow 4\left(x^2+x+\frac{1}{4}\right)+y^2=1 \Leftrightarrow 4\left(x+\frac{1}{2}\right)^2+y^2=1 \Leftrightarrow \frac{\left(x+\frac{1}{2}\right)^2}{1/4}+y^2=1$. This is an equation of an *ellipse* with vertices $\left(-\frac{1}{2},0\pm 1\right)=\left(-\frac{1}{2},\pm 1\right)$. The foci are at $\left(-\frac{1}{2},0\pm\sqrt{1-\frac{1}{4}}\right)=\left(-\frac{1}{2},\pm\sqrt{3}/2\right)$.

31. The parabola with vertex $(0,0)$ and focus $(0,-2)$ opens downward and has $p=-2$, so its equation is $x^2=4py=-8y$.

32. The parabola with vertex $(1,0)$ and directrix $x=-5$ opens to the right and has $p=6$, so its equation is $y^2=4p(x-1)=24(x-1)$.

33. The distance from the focus $(-4,0)$ to the directrix $x=2$ is $2-(-4)=6$, so the distance from the focus to the vertex is $\frac{1}{2}(6)=3$ and the vertex is $(-1,0)$. Since the focus is to the left of the vertex, $p=-3$. An equation is $y^2=4p(x+1) \Rightarrow y^2=-12(x+1)$.

34. The distance from the focus (3,6) to the vertex $(3,2)$ is $6-2=4$. Since the focus is above the vertex, $p=4$. An equation is $(x-3)^2=4p(y-2) \Rightarrow (x-3)^2=16(y-2)$.

35. The parabola must have equation $y^2=4px$, so $(-4)^2=4p(1) \Rightarrow p=4 \Rightarrow y^2=16x$.

36. Vertical axis $\Rightarrow (x-h)^2=4p(y-k)$. Substituting $(-2,3)$ and $(0,3)$ gives $(-2-h)^2=4p(3-k)$ and $(-h)^2=4p(3-k) \Rightarrow (-2-h)^2=(-h)^2 \Rightarrow 4+4h+h^2=h^2 \Rightarrow h=-1 \Rightarrow 1=4p(3-k)$. Substituting $(1,9)$ gives $[1-(-1)]^2=4p(9-k) \Rightarrow 4=4p(9-k)$. Solving for p from these equations gives $p=\frac{1}{4(3-k)}=\frac{1}{9-k} \Rightarrow$ $4(3-k)=9-k \Rightarrow k=1 \Rightarrow p=\frac{1}{8} \Rightarrow (x+1)^2=\frac{1}{2}(y-1) \Rightarrow 2x^2+4x-y+3=0$.

37. The ellipse with foci $(\pm 2,0)$ and vertices $(\pm 5,0)$ has center $(0,0)$ and a horizontal major axis, with $a=5$ and $c=2$, so $b=\sqrt{a^2-c^2}=\sqrt{21}$. An equation is $\frac{x^2}{25}+\frac{y^2}{21}=1$.

38. The ellipse with foci $(0,\pm 5)$ and vertices $(0,\pm 13)$ has center $(0,0)$ and a vertical major axis, with $c=5$ and $a=13$, so $b=\sqrt{a^2-c^2}=12$. An equation is $\frac{x^2}{144}+\frac{y^2}{169}=1$.

39. Since the vertices are $(0,0)$ and (0,8) , the ellipse has center $(0,4)$ with a vertical axis and $a=4$. The foci at $(0,2)$ and $(0,6)$ are 2 units from the center, so $c=2$ and $b=\sqrt{a^2-c^2}=\sqrt{4^2-2^2}=\sqrt{12}$. An equation is $\frac{(x-0)^2}{b^2}+\frac{(y-4)^2}{a^2}=1 \Rightarrow \frac{x^2}{12}+\frac{(y-4)^2}{16}=1$.

40. Since the foci are (0,–1) and (8,–1) , the ellipse has center $(4,-1)$ with a horizontal axis and $c=4$. The vertex $(9,-1)$ is 5 units from the center, so $a=5$ and $b=\sqrt{a^2-c^2}=\sqrt{5^2-4^2}=\sqrt{9}$. An equation is $\frac{(x-4)^2}{a^2}+\frac{(y+1)^2}{b^2}=1 \Rightarrow \frac{(x-4)^2}{25}+\frac{(y+1)^2}{9}=1$.

41. Center $(2,2)$, $c=2$, $a=3 \Rightarrow b=\sqrt{5} \Rightarrow \frac{1}{9}(x-2)^2+\frac{1}{5}(y-2)^2=1$

42. Center $(0,0)$, $c=2$, major axis horizontal $\Rightarrow$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $b^2=a^2-c^2=a^2-4$. Since the ellipse passes through $(2,1)$, we have

$2a=\left|PF_1\right|+\left|PF_2\right|=\sqrt{17}+1 \Rightarrow a^2=\frac{9+\sqrt{17}}{2}$ and $b^2=\frac{1+\sqrt{17}}{2}$, so the ellipse has equation

$\frac{2x^2}{9+\sqrt{17}}+\frac{2y^2}{1+\sqrt{17}}=1$.

43. Center $(0,0)$, vertical axis, $c=3$, $a=1 \Rightarrow b=\sqrt{8}=2\sqrt{2} \Rightarrow y^2-\frac{1}{8}x^2=1$

44. Center $(0,0)$, horizontal axis, $c=6$, $a=4 \Rightarrow b=2\sqrt{5} \Rightarrow \frac{1}{16}x^2-\frac{1}{20}y^2=1$

45. Center $(4,3)$, horizontal axis, $c=3$, $a=2 \Rightarrow b=\sqrt{5} \Rightarrow \frac{1}{4}(x-4)^2-\frac{1}{5}(y-3)^2=1$

46. Center $(2,3)$, vertical axis, $c=5$, $a=3 \Rightarrow b=4 \Rightarrow \frac{1}{9}(y-3)^2-\frac{1}{16}(x-2)^2=1$

47. Center $(0,0)$, horizontal axis, $a=3$, $\frac{b}{a}=2 \Rightarrow b=6 \Rightarrow \frac{1}{9}x^2-\frac{1}{36}y^2=1$

48. Center $(4,2)$, horizontal axis, asymptotes $y-2=\pm(x-4) \Rightarrow c=2$, $b/a=1 \Rightarrow a=b \Rightarrow c^2=4=a^2+b^2=2a^2 \Rightarrow a^2=2 \Rightarrow \frac{1}{2}(x-4)^2-\frac{1}{2}(y-2)^2=1$

49. In Figure 8, we see that the point on the ellipse closest to a focus is the closer vertex (which is a distance $a-c$ from it) while the farthest point is the other vertex (at a distance of $a+c$). So for this lunar orbit, $(a-c)+(a+c)=2a=(1728+110)+(1728+314)$, or $a=1940$; and $(a+c)-(a-c)=2c=314-110$, or $c=102$. Thus, $b^2=a^2-c^2=3 , 753 , 196$, and the equation is $\frac{x^2}{3,763,600}+\frac{y^2}{3,753,196}=1$.

50. **(a)** Choose V to be the origin, with $x-$ axis through V and F . \ Then F is $(p,0)$, A is $(p,5)$, so substituting A into the equation $y^2=4px$ gives $25=4p^2$ so $p=\frac{5}{2}$ and $y^2=10x$.

(b) $x=11 \Rightarrow y=\sqrt{110} \Rightarrow |CD|=2\sqrt{110}$

51. **(a)** Set up the coordinate system so that A is $(-200,0)$ and B is $(200,0)$.
$|PA|-|PB|=(1200)\ (980)=1 , 176 , 000$ ft

$=\frac{2450}{11}$ mi $=2a \Rightarrow a=\frac{1225}{11}$, and $c=200$ so $b^2=c^2-a^2=\frac{3,339,375}{121} \Rightarrow \frac{121x^2}{1,500,625}-\frac{121y^2}{3,339,375}=1$.

(b) Due north of $B \Rightarrow x=200 \Rightarrow \frac{(121)(200)^2}{1,500,625}-\frac{121y^2}{3,339,375}=1 \Rightarrow y=\frac{133,575}{539} \approx 248$ mi

52. $\left|PF_1\right|-\left|PF_2\right|=\pm 2a \Leftrightarrow \sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a \Leftrightarrow \sqrt{(x+c)^2+y^2}=\sqrt{(x-c)^2+y^2}\pm 2a \Leftrightarrow$
$(x+c)^2+y^2=(x-c)^2+y^2+4a^2\pm 4a\sqrt{(x-c)^2+y^2} \Leftrightarrow 4cx-4a^2=\pm 4a\sqrt{(x-c)^2+y^2} \Leftrightarrow$
$c^2x^2-2a^2cx+a^4=a^2\left(x^2-2cx+c^2+y^2\right) \Leftrightarrow \left(c^2-a^2\right)x^2-a^2y^2=a^2\left(c^2-a^2\right) \Leftrightarrow b^2x^2-a^2y^2=a^2b^2$ (where $b^2=c^2-a^2$) $\Leftrightarrow \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

53. The function whose graph is the upper branch of this hyperbola is concave upward. The function is $y=f(x)=a\sqrt{1+\frac{x^2}{b^2}}=\frac{a}{b}\sqrt{b^2+x^2}$, so $y'=\frac{a}{b}x\left(b^2+x^2\right)^{-1/2}$ and
$y''=\frac{a}{b}\left[\left(b^2+x^2\right)^{-1/2}-x^2\left(b^2+x^2\right)^{-3/2}\right]=ab\left(b^2+x^2\right)^{-3/2}>0$ for all x , and so f is concave upward.

54. We can follow exactly the same sequence of steps as in the derivation of Formula 4, except we use the points $(1,1)$ and $(-1,-1)$ in the distance formula (first equation of that derivation) so $\sqrt{(x-1)^2+(y-1)^2}+\sqrt{(x+1)^2+(y+1)^2}=4$ will lead (after moving the second term to the right, squaring, and simplifying) to $2\sqrt{(x+1)^2+(y+1)^2}=x+y+4$, which, after squaring and simplifying again, leads to $3x^2-2xy+3y^2=8$.

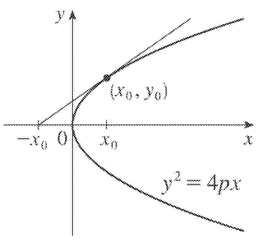
55. **(a)** If $k>16$, then $k-16>0$, and $\frac{x^2}{k}+\frac{y^2}{k-16}=1$ is an *ellipse* since it is the sum of two squares on the left side.

(b) If $0<k<16$, then $k-16<0$, and $\frac{x^2}{k}+\frac{y^2}{k-16}=1$ is a *hyperbola* since it is the difference of two squares on the left side.
(c) If $k<0$, then $k-16<0$, and there is *no curve* since the left side is the sum of two negative terms, which cannot equal 1.
(d) In case (a), $a^2=k$, $b^2=k-16$, and $c^2=a^2-b^2=16$, so the foci are at $(\pm 4,0)$. In case (b), $k-16<0$, so

$a^2=k$, $b^2=16-k$, and $c^2=a^2+b^2=16$, and so again the foci are at $(\pm 4,0)$.

56. **(a)** $y^2=4px \Rightarrow 2yy'=4p \Rightarrow y'=\frac{2p}{y}$, so the tangent line is $y-y_0=\frac{2p}{y_0}(x-x_0) \Rightarrow yy_0-y_0^2=2p(x-x_0) \Leftrightarrow$

$yy_0-4px_0=2px-2px_0 \Rightarrow yy_0=2p(x+x_0)$.



(b) The $x-$ intercept is $-x_0$.

57. Use the parametrization $x=2\cos t$, $y=\sin t$, $0\leq t\leq 2\pi$ to get

$$L=4\int_0^{\pi/2}\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt=4\int_0^{\pi/2}\sqrt{4\sin^2 t+\cos^2 t}\,dt=4\int_0^{\pi/2}\sqrt{3\sin^2 t+1}\,dt$$

Using Simpson's Rule with $n=10$, $\Delta t=\frac{\pi/2-0}{10}=\frac{\pi}{20}$, and $f(t)=\sqrt{3\sin^2 t+1}$, we get

$$L\approx \frac{4}{3}\left(\frac{\pi}{20}\right)\left[f(0)+4f\left(\frac{\pi}{20}\right)+2f\left(\frac{2\pi}{20}\right)+\cdots+2f\left(\frac{8\pi}{20}\right)+4f\left(\frac{9\pi}{20}\right)+f\left(\frac{\pi}{2}\right)\right]\approx 9.69$$

58. The length of the major axis is $2a$, so $a=\frac{1}{2}(1.18\times 10^{10})=5.9\times 10^9$. The length of the minor axis is $2b$, so $b=\frac{1}{2}(1.14\times 10^{10})=5.7\times 10^9$. An equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, or converting into parametric equations, $x=a\cos\theta$ and $y=b\sin\theta$. So

$$L=4\int_0^{\pi/2}\sqrt{(dx/d\theta)^2+(dy/d\theta)^2}\,d\theta=4\int_0^{\pi/2}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\,d\theta$$

Using Simpson's Rule with $n=10$, $\Delta\theta=\frac{\pi/2-0}{10}=\frac{\pi}{20}$, and $f(\theta)=\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$, we get

$L\approx 4\cdot S_{10}$

$$= 4\cdot\frac{\pi}{20\cdot 3}\left[f(0)+4f\left(\frac{\pi}{20}\right)+2f\left(\frac{2\pi}{20}\right)+\cdots+2f\left(\frac{8\pi}{20}\right)+4f\left(\frac{9\pi}{20}\right)+f\left(\frac{\pi}{2}\right)\right]$$
$$\approx 3.64\times 10^{10}\text{km}$$

59. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\Rightarrow \frac{2x}{a^2}+\frac{2yy'}{b^2}=0\Rightarrow y'=-\frac{b^2x}{a^2y}$ ($y\neq 0$). Thus, the slope of the tangent line at P is $-\frac{b^2x_1}{a^2y_1}$. The slope of F_1P is $\frac{y_1}{x_1+c}$ and of F_2P is $\frac{y_1}{x_1-c}$. By the formula from Problems Plus, we have

$$\tan\alpha=\frac{\frac{y_1}{x_1+c}+\frac{b^2x_1}{a^2y_1}}{1-\frac{b^2x_1y_1}{a^2y_1(x_1+c)}}=\frac{a^2y_1^2+b^2x_1(x_1+c)}{a^2y_1(x_1+c)-b^2x_1y_1}=\frac{a^2b^2+b^2cx_1}{c^2x_1y_1+a^2cy_1}\left[\begin{array}{r}\text{using } b^2x_1^2+a^2y_1^2=a^2b^2\\ \text{and } a^2-b^2=c^2\end{array}\right]=$$

$$\frac{b^2(cx_1+a^2)}{cy_1(cx_1+a^2)}=\frac{b^2}{cy_1}\quad\text{and}$$

$$\tan\beta=\frac{-\frac{y_1}{x_1-c}-\frac{b^2x_1}{a^2y_1}}{1-\frac{b^2x_1y_1}{a^2y_1(x_1-c)}}=\frac{-a^2y_1^2-b^2x_1\left(x_1-c\right)}{a^2y_1(x_1-c)-b^2x_1y_1}=\frac{-a^2b^2+b^2cx_1}{c^2x_1y_1-a^2cy_1}=\frac{b^2(cx_1-a^2)}{cy_1(cx_1-a^2)}=\frac{b^2}{cy_1}\quad\text{So } \alpha=\beta$$

60. The slopes of the line segments F_1P and F_2P are $\frac{y_1}{x_1+c}$ and $\frac{y_1}{x_1-c}$, where P is $\left(x_1,y_1\right)$.

Differentiating implicitly, $\frac{2x}{a^2}-\frac{2yy'}{b^2}=0\Rightarrow y'=\frac{b^2x}{a^2y}\Rightarrow$ the slope of the tangent at P is $\frac{b^2x_1}{a^2y_1}$, so by the formula from Problems Plus,

$$\tan\alpha = \frac{\dfrac{b^2x_1}{a^2y_1} - \dfrac{y_1}{x_1+c}}{1+\dfrac{b^2x_1y_1}{a^2y_1(x_1+c)}} = \frac{b^2x_1(x_1+c)-a^2y_1^2}{a^2y_1(x_1+c)+b^2x_1y_1} = \frac{b^2(cx_1+a^2)}{cy_1(cx_1+a^2)}\left[\begin{array}{r}\text{using } x_1^2/a^2-y_1^2/b^2=1\\ \text{and } a^2+b^2=c^2\end{array}\right] = \frac{b^2}{cy_1}$$

and

$$\tan\beta = \frac{-\dfrac{b^2x_1}{a^2y_1} + \dfrac{y_1}{x_1-c}}{1+\dfrac{b^2x_1y_1}{a^2y_1(x_1-c)}} = \frac{-b^2x_1(x_1-c)+a^2y_1^2}{a^2y_1(x_1-c)+b^2x_1y_1} = \frac{b^2(cx_1-a^2)}{cy_1(cx_1-a^2)} = \frac{b^2}{cy_1}$$

So $\alpha=\beta$.

1. The directrix $y=6$ is above the focus at the origin, so we use the form with " $+e\sin\theta$ " in the denominator. (See Theorem 6 and Figure 2.) $r=\dfrac{ed}{1+e\sin\theta}=\dfrac{\frac{7}{4}\cdot 6}{1+\frac{7}{4}\sin\theta}=\dfrac{42}{4+7\sin\theta}$

2. The directrix $x=4$ is to the right of the focus at the origin, so we use the form with " $+e\cos\theta$ " in the denominator. $e=1$ for a parabola, so an equation is $r=\dfrac{ed}{1+e\cos\theta}=\dfrac{1\cdot 4}{1+1\cos\theta}=\dfrac{4}{1+\cos\theta}$

3. The directrix $x=-5$ is to the left of the focus at the origin, so we use the form with " $-e\cos\theta$ " in the denominator. $r=\dfrac{ed}{1-e\cos\theta}=\dfrac{\frac{3}{4}\cdot 5}{1-\frac{3}{4}\cos\theta}=\dfrac{15}{4-3\cos\theta}$

4. The directrix $y=-2$ is below the focus at the origin, so we use the form with " $-e\sin\theta$ " in the denominator. $r=\dfrac{ed}{1-e\sin\theta}=\dfrac{2\cdot 2}{1-2\sin\theta}=\dfrac{4}{1-2\sin\theta}$

5. The vertex $(4,3\pi/2)$ is 4 units below the focus at the origin, so the directrix is 8 units below the focus $(d=8)$, and we use the form with " $-e\sin\theta$ " in the denominator. $e=1$ for a parabola, so an equation is $r=\dfrac{ed}{1-e\sin\theta}=\dfrac{1(8)}{1-1\sin\theta}=\dfrac{8}{1-\sin\theta}$.

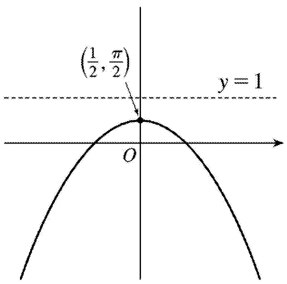
6. The vertex $P(1,\pi/2)$ is 1 unit above the focus F at the origin, so $|PF|=1$ and we use the form with " $+e\sin\theta$ " in the denominator. The distance from the focus to the directrix l is d , so $e=\dfrac{|PF|}{|Pl|}\Rightarrow 0.8=\dfrac{1}{d-1}\Rightarrow 0.8d-0.8=1\Rightarrow 0.8d=1.8\Rightarrow d=2.25$.
An equation is $r=\dfrac{ed}{1+e\sin\theta}=\dfrac{0.8(2.25)}{1+0.8\sin\theta}\cdot\dfrac{5}{5}=\dfrac{9}{5+4\sin\theta}$.

7. The directrix $r=4\sec\theta$ (equivalent to $r\cos\theta=4$ or $x=4$) is to the right of the focus at the origin, so we will use the form with " $+e\cos\theta$ " in the denominator. The distance from the focus to the directrix is $d=4$, so an equation is $r=\dfrac{ed}{1+e\cos\theta}=\dfrac{0.5(4)}{1+0.5\cos\theta}\cdot\dfrac{2}{2}=\dfrac{4}{2+\cos\theta}$.

8. The directrix $r=-6\csc\theta$ (equivalent to $r\sin\theta=-6$ or $y=-6$) is below the focus at the origin, so we will use the form with " $-e\sin\theta$ " in the denominator. The distance from the focus to the directrix is $d=6$, so an equation is $r=\dfrac{ed}{1-e\sin\theta}=\dfrac{3(6)}{1-3\sin\theta}=\dfrac{18}{1-3\sin\theta}$.

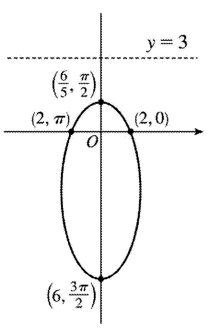
9. $r=\dfrac{1}{1+\sin\theta}=\dfrac{ed}{1+e\sin\theta}$, where $d=e=1$.

(a) Eccentricity $=e=1$
(b) Since $e=1$, the conic is a parabola.
(c) Since " $+e\sin\theta$ '' appears in the denominator, the directrix is above the focus at the origin. $d=|Fl|=1$, so an equation of the directrix is $y=1$.
(d) The vertex is at $\left(\frac{1}{2},\frac{\pi}{2}\right)$, midway between the focus and the directrix.



10. $r=\dfrac{6}{3+2\sin\theta}=\dfrac{2}{1+\frac{2}{3}\sin\theta}=\dfrac{\frac{2}{3}\cdot 3}{1+\frac{2}{3}\sin\theta}$

(a) $e=\frac{2}{3}$
(b) Ellipse
(c) $y=3$
(d) Vertices $\left(\frac{6}{5},\frac{\pi}{2}\right)$ and $\left(6,\frac{3\pi}{2}\right)$; center $\left(\frac{12}{5},\frac{3\pi}{2}\right)$



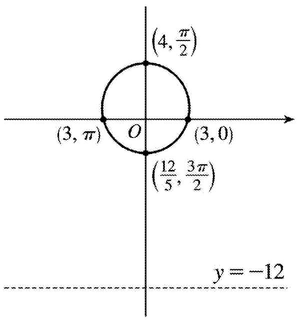
11. $r=\dfrac{12}{4-\sin\theta}\cdot\dfrac{1/4}{1/4}=\dfrac{3}{1-\frac{1}{4}\sin\theta}$, where $e=\frac{1}{4}$ and $ed=3\Rightarrow d=12$.

(a) Eccentricity $=e=\frac{1}{4}$
(b) Since

$e=\frac{1}{4}<1$, the conic is an ellipse.

(c) Since " $-e\sin\theta$ " appears in the denominator, the directrix is below the focus at the origin. $d=|Fl|=12$, so an equation of the directrix is $y=-12$.

(d) The vertices are $\left(4,\frac{\pi}{2}\right)$ and $\left(\frac{12}{5},\frac{3\pi}{2}\right)$, so the center is midway between them, that is, $\left(\frac{4}{5},\frac{\pi}{2}\right)$.



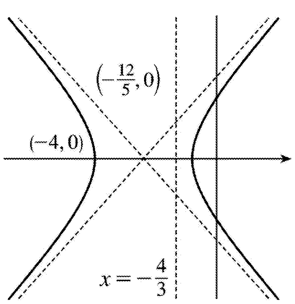
12. $r=\dfrac{4}{2-3\cos\theta}=\dfrac{2}{1-\frac{3}{2}\cos\theta}=\dfrac{\frac{3}{2}\cdot\frac{4}{3}}{1-\frac{3}{2}\cos\theta}$

(a) $e=\frac{3}{2}$

(b) Hyperbola

(c) $x=-\frac{4}{3}$

(d) The vertices are $(-4,0)$ and $\left(\frac{4}{5},\pi\right)=\left(-\frac{4}{5},0\right)$, so the center is $\left(-\frac{12}{5},0\right)$. The asymptotes are parallel to $\theta=\pm\cos^{-1}\frac{2}{3}$. [Their slopes are $\pm\tan\left(\cos^{-1}\frac{2}{3}\right)=\pm\frac{\sqrt{5}}{2}$]



13.

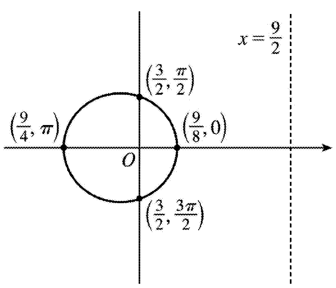
$r=\frac{9}{6+2\cos\theta}\cdot\frac{1/6}{1/6}=\frac{3/2}{1+\frac{1}{3}\cos\theta}$, where $e=\frac{1}{3}$ and $ed=\frac{3}{2}\Rightarrow d=\frac{9}{2}$.

(a) Eccentricity $=e=\frac{1}{3}$

(b) Since $e=\frac{1}{3}<1$, the conic is an ellipse.

(c) Since " $+e\cos\theta$ ’’ appears in the denominator, the directrix is to the right of the focus at the origin. $d=|Fl|=\frac{9}{2}$, so an equation of the directrix is $x=\frac{9}{2}$.

(d) The vertices are $\left(\frac{9}{8},0\right)$ and $\left(\frac{9}{4},\pi\right)$, so the center is midway between them, that is, $\left(\frac{9}{16},\pi\right)$.



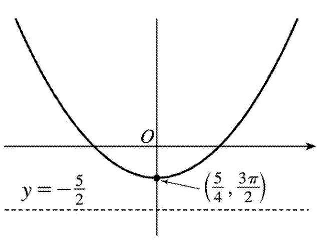
14. $r=\frac{5}{2-2\sin\theta}=\frac{\frac{5}{2}}{1-\sin\theta}$

(a) $e=1$

(b) Parabola

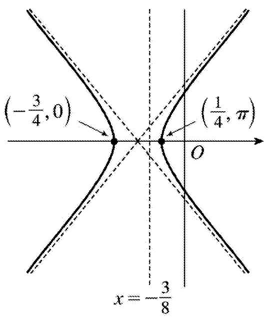
(c) $y=-\frac{5}{2}$

(d) The focus is $(0,0)$, so the vertex is $\left(\frac{5}{4},\frac{3\pi}{2}\right)$ and the parabola opens up.



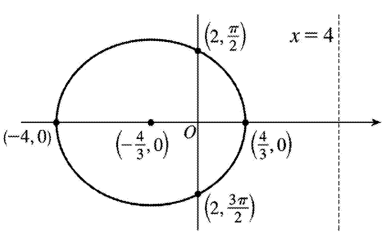
15. $r=\frac{3}{4-8\cos\theta}\cdot\frac{1/4}{1/4}=\frac{3/4}{1-2\cos\theta}$, where $e=2$ and $ed=\frac{3}{4}\Rightarrow d=\frac{3}{8}$.

(a) Eccentricity $=e=2$
(b) Since $e=2>1$, the conic is a hyperbola.
(c) Since " $-e\cos\theta$ '' appears in the denominator, the directrix is to the left of the focus at the origin. $d=|Fl|=\frac{3}{8}$, so an equation of the directrix is $x=-\frac{3}{8}$.
(d) The vertices are $\left(-\frac{3}{4},0\right)$ and $\left(\frac{1}{4},\pi\right)$, so the center is midway between them, that is, $\left(\frac{1}{2},\pi\right)$.



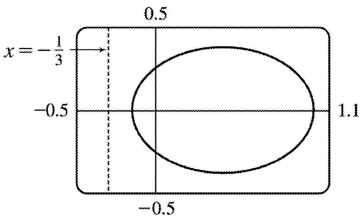
16. $r=\frac{4}{2+\cos\theta}=\frac{2}{1+\frac{1}{2}\cos\theta}=\frac{\frac{1}{2}\cdot 4}{1+\frac{1}{2}\cos\theta}$

(a) $e=\frac{1}{2}$
(b) Ellipse
(c) $x=4$
(d) The vertices are $\left(\frac{4}{3},0\right)$ and $(4,\pi)=(-4,0)$, so the center is $\left(-\frac{4}{3},0\right)$.

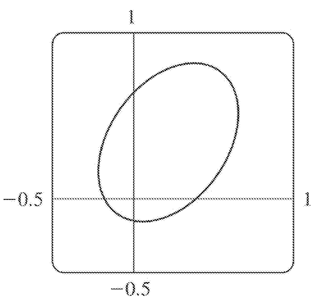


17. **(a)** The equation is $r=\frac{1}{4-3\cos\theta}=\frac{1/4}{1-\frac{3}{4}\cos\theta}$, so $e=\frac{3}{4}$ and $ed=\frac{1}{4}\Rightarrow d=\frac{1}{3}$. The conic is an ellipse, and the equation of its directrix is $x=r\cos\theta=-\frac{1}{3}\Rightarrow r=-\frac{1}{3\cos\theta}$. We must be careful in our

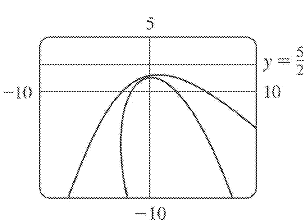
choice of parameter values in this equation ($-1\leq\theta\leq 1$ works well).



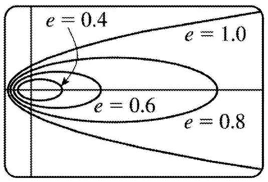
(b) The equation is obtained by replacing θ with $\theta-\dfrac{\pi}{3}$ in the equation of the original conic (see Example 4), so $r=\dfrac{1}{4-3\cos\left(\theta-\dfrac{\pi}{3}\right)}$.



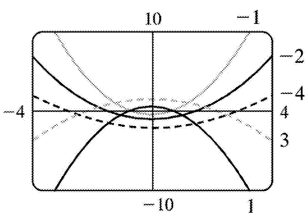
18. $r=\dfrac{5}{2+2\sin\theta}=\dfrac{5/2}{1+\sin\theta}$, so $e=1$ and $d=\dfrac{5}{2}$. The equation of the directrix is $y=r\sin\theta=\dfrac{5}{2}\Rightarrow$ $r=\dfrac{5}{2\sin\theta}$. If the parabola is rotated about its focus (the origin) through $\dfrac{\pi}{6}$, its equation is the same as that of the original, with θ replaced by $\theta-\dfrac{\pi}{6}$ (see Example 4), so $r=\dfrac{5}{2+2\sin\,(\theta-\pi/6)}$. In graphing each of these curves, we must be careful to select parameter ranges which prevent the denominator from vanishing while still showing enough of the curve.



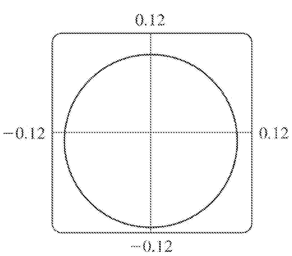
19. For $e<1$ the curve is an ellipse. It is nearly circular when e is close to 0 . As e increases, the graph is stretched out to the right, and grows larger (that is, its right–hand focus moves to the right while its left–hand focus remains at the origin.) At $e=1$, the curve becomes a parabola with focus at the origin.



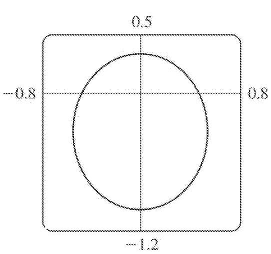
20. **(a)** The value of *d* does not seem to affect the shape of the conic (a parabola) at all, just its size, position, and orientation (for $d<0$ it opens upward, for $d>0$ it opens downward).



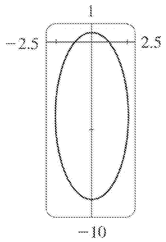
(b) We consider only positive values of e . When $0<e<1$, the conic is an ellipse. As $e \to 0^+$, the graph approaches perfect roundness and zero size. As e increases, the ellipse becomes more elongated, until at $e=1$ it turns into a parabola. For $e>1$, the conic is a hyperbola, which moves downward and gets broader as e continues to increase.



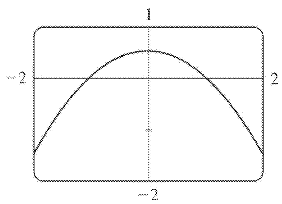
$e=0.1$



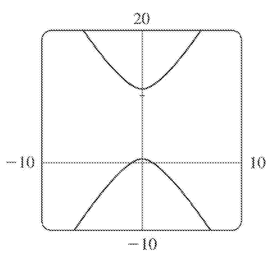
$e=0.5$



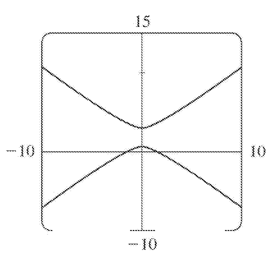
e=0.9



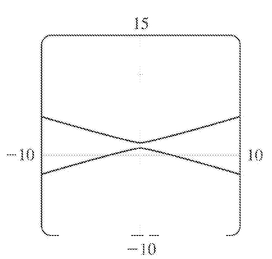
e=1



e=1.1

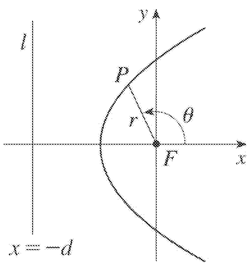


e=1.5

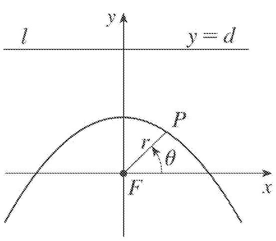


$e=10$

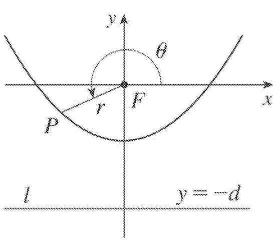
21. $|PF|=e|Pl| \Rightarrow r=e[d-r\cos(\pi-\theta)]=e(d+r\cos\theta) \Rightarrow r(1-e\cos\theta)=ed \Rightarrow r=\dfrac{ed}{1-e\cos\theta}$



22. $|PF|=e|Pl| \Rightarrow r=e[d-r\sin\theta] \Rightarrow r(1+e\sin\theta)=ed \Rightarrow r=\dfrac{ed}{1+e\sin\theta}$



23. $|PF|=e|Pl| \Rightarrow r=e[d-r\sin(\theta-\pi)]=e(d+r\sin\theta) \Rightarrow r(1-e\sin\theta)=ed \Rightarrow r=\dfrac{ed}{1-e\sin\theta}$



24. The parabolas intersect at the two points where $\dfrac{c}{1+\cos\theta}=\dfrac{d}{1-\cos\theta} \Rightarrow \cos\theta=\dfrac{c-d}{c+d} \Rightarrow r=\dfrac{c+d}{2}$.

For the first parabola, $\dfrac{dr}{d\theta}=\dfrac{c\sin\theta}{(1+\cos\theta)^2}$, so

$$\frac{dy}{dx}=\frac{(dr/d\theta)\sin\theta+r\cos\theta}{(dr/d\theta)\cos\theta-r\sin\theta}=\frac{c\sin^2\theta+c\cos\theta(1+\cos\theta)}{c\sin\theta\cos\theta-c\sin\theta(1+\cos\theta)}=\frac{1+\cos\theta}{-\sin\theta}$$

and similarly for the second,

$\frac{dy}{dx}=\frac{1-\cos\theta}{\sin\theta}=\frac{\sin\theta}{1+\cos\theta}$. Since the product of these slopes is -1 , the parabolas intersect at right angles.

25. **(a)** If the directrix is $x=-d$, then $r=\frac{ed}{1-e\cos\theta}$, and, from (4), $a^2=\frac{e^2d^2}{\left(1-e^2\right)^2}\Rightarrow ed=a(1-e^2)$.

Therefore, $r=\frac{a(1-e^2)}{1-e\cos\theta}$.

(b) $e=0.017$ and the major axis $=2a=2.99\times10^8\Rightarrow a=1.495\times10^8$. Therefore

$r=\frac{1.495\times10^8\left[1-(0.017)^2\right]}{1-0.017\cos\theta}\approx\frac{1.49\times10^8}{1-0.017\cos\theta}$.

26. **(a)** The Sun is at point F in Figure 1 so that perihelion is in the positive x –direction and aphelion is in the negative x –direction. At perihelion, $\theta=0$, so $r=\frac{a\left(1-e^2\right)}{1+e\cos 0}=\frac{a(1-e)(1+e)}{1+e}=a(1-e)$.

At aphelion, $\theta=\pi$, so $r=\frac{a\left(1-e^2\right)}{1+e\cos\pi}=\frac{a(1-e)(1+e)}{1-e}=a(1+e)$.

(b) At perihelion, $r=a(1-e)\approx(1.495\times10^8)(1-0.017)\approx1.47\times10^8$ km.

At aphelion, $r=a(1+e)\approx\left(1.495\times10^8\right)(1+0.017)\approx1.52\times10^8$ km.

27. Here $2a=$ length of major axis $=36.18$ AU $\Rightarrow a=18.09$ AU and $e=0.97$. By Exercise 25(a), the equation of the orbit is $r=\frac{18.09\left[1-(0.97)^2\right]}{1-0.97\cos\theta}\approx\frac{1.07}{1-0.97\cos\theta}$. By Exercise 26(a), the maximum distance from the comet to the sun is $18.09(1+0.97)\approx35.64$ AU or about 3.314 billion miles.

28. Here $2a=$ length of major axis $=356.5$ AU $\Rightarrow a=178.25$ AU and $e=0.9951$. By Exercise 25(a), the equation of the orbit is $r=\frac{178.25\left[1-(0.9951)^2\right]}{1-0.9951\cos\theta}\approx\frac{1.7426}{1-0.9951\cos\theta}$. By Exercise 26(a), the minimum distance from the comet to the sun is $178.25\,(1-0.9951)\approx0.8734$ AU or about 81 million miles.

29. The minimum distance is at perihelion, where

$4.6\times10^7=r=a(1-e)=a(1-0.206)=a(0.794)\Rightarrow a=4.6\times10^7/0.794$. So the maximum distance, which is at aphelion, is $r=a(1+e)=\left(4.6\times10^7/0.794\right)(1.206)\approx7.0\times10^7$ km.

30. At perihelion, $r=a(1-e)=4.43\times10^9$, and at aphelion, $r=a(1+e)=7.37\times10^9$. Adding, we get

$2a=11.80\times 10^9$, so $a=5.90\times 10^9$ km. Therefore $1+e=a(1+e)/a=\dfrac{7.37}{5.90}\approx 1.249$ and $e\approx 0.249$.

31. From Exercise 29, we have $e=0.206$ and $a(1-e)=4.6\times 10^7$ km. Thus, $a=4.6\times 10^7/0.794$. From Exercise 25, we can write the equation of Mercury's orbit as $r=a\dfrac{1-e^2}{1-e\cos\theta}$. So since

$\dfrac{dr}{d\theta}=\dfrac{-a(1-e^2)e\sin\theta}{(1-e\cos\theta)^2}\Rightarrow$

$$r^2+\left(\frac{dr}{d\theta}\right)^2=\frac{a^2\left(1-e^2\right)^2}{(1-e\cos\theta)^2}+\frac{a^2\left(1-e^2\right)^2e^2\sin^2\theta}{(1-e\cos\theta)^4}=\frac{a^2\left(1-e^2\right)^2}{(1-e\cos\theta)^4}\left(1-2e\cos\theta+e^2\right)$$

the length of the orbit is

$$L=\int_0^{2\pi}\sqrt{r^2+(dr/d\theta)^2}\,d\theta=a\left(1-e^2\right)\int_0^{2\pi}\frac{\sqrt{1+e^2-2e\cos\theta}}{(1-e\cos\theta)^2}\,d\theta\approx 3.6\times 10^8\text{km}$$

This seems reasonable, since Mercury's orbit is nearly circular, and the circumference of a circle of radius a is $2\pi a\approx 3.6\times 10^8$ km.

1. **(a)** A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.
(b) The terms a_n approach 8 as n becomes large. In fact, we can make a_n as close to 8 as we like by taking n sufficiently large.
(c) The terms a_n become large as n becomes large. In fact, we can make a_n as large as we like by taking n sufficiently large.

2. **(a)** From Definition 1, a convergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ exists. Examples: $\{1/n\}$, $\left\{1/2^n\right\}$
(b) A divergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ *does not* exist. Examples: $\{n\}$, $\{\sin n\}$

3. $a_n=1-(0.2)^n$, so the sequence is $\{0.8,0.96,0.992,0.9984,0.99968,\dots\}$.

4. $a_n=\dfrac{n+1}{3n-1}$, so the sequence is $\left\{\frac{2}{2},\frac{3}{5},\frac{4}{8},\frac{5}{11},\frac{6}{14},\dots\right\}=\left\{1,\frac{3}{5},\frac{1}{2},\frac{5}{11},\frac{3}{7},\dots\right\}$.

5. $a_n=\dfrac{3(-1)^n}{n!}$, so the sequence is $\left\{\frac{-3}{1},\frac{3}{2},\frac{-3}{6},\frac{3}{24},\frac{-3}{120},\dots\right\}=\left\{-3,\frac{3}{2},-\frac{1}{2},\frac{1}{8},-\frac{1}{40},\dots\right\}$.

6. $a_n=2\cdot 4\cdot 6\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ (2n)$, so the sequence is $\{2,2\cdot 4,2\cdot 4\cdot 6,2\cdot 4\cdot 6\cdot 8,2\cdot 4\cdot 6\cdot 8\cdot 10,\dots\}=\{2,8,48,384,3840,\dots\}$.

7. $a_1=3$, $a_{n+1}=2a_n-1$. Each term is defined in terms of the preceding term.
$a_2=2a_1-1=2(3)-1=5$. $a_3=2a_2-1=2(5)-1=9$. $a_4=2a_3-1=2(9)-1=17$. $a_5=2a_4-1=2(17)-1=33$. The sequence is $\{3,5,9,17,33,\dots\}$.

8. $a_1=4$, $a_{n+1}=\dfrac{a_n}{a_n-1}$. Each term is defined in terms of the preceding term.

$a_2=\dfrac{a_1}{a_1-1}=\dfrac{4}{4-1}=\dfrac{4}{3}$. $a_3=\dfrac{a_2}{a_2-1}=\dfrac{4/3}{\frac{4}{3}-1}=\dfrac{4/3}{1/3}=4$. Since $a_3=a_1$, we can see that the terms of the sequence will alternately equal 4 and 4/3 , so the sequence is $\left\{4,\frac{4}{3},4,\frac{4}{3},4,\dots\right\}$.

9. The numerators are all 1 and the denominators are powers of 2 , so $a_n = \frac{1}{2^n}$.

10. The numerators are all 1 and the denominators are multiples of 2 , so $a_n = \frac{1}{2n}$.

11. $\{2,7,12,17,\ldots\}$. Each term is larger than the preceding one by 5 , so $a_n = a_1 + d(n-1) = 2+5(n-1) = 5n-3$.

12. $\left\{ -\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \ldots \right\}$. The numerator of the n th term is n and its denominator is $(n+1)^2$. Including the alternating signs, we get $a_n = (-1)^n \frac{n}{(n+1)^2}$.

13. $\left\{ 1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \ldots \right\}$. Each term is $-\frac{2}{3}$ times the preceding one, so $a_n = \left(-\frac{2}{3} \right)^{n-1}$.

14. $\{5,1,5,1,5,1,\ldots\}$. The average of 5 and 1 is 3 , so we can think of the sequence as alternately adding 2 and -2 to 3 . Thus, $a_n = 3 + (-1)^{n+1} \cdot 2$.

15. $a_n = n(n-1)$. $a_n \to \infty$ as $n \to \infty$, so the sequence diverges.

16. $a_n = \frac{n+1}{3n-1} = \frac{1+1/n}{3-1/n}$, so $a_n \to \frac{1+0}{3-0} = \frac{1}{3}$ as $n \to \infty$. Converges

17. $a_n = \frac{3+5n^2}{n+n^2} = \frac{\left(3+5n^2\right)/n^2}{\left(n+n^2\right)/n^2} = \frac{5+3/n^2}{1+1/n}$, so $a_n \to \frac{5+0}{1+0} = 5$ as $n \to \infty$. Converges

18. $a_n = \frac{\sqrt{n}}{1+\sqrt{n}} = \frac{1}{1/\sqrt{n}+1}$, so $a_n \to \frac{1}{0+1} = 1$ as $n \to \infty$. Converges

19. $a_n = \frac{2^n}{3^{n+1}} = \frac{1}{3} \left(\frac{2}{3} \right)^n$, so $\lim_{n\to\infty} a_n = \frac{1}{3} \lim_{n\to\infty} \left(\frac{2}{3} \right)^n = \frac{1}{3} \cdot 0 = 0$ by (8) with $r = \frac{2}{3}$. Converges

20. $a_n = \frac{n}{1+\sqrt{n}} = \frac{\sqrt{n}}{1/\sqrt{n}+1}$. The numerator approaches ∞ and the denominator approaches $0+1=1$ as

$n\to\infty$, so $a_n\to\infty$ as $n\to\infty$ and the sequence diverges.

21. $a_n=\dfrac{(-1)^{n-1}n}{n^2+1}=\dfrac{(-1)^{n-1}}{n+1/n}$, so $0\leq\left|a_n\right|=\dfrac{1}{n+1/n}\leq\dfrac{1}{n}\to 0$ as $n\to\infty$, so $a_n\to 0$ by the Squeeze Theorem and Theorem 6. Converges

22. $a_n=\dfrac{(-1)^n n^3}{n^3+2n^2+1}$. Now $\left|a_n\right|=\dfrac{n^3}{n^3+2n^2+1}=\dfrac{1}{1+\dfrac{2}{n}+\dfrac{1}{n^3}}\to 1$ as $n\to\infty$, but the terms of the sequence $\{a_n\}$ alternate in sign, so the sequence $a_1,a_3,a_5,\dots$ converges to -1 and the sequence $a_2,a_4,a_6,\dots$ converges to $+1$. This shows that the given sequence diverges since its terms don't approach a single real number.

23. $a_n=\cos(n/2)$. This sequence diverges since the terms don't approach any particular real number as $n\to\infty$. The terms take on values between -1 and 1 .

24. $a_n=\cos(2/n)$. As $n\to\infty$, $2/n\to 0$, so $\cos(2/n)\to\cos 0=1$. Converges

25. $a_n=\dfrac{(2n-1)!}{(2n+1)!}=\dfrac{(2n-1)!}{(2n+1)(2n)(2n-1)!}=\dfrac{1}{(2n+1)(2n)}\to 0$ as $n\to\infty$. Converges

26. $2n\to\infty$ as $n\to\infty$, so since $\lim\limits_{x\to\infty}\arctan x=\dfrac{\pi}{2}$, we have $\lim\limits_{n\to\infty}\arctan 2n=\dfrac{\pi}{2}$. Converges

27. $a_n=\dfrac{e^n+e^{-n}}{e^{2n}-1}\cdot\dfrac{e^{-n}}{e^{-n}}=\dfrac{1+e^{-2n}}{e^n-e^{-n}}\to\dfrac{1+0}{e^n-0}\to 0$ as $n\to\infty$. Converges

28. $a_n=\dfrac{\ln n}{\ln 2n}=\dfrac{\ln n}{\ln 2+\ln n}=\dfrac{1}{\dfrac{\ln 2}{\ln n}+1}\to\dfrac{1}{0+1}\to 1$ as $n\to\infty$. Converges

29. $a_n=n^2e^{-n}=\dfrac{n^2}{e^n}$. Since $\lim\limits_{x\to\infty}\dfrac{x^2}{e^x}=\lim\limits_{x\to\infty}\dfrac{2x}{e^x}=\lim\limits_{x\to\infty}\dfrac{2}{e^x}=0$, it follows from Theorem 3 that $\lim\limits_{n\to\infty}a_n=0$. Converges

30. $a_n = n\cos n\pi = n(-1)^n$. Since $\left|a_n\right| = n \to \infty$ as $n \to \infty$, the given sequence diverges.

31. $0 \le \dfrac{\cos^2 n}{2^n} \le \dfrac{1}{2^n}$, so since $\lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$, $\left\{ \dfrac{\cos^2 n}{2^n} \right\}$ converges to 0 by the Squeeze Theorem.

32. $a_n = \ln(n+1) - \ln n = \ln\left(\dfrac{n+1}{n}\right) = \ln\left(1 + \dfrac{1}{n}\right) \to \ln(1) = 0$ as $n \to \infty$. Converges

33. $a_n = n\sin(1/n) = \dfrac{\sin(1/n)}{1/n}$. Since $\lim\limits_{x\to\infty} \dfrac{\sin(1/x)}{1/x} = \lim\limits_{t\to 0^+} \dfrac{\sin t}{t}$ [] where $t = 1/x$]$=1$, it follows from Theorem 3 that $\left\{ a_n \right\}$ converges to 1.

34. $a_n = \sqrt{n} - \sqrt{n^2 - 1} = \sqrt{n^2 \cdot \dfrac{1}{n}} - \sqrt{n^2\left(1 - \dfrac{1}{n^2}\right)} = n\left(\dfrac{1}{\sqrt{n}} - \sqrt{1 - \dfrac{1}{n^2}}\right) \to n(0-1) \to -n$ as $n \to \infty$, so $a_n \to -\infty$ as $n \to \infty$. Diverges

35. $a_n = \left(1 + \dfrac{2}{n}\right)^{1/n} \Rightarrow \ln a_n = \dfrac{1}{n}\ln\left(1 + \dfrac{2}{n}\right)$. As $n \to \infty$, $\dfrac{1}{n} \to 0$ and $\ln\left(1 + \dfrac{2}{n}\right) \to 0$, so $\ln a_n \to 0$. Thus, $a_n \to e^0 = 1$ as $n \to \infty$. Converges

36. $a_n = \dfrac{\sin 2n}{1+\sqrt{n}}$. $\left|a_n\right| \le \dfrac{1}{1+\sqrt{n}}$ and $\lim\limits_{n\to\infty} \dfrac{1}{1+\sqrt{n}} = 0$, so $\dfrac{-1}{1+\sqrt{n}} \le a_n \le \dfrac{1}{1+\sqrt{n}} \Rightarrow \lim\limits_{n\to\infty} a_n = 0$ by the Squeeze Theorem. Converges

37. $\{0,1,0,0,1,0,0,0,1,\ldots\}$ diverges since the sequence takes on only two values, 0 and 1, and never stays arbitrarily close to either one (or any other value) for n sufficiently large.

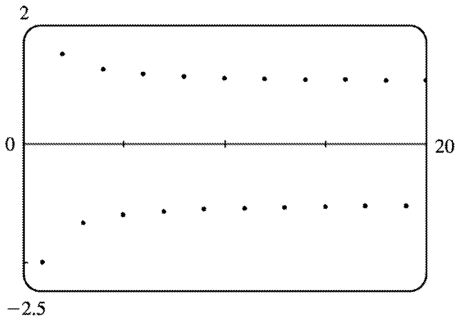
38. $\left\{ \dfrac{1}{1}, \dfrac{1}{3}, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{3}, \dfrac{1}{5}, \dfrac{1}{4}, \dfrac{1}{6}, \ldots \right\}$. $a_{2n-1} = \dfrac{1}{n}$ and $a_{2n} = \dfrac{1}{n+2}$ for all positive integers n . $\lim\limits_{n\to\infty} a_n = 0$ since $\lim\limits_{n\to\infty} a_{2n-1} = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$ and $\lim\limits_{n\to\infty} a_{2n} = \lim\limits_{n\to\infty} \dfrac{1}{n+2} = 0$. For n sufficiently large, a_n can be made as close to 0 as we like. Converges

39. $a_n = \dfrac{n!}{2^n} = \dfrac{1}{2} \cdot \dfrac{2}{2} \cdot \dfrac{3}{2} \cdot \cdot \cdot \cdot \cdot \dfrac{(n-1)}{2} \cdot \dfrac{n}{2} \ge \dfrac{1}{2} \cdot \dfrac{n}{2} = \dfrac{n}{4} \to \infty$ as $n \to \infty$, so $\left\{ a_n \right\}$ diverges.

40.

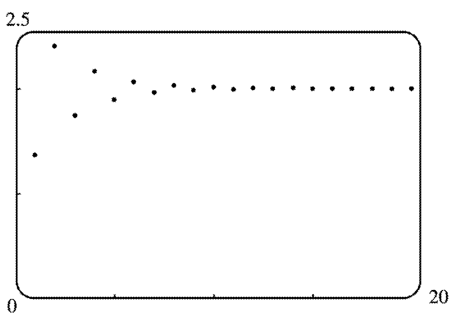
$0 < \left| a_n \right| = \frac{3^n}{n!} = \frac{3}{1} \cdot \frac{3}{2} \cdot \frac{3}{3} \cdot \ \cdots \ \cdot \frac{3}{(n-1)} \cdot \frac{3}{n} \le \frac{3}{1} \cdot \frac{3}{2} \cdot \frac{3}{n} = \frac{27}{2n} \to 0$ as $n \to \infty$, so by the Squeeze Theorem and Theorem 6, $\left\{ (-3)^n / n \right\}$ converges to 0 .

41.



From the graph, we see that the sequence $\left\{ (-1)^n \frac{n+1}{n} \right\}$ is divergent, since it oscillates between 1 and -1 (approximately).

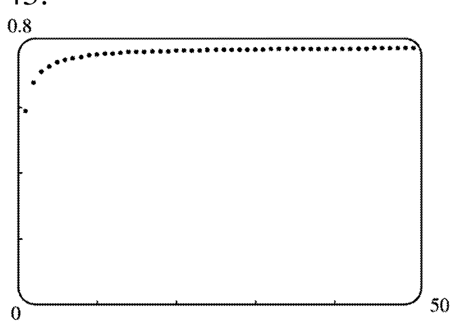
42.



From the graph, it appears that the sequence converges to 2 .

$\left\{ \left(-\frac{2}{\pi} \right)^n \right\}$ converges to 0 by (6), and hence $\left\{ 2 + \left(-\frac{2}{\pi} \right)^n \right\}$ converges to $2+0=2$.

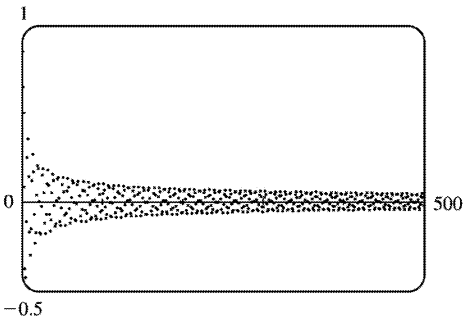
43.



From the graph, it appears that the sequence converges to about 0.78 .

$\lim_{n \to \infty} \frac{2n}{2n+1} = \lim_{n \to \infty} \frac{2}{2+1/n} = 1$, so $\lim_{n \to \infty} \arctan\left(\frac{2n}{2n+1} \right) = \arctan 1 = \frac{\pi}{4}$.

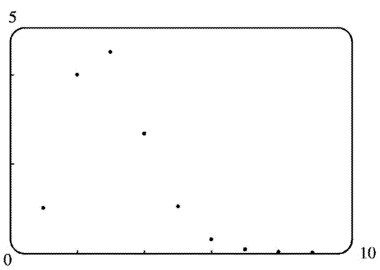
44.



From the graph, it appears that the sequence converges (slowly) to 0 .

$0 \le \dfrac{|\sin n|}{\sqrt{n}} \le \dfrac{1}{\sqrt{n}} \to 0$ as $n \to \infty$, so by the Squeeze Theorem and Theorem 6, $\left\{ \dfrac{\sin n}{\sqrt{n}} \right\}$ converges to 0 .

45.



From the graph, it appears that the sequence converges to 0 .

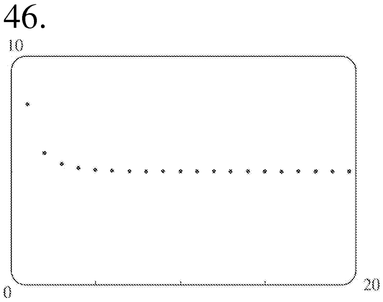
$$0 < a_n = \frac{n^3}{n!} = \frac{n}{n} \cdot \frac{n}{(n-1)} \cdot \frac{n}{(n-2)} \cdot \frac{1}{(n-3)} \cdot \;\cdot\;\cdot\;\cdot\;\cdot\; \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1}$$

$$\le \frac{n^2}{(n-1)(n-2)(n-3)} \qquad [\text{for } n \ge 4]$$

$$= \frac{1/n}{(1-1/n)\,(1-2/n)\,(1-3/n)} \to 0 \text{ as } n \to \infty$$

So by the Squeeze Theorem, $\left\{ n^3/\text{n}! \right\}$ converges to 0 .

46.



From the graph, it appears that the sequence converges to 5.

$$5=\sqrt[n]{5^n} \le \sqrt[n]{3^n+5^n} \le \sqrt[n]{5^n+5^n} = \sqrt[n]{2}\sqrt[n]{5^n}$$
$$= \sqrt[n]{2}\cdot 5 \to 5 \text{ as } n\to\infty \; \lim_{n\to\infty} 2^{1/n}=2^0=1]$$

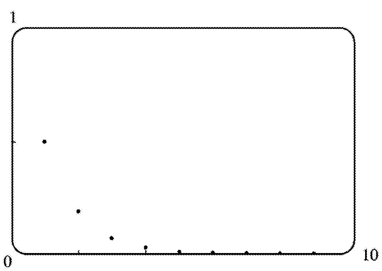
Hence, $a_n \to 5$ by the Squeeze Theorem.

Alternate Solution: Let $y=\left(3^x+5^x\right)^{1/x}$. Then

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(3^x+5^x\right)}{x} = \lim_{x\to\infty} \frac{3^x \ln 3+5^x \ln 5}{3^x+5^x}$$
$$= \lim_{x\to\infty} \frac{\left(\frac{3}{5}\right)^x \ln 3+\ln 5}{\left(\frac{3}{5}\right)^x+1} = \ln 5$$

so $\lim_{x\to\infty} y=e^{\ln 5}=5$, and so $\left\{\sqrt[n]{3^n+5^n}\right\}$ converges to 5 .

47.

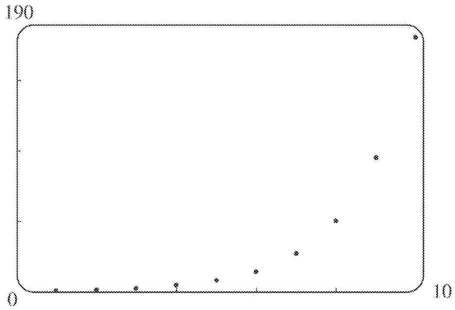


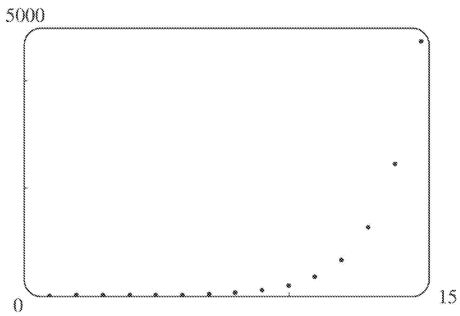
From the graph, it appears that the sequence approaches 0 .

$$0<a_n = \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{(2n)^n} = \frac{1}{2n}\cdot\frac{3}{2n}\cdot\frac{5}{2n}\cdot \cdots \cdot \frac{2n-1}{2n}$$
$$\le \frac{1}{2n}\cdot(1)\cdot(1)\cdot \cdots \cdot(1)=\frac{1}{2n}\to 0 \quad \text{as} \quad n\to\infty$$

So by the Squeeze Theorem, $\left\{\frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{(2n)^n}\right\}$ converges to 0 .

48.





From the graphs, it seems that the sequence diverges. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdot \cdot \cdot \cdot (2n-1)}{n!}$. We first prove by induction that $a_n \geq \left(\frac{3}{2} \right)^{n-1}$ for all n . This is clearly true for $n=1$, so let $P(n)$ be the statement that the above is true for n . We must show it is then true for $n+1$.
$a_{n+1} = a_n \cdot \dfrac{2n+1}{n+1} \geq \left(\frac{3}{2} \right)^{n-1} \cdot \dfrac{2n+1}{n+1}$ (induction hypothesis). But $\dfrac{2n+1}{n+1} \geq \dfrac{3}{2}$, and so we get that $a_{n+1} \geq \left(\frac{3}{2} \right)^{n-1} \cdot \frac{3}{2} = \left(\frac{3}{2} \right)^{n}$ which is $P(n+1)$. Thus, we have proved our first assertion, so since $\left\{ \left(\frac{3}{2} \right)^{n-1} \right\}$ diverges (by (8)), so does the given sequence $\{ a_n \}$.

49. **(a)** $a_n = 1000(1.06)^n \Rightarrow a_1 = 1060$, $a_2 = 1123.60$, $a_3 = 1191.02$, $a_4 = 1262.48$, and $a_5 = 1338.23$.

(b) $\lim\limits_{n\to\infty} a_n = 1000 \lim\limits_{n\to\infty} (1.06)^n$, so the sequence diverges by (8) with $r = 1.06 > 1$.

50. $a_{n+1} = \begin{cases} \frac{1}{2} a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$ When $a_1 = 11$, the first 40 terms are 11 , 34 , 17 , 52 , 26 , 13 , 40 , 20 , 10 , 5 , 16 , 8 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 . When $a_1 = 25$, the first 40 terms are 25 , 76 , 38 , 19 , 58 , 29 , 88 , 44 , 22 , 11 , 34 , 17 , 52 , 26 , 13 , 40 , 20 , 10 , 5 , 16 , 8 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 . The famous Collatz conjecture is that this sequence always reaches 1 , regardless of the

starting point a_1 .

51. If $|r| \geq 1$, then $\{r^n\}$ diverges by (8), so $\{nr^n\}$ diverges also, since $|nr^n| = n|r^n| \geq |r^n|$. If $|r|<1$ then $\lim_{x\to\infty} xr^x = \lim_{x\to\infty} \frac{x}{r^{-x}} = \lim_{x\to\infty} \frac{1}{(-\ln r)r^{-x}} = \lim_{x\to\infty} \frac{r^x}{-\ln r} = 0$, so $\lim_{n\to\infty} nr^n = 0$, and hence $\{nr^n\}$ converges whenever $|r|<1$.

52. **(a)** Let $\lim_{n\to\infty} a_n = L$. By Definition 1, this means that for every $\varepsilon>0$ there is an integer N such that $|a_n - L| < \varepsilon$ whenever $n>N$. Thus, $|a_{n+1} - L| < \varepsilon$ whenever $n+1>N \Leftrightarrow n>N-1$. It follows that $\lim_{n\to\infty} a_{n+1} = L$ and so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$.

(b) If $L = \lim_{n\to\infty} a_n$ then $\lim_{n\to\infty} a_{n+1} = L$ also, so L must satisfy $L = 1/(1+L) \Rightarrow L^2 + L - 1 = 0 \Rightarrow L = \frac{-1+\sqrt{5}}{2}$ (since L has to be non–negative if it exists).

53. Since $\{a_n\}$ is a decreasing sequence, $a_n > a_{n+1}$ for all $n \geq 1$. Because all of its terms lie between 5 and 8 , $\{a_n\}$ is a bounded sequence. By the Monotonic Sequence Theorem, $\{a_n\}$ is convergent; that is, $\{a_n\}$ has a limit L . L must be less than 8 since $\{a_n\}$ is decreasing, so $5 \leq L < 8$.

54. $a_n = 1/5^n$ defines a decreasing geometric sequence since $a_{n+1} = \frac{1}{5} a_n < a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$.

55. $a_n = \frac{1}{2n+3}$ is decreasing since $a_{n+1} = \frac{1}{2(n+1)+3} = \frac{1}{2n+5} < \frac{1}{2n+3} = a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$. Note that $a_1 = \frac{1}{5}$.

56. $a_n = \frac{2n-3}{3n+4}$ defines an increasing sequence since for $f(x) = \frac{2x-3}{3x+4}$,
$f'(x) = \frac{(3x+4)(2)-(2x-3)(3)}{(3x+4)^2} = \frac{17}{(3x+4)^2} > 0$. The sequence is bounded since $a_n \geq a_1 = -\frac{1}{7}$ for $n \geq 1$, and $a_n < \frac{2n-3}{3n} < \frac{2n}{3n} = \frac{2}{3}$ for $n \geq 1$.

57. $a_n = \cos(n\pi/2)$ is not monotonic. The first few terms are $0, -1, 0, 1, 0, -1, 0, 1, \ldots$. In fact, the sequence consists of the terms $0, -1, 0, 1$ repeated over and over again in that order. The sequence is bounded since $|a_n| \le 1$ for all $n \ge 1$.

58. $a_n = ne^{-n}$ defines a positive decreasing sequence since the function $f(x) = xe^{-x}$ is decreasing for $x > 1$. $[f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) < 0]$ for $x > 1$.] The sequence is bounded above by $a_1 = \frac{1}{e}$ and below by 0.

59. $a_n = \frac{n}{n^2+1}$ defines a decreasing sequence since for $f(x) = \frac{x}{x^2+1}$,
$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \le 0$ for $x \ge 1$. The sequence is bounded since $0 < a_n \le \frac{1}{2}$ for all $n \ge 1$.

60. $a_n = n + \frac{1}{n}$ defines an increasing sequence since the function $g(x) = x + \frac{1}{x}$ is increasing for $x > 1$. $[g'(x) = 1 - 1/x^2 > 0]$ for $x > 1$.] The sequence is unbounded since $a_n \to \infty$ as $n \to \infty$. (It is, however, bounded below by $a_1 = 2$.)

61. $a_1 = 2^{1/2}$, $a_2 = 2^{3/4}$, $a_3 = 2^{7/8}$, $\ldots$, so $a_n = 2^{(2^n-1)/2^n} = 2^{1-(1/2^n)}$. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^{1-(1/2^n)} = 2^1 = 2$.
Alternate solution : Let $L = \lim_{n\to\infty} a_n$. (We could show the limit exists by showing that $\{a_n\}$ is bounded and increasing.) Then L must satisfy $L = \sqrt{2 \cdot L} \Rightarrow L^2 = 2L \Rightarrow L(L-2) = 0$. $L \ne 0$ since the sequence increases, so $L = 2$.

62. **(a)** Let P_n be the statement that $a_{n+1} \ge a_n$ and $a_n \le 3$. P_1 is obviously true. We will assume that P_n is true and then show that as a consequence P_{n+1} must also be true. $a_{n+2} \ge a_{n+1} \Leftrightarrow \sqrt{2+a_{n+1}} \ge \sqrt{2+a_n}$ $\Leftrightarrow 2+a_{n+1} \ge 2+a_n \Leftrightarrow a_{n+1} \ge a_n$, which is the induction hypothesis. $a_{n+1} \le 3 \Leftrightarrow \sqrt{2+a_n} \le 3 \Leftrightarrow 2+a_n \le 9 \Leftrightarrow$ $a_n \le 7$, which is certainly true because we are assuming that $a_n \le 3$. So P_n is true for all n , and so $a_1 \le a_n \le 3$ (showing that the sequence is bounded), and hence by the Monotonic Sequence Theorem, $\lim_{n\to\infty} a_n$ exists.
(b) If

$L=\lim_{n\to\infty} a_n$, then $\lim_{n\to\infty} a_{n+1}=L$ also, so $L=\sqrt{2+L} \Rightarrow L^2=2+L \Leftrightarrow L^2-L-2=0 \Leftrightarrow (L+1)(L-2)=0 \Leftrightarrow L=2$ (since L can't be negative).

63. We show by induction that $\{a_n\}$ is increasing and bounded above by 3 .
Let P_n be the proposition that $a_{n+1}>a_n$ and $0<a_n<3$. Clearly P_1 is true. Assume that P_n is true. Then $a_{n+1}>a_n \Rightarrow \frac{1}{a_{n+1}}<\frac{1}{a_n} \Rightarrow -\frac{1}{a_{n+1}}>-\frac{1}{a_n}$.
Now $a_{n+2}=3-\frac{1}{a_{n+1}}>3-\frac{1}{a_n}=a_{n+1} \Leftrightarrow P_{n+1}$. This proves that $\{a_n\}$ is increasing and bounded above by 3 , so $1=a_1<a_n<3$, that is, $\{a_n\}$ is bounded, and hence convergent by the Monotonic Sequence Theorem.
If $L=\lim_{n\to\infty} a_n$, then $\lim_{n\to\infty} a_{n+1}=L$ also, so L must satisfy $L=3-1/L \Rightarrow L^2-3L+1=0 \Rightarrow L=\frac{3\pm\sqrt{5}}{2}$. But $L>1$, so $L=\frac{3+\sqrt{5}}{2}$.

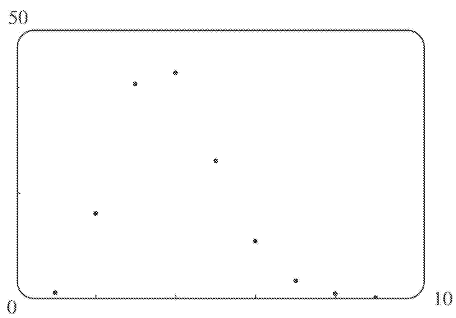
64. We use induction. Let P_n be the statement that $0<a_{n+1}\le a_n\le 2$. Clearly P_1 is true, since $a_2=1/(3-2)=1$. Now assume that P_n is true. Then $a_{n+1}\le a_n \Rightarrow -a_{n+1}\ge -a_n \Rightarrow 3-a_{n+1}\ge 3-a_n \Rightarrow$ $a_{n+2}=\frac{1}{3-a_{n+1}}\le\frac{1}{3-a_n}=a_{n+1}$. Also $a_{n+2}>0$ (since $3-a_{n+1}$ is positive) and $a_{n+1}\le 2$ by the induction hypothesis, so P_{n+1} is true.
To find the limit, we use the fact that $\lim_{n\to\infty} a_n=\lim_{n\to\infty} a_{n+1} \Rightarrow L=\frac{1}{3-L} \Rightarrow L^2-3L+1=0 \Rightarrow L=\frac{3\pm\sqrt{5}}{2}$. But $L\le 2$, so we must have $L=\frac{3-\sqrt{5}}{2}$.

65. **(a)** Let a_n be the number of rabbit pairs in the n th month. Clearly $a_1=1=a_2$. In the n th month, each pair that is 2 or more months old (that is, a_{n-2} pairs) will produce a new pair to add to the a_{n-1} pairs already present. Thus, $a_n=a_{n-1}+a_{n-2}$, so that $\{a_n\}=\{f_n\}$, the Fibonacci sequence.
(b) $a_n=\frac{f_{n+1}}{f_n} \Rightarrow a_{n-1}=\frac{f_n}{f_{n-1}}=\frac{f_{n-1}+f_{n-2}}{f_{n-1}}=1+\frac{f_{n-2}}{f_{n-1}}=1+\frac{1}{f_{n-1}/f_{n-2}}=1+\frac{1}{a_{n-2}}$. If $L=\lim_{n\to\infty} a_n$, then

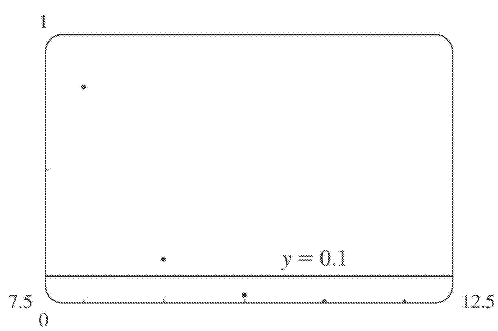
$L = \lim_{n\to\infty} a_{n-1}$ and $L = \lim_{n\to\infty} a_{n-2}$, so L must satisfy $L = 1 + \frac{1}{L} \Rightarrow L^2 - L - 1 = 0 \Rightarrow L = \frac{1+\sqrt{5}}{2}$ (since L must be positive).

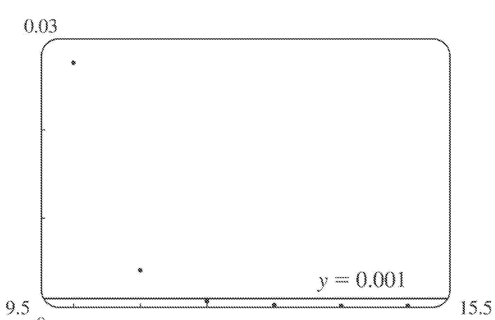
66. **(a)** If f is continuous, then $f(L) = f\left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} f\left(a_n\right) = \lim_{n\to\infty} a_{n+1} = L$ by Exercise 52(a).

(b) By repeatedly pressing the cosine key on the calculator (that is, taking cosine of the previous answer) until the displayed value stabilizes, we see that $L \approx 0.73909$.

67. **(a)** 

From the graph, it appears that the sequence $\left\{ \frac{n^5}{n!} \right\}$ converges to 0 , that is, $\lim_{n\to\infty} \frac{n^5}{n!} = 0$.

(b) 



From the first graph, it seems that the smallest possible value of N corresponding to $\varepsilon = 0.1$ is 9 , since $n^5/n! < 0.1$ whenever $n \geq 10$, but $9^5/9! > 0.1$. From the second graph, it seems that for $\varepsilon = 0.001$, the smallest possible value for N is 11 .

68. Let $\varepsilon>0$ and let N be any positive integer larger than $\ln(\varepsilon)/\ln|r|$. If $n>N$ then $n>\ln(\varepsilon)/\ln|r|$ $\Rightarrow n\ln|r|<\ln\varepsilon \Rightarrow \ln\left(|r|^n\right)<\ln\varepsilon \Rightarrow |r|^n<\varepsilon \Rightarrow \left|r^n-0\right|<\varepsilon$, and so by Definition 1, $\lim_{n\to\infty} r^n=0$.

69. If $\lim_{n\to\infty}\left|a_n\right|=0$ then $\lim_{n\to\infty}-\left|a_n\right|=0$, and since $-\left|a_n\right|\le a_n\le\left|a_n\right|$, we have that $\lim_{n\to\infty} a_n=0$ by the Squeeze Theorem.

70. **(a)**

$$\begin{aligned}\frac{b^{n+1}-a^{n+1}}{b-a} &= b^n+b^{n-1}a+b^{n-2}a^2+b^{n-3}a^3+\cdot\cdot\cdot+ba^{n-1}+a^n \\ &< b^n+b^{n-1}b+b^{n-2}b^2+b^{n-3}b^3+\cdot\cdot\cdot+bb^{n-1}+b^n=(n+1)b^n\end{aligned}$$

(b) Since $b-a>0$, we have $b^{n+1}-a^{n+1}<(n+1)b^n(b-a)\Rightarrow b^{n+1}-(n+1)b^n(b-a)<a^{n+1}\Rightarrow b^n\left[(n+1)a-nb\right]<a^{n+1}$.

(c) With this substitution, $(n+1)a-nb=1$, and so $b^n=\left(1+\frac{1}{n}\right)^n<a^{n+1}=\left(1+\frac{1}{n+1}\right)^{n+1}$.

(d) With this substitution, we get $\left(1+\frac{1}{2n}\right)^n\left(\frac{1}{2}\right)<1\Rightarrow\left(1+\frac{1}{2n}\right)^n<2\Rightarrow\left(1+\frac{1}{2n}\right)^{2n}<4$.

(e) $a_n<a_{2n}$ since $\left\{a_n\right\}$ is increasing, so $a_n<a_{2n}<4$.

(f) Since $\left\{a_n\right\}$ is increasing and bounded above by 4 , $a_1\le a_n\le 4$, and so $\left\{a_n\right\}$ is bounded and monotonic, and hence has a limit by Theorem 11.

71. **(a)** First we show that $a>a_1>b_1>b$.

$a_1-b_1=\frac{a+b}{2}-\sqrt{ab}=\frac{1}{2}\left(a-2\sqrt{ab}+b\right)=\frac{1}{2}\left(\sqrt{a}-\sqrt{b}\right)^2>0$ (since $a>b$) $\Rightarrow a_1>b_1$. Also $a-a_1=a-\frac{1}{2}(a+b)=\frac{1}{2}(a-b)>0$ and $b-b_1=b-\sqrt{ab}=\sqrt{b}\left(\sqrt{b}-\sqrt{a}\right)<0$, so $a>a_1>b_1>b$. In the same way we can show that $a_1>a_2>b_2>b_1$ and so the given assertion is true for $n=1$. Suppose it is true for $n=k$, that is, $a_k>a_{k+1}>b_{k+1}>b_k$. Then

$$\begin{aligned}a_{k+2}-b_{k+2} &= \frac{1}{2}\left(a_{k+1}+b_{k+1}\right)-\sqrt{a_{k+1}b_{k+1}}=\frac{1}{2}\left(a_{k+1}-2\sqrt{a_{k+1}b_{k+1}}+b_{k+1}\right) \\ &= \frac{1}{2}\left(\sqrt{a_{k+1}}-\sqrt{b_{k+1}}\right)^2>0\end{aligned}$$

$$a_{k+1}-a_{k+2} = a_{k+1}-\frac{1}{2}\left(a_{k+1}+b_{k+1}\right)=\frac{1}{2}\left(a_{k+1}-b_{k+1}\right)>0$$

and $b_{k+1}-b_{k+2}=b_{k+1}-\sqrt{a_{k+1}b_{k+1}}=\sqrt{b_{k+1}}\left(\sqrt{b_{k+1}}-\sqrt{a_{k+1}}\right)<0\Rightarrow a_{k+1}>a_{k+2}>b_{k+2}>b_{k+1}$, so the assertion is true for $n=k+1$. Thus, it is true for all n by mathematical induction.

(b) From part (a) we have $a>a_n>a_{n+1}>b_{n+1}>b_n>b$, which shows that both sequences, $\{a_n\}$ and $\{b_n\}$, are monotonic and bounded. So they are both convergent by the Monotonic Sequence Theorem.

(c) Let $\lim_{n\to\infty} a_n=\alpha$ and $\lim_{n\to\infty} b_n=\beta$. Then $\lim_{n\to\infty} a_{n+1}=\lim_{n\to\infty}\frac{a_n+b_n}{2}\Rightarrow\alpha=\frac{\alpha+\beta}{2}\Rightarrow$ $2\alpha=\alpha+\beta\Rightarrow\alpha=\beta$.

72. **(a)** Let $\varepsilon>0$. Since $\lim_{n\to\infty} a_{2n}=L$, there exists N_1 such that $\left|a_{2n}-L\right|<\varepsilon$ for $n>N_1$. Since $\lim_{n\to\infty} a_{2n+1}=L$, there exists N_2 such that $\left|a_{2n+1}-L\right|<\varepsilon$ for $n>N_2$. Let $N=\max\left\{2N_1,2N_2+1\right\}$ and let $n>N$. If n is even, then $n=2m$ where $m>N_1$, so $\left|a_n-L\right|=\left|a_{2m}-L\right|<\varepsilon$. If n is odd, then $n=2m+1$, where $m>N_2$, so $\left|a_n-L\right|=\left|a_{2m+1}-L\right|<\varepsilon$. Therefore $\lim_{n\to\infty} a_n=L$.

(b) $a_1=1$, $a_2=1+\frac{1}{1+1}=\frac{3}{2}=1.5$, $a_3=1+\frac{1}{5/2}=\frac{7}{5}=1.4$, $a_4=1+\frac{1}{12/5}=\frac{17}{12}=1.41\overline{6}$, $a_5=1+\frac{1}{29/12}=\frac{41}{29}\approx 1.413793$, $a_6=1+\frac{1}{70/29}=\frac{99}{70}\approx 1.414286$, $a_7=1+\frac{1}{169/70}=\frac{239}{169}\approx 1.414201$, $a_8=1+\frac{1}{408/169}=\frac{577}{408}\approx 1.414216$. Notice that $a_1<a_3<a_5<a_7$ and $a_2>a_4>a_6>a_8$. It appears that the odd terms are increasing and the even terms are decreasing. Let's prove that $a_{2n-2}>a_{2n}$ and $a_{2n-1}<a_{2n+1}$ by mathematical induction. Suppose that $a_{2k-2}>a_{2k}$. Then $1+a_{2k-2}>1+a_{2k}\Rightarrow\frac{1}{1+a_{2k-2}}<\frac{1}{1+a_{2k}}\Rightarrow$ $1+\frac{1}{1+a_{2k-2}}<1+\frac{1}{1+a_{2k}}\Rightarrow a_{2k-1}<a_{2k+1}\Rightarrow 1+a_{2k-1}<1+a_{2k+1}\Rightarrow\frac{1}{1+a_{2k-1}}>\frac{1}{1+a_{2k+1}}\Rightarrow$ $1+\frac{1}{1+a_{2k-1}}>1+\frac{1}{1+a_{2k+1}}\Rightarrow a_{2k}>a_{2k+2}$. We have thus shown, by induction, that the odd terms are increasing and the even terms are decreasing. Also all terms lie between 1 and 2 , so both $\{a_n\}$ and $\{b_n\}$ are bounded monotonic sequences and are therefore convergent by Theorem 11. Let $\lim_{n\to\infty} a_{2n}=L$. Then $\lim_{n\to\infty} a_{2n+2}=L$ also. We have $a_{n+2}=1+\frac{1}{1+1+1/(1+a_n)}=1+\frac{1}{(3+2a_n)/(1+a_n)}=\frac{4+3a_n}{3+2a_n}$, so

$a_{2n+2}=\dfrac{4+3a_{2n}}{3+2a_{2n}}$. Taking limits of both sides, we get $L=\dfrac{4+3L}{3+2L}\Rightarrow 3L+2L^2=4+3L\Rightarrow L^2=2\Rightarrow L=\sqrt{2}$ (since $L>0$). Thus, $\lim_{n\to\infty} a_{2n}=\sqrt{2}$. Similarly we find that $\lim_{n\to\infty} a_{2n+1}=\sqrt{2}$. So, by part (a), $\lim_{n\to\infty} a_n=\sqrt{2}$.

73. **(a)** Suppose $\{p_n\}$ converges to p . Then

$$p_{n+1}=\frac{bp_n}{a+p_n}\Rightarrow \lim_{n\to\infty} p_{n+1}=\frac{b\lim_{n\to\infty} p_n}{a+\lim_{n\to\infty} p_n}\Rightarrow p=\frac{bp}{a+p}\Rightarrow p^2+ap=bp\Rightarrow p(p+a-b)=0\Rightarrow p=0 \text{ or } p=b-a .$$

(b) $p_{n+1}=\dfrac{bp_n}{a+p_n}=\dfrac{\frac{b}{a}p_n}{1+\frac{p_n}{a}}<\dfrac{b}{a}p_n$ since $1+\dfrac{p_n}{a}>1$.

(c) By part (b), $p_1<\left(\frac{b}{a}\right)p_0$, $p_2<\left(\frac{b}{a}\right)p_1<\left(\frac{b}{a}\right)^2p_0$, $p_3<\left(\frac{b}{a}\right)p_2<\left(\frac{b}{a}\right)^3p_0$, etc. In general, $p_n<\left(\frac{b}{a}\right)^np_0$, so $\lim_{n\to\infty} p_n\le\lim_{n\to\infty}\left(\frac{b}{a}\right)^n\cdot p_0=0$ since $b<a$.

(d) Let $a<b$. We first show, by induction, that if $p_0<b-a$, then $p_n<b-a$ and $p_{n+1}>p_n$.

For $n=0$, we have $p_1-p_0=\dfrac{bp_0}{a+p_0}-p_0=\dfrac{p_0(b-a-p_0)}{a+p_0}>0$ since $p_0<b-a$. So $p_1>p_0$.

Now we suppose the assertion is true for $n=k$, that is, $p_k<b-a$ and $p_{k+1}>p_k$. Then

$b-a-p_{k+1}=b-a-\dfrac{bp_k}{a+p_k}=\dfrac{a(b-a)+bp_k-ap_k-bp_k}{a+p_k}=\dfrac{a(b-a-p_k)}{a+p_k}>0$ because $p_k<b-a$. So $p_{k+1}<b-a$. And

$p_{k+2}-p_{k+1}=\dfrac{bp_{k+1}}{a+p_{k+1}}-p_{k+1}=\dfrac{p_{k+1}(b-a-p_{k+1})}{a+p_{k+1}}>0$ since $p_{k+1}<b-a$. Therefore, $p_{k+2}>p_{k+1}$. Thus, the assertion is true for $n=k+1$. It is therefore true for all n by mathematical induction. A similar proof by induction shows that if $p_0>b-a$, then $p_n>b-a$ and $\{p_n\}$ is decreasing. In either case the sequence $\{p_n\}$ is bounded and monotonic, so it is convergent by the Monotonic Sequence Theorem. It then follows from part (a) that $\lim_{n\to\infty} p_n=b-a$.

1. Using Theorem 5 with $\sum_{n=0}^{\infty} b_n (x-5)^n$, $b_n = \dfrac{f^{(n)}(a)}{n!}$, so $b_8 = \dfrac{f^{(8)}(5)}{8!}$.

2. **(a)** Using Formula 6, a power series expansion of f at 1 must have the form $f(1)+f'(1)(x-1)+\cdots$. Comparing to the given series, $1.6-0.8(x-1)+\cdots$, we must have $f'(1)=-0.8$. But from the graph, $f'(1)$ is positive. Hence, the given series is *not* the Taylor series of f centered at 1 .

(b) A power series expansion of f at 2 must have the form $f(2)+f'(2)(x-2)+\frac{1}{2}f''(2)(x-2)^2+\cdots$. Comparing to the given series, $2.8+0.5(x-2)+1.5(x-2)^2-0.1(x-2)^3+\cdots$, we must have $\frac{1}{2}f''(2)=1.5$; that is, $f''(2)$ is positive. But from the graph, f is concave downward near $x=2$, so $f''(2)$ must be negative. Hence, the given series is *not* the Taylor series of f centered at 2 .

3.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1
⋮	⋮	⋮

We use Equation 7 with $f(x)=\cos x$.

$$\cos x = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\cdots$$

$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

If $a_n = \dfrac{(-1)^n x^{2n}}{(2n)!}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right| = x^2\lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} = 0<1 \text{ for all } x .$$

So $R=\infty$ (Ratio Test).

4.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin 2x$	0
1	$2\cos 2x$	2
2	$-2^2\sin 2x$	0
3	$-2^3\cos 2x$	-2^3
4	$2^4\sin 2x$	0
⋮	⋮	⋮

$f^{(n)}(0)=0$ if n is even and $f^{(2n+1)}(0)=(-1)^n 2^{2n+1}$, so

$$\sin 2x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2^2 |x|^2}{(2n+3)(2n+2)} = 0 < 1 \text{ for all } x,$$

so $R=\infty$ (Ratio Test).

5.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-3}$	1
1	$-3(1+x)^{-4}$	-3
2	$12(1+x)^{-5}$	12
3	$-60(1+x)^{-6}$	-60
4	$360(1+x)^{-7}$	360
⋮	⋮	⋮

$$\begin{aligned}(1+x)^{-3} &= f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\cdots \\ &= 1-3x+\frac{4\cdot 3}{2!}x^2-\frac{5\cdot 4\cdot 3}{3!}x^3+\frac{6\cdot 5\cdot 4\cdot 3}{4!}x^4-\cdots \\ &= 1-3x+\frac{4\cdot 3\cdot 2}{2\cdot 2!}x^2-\frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 3!}x^3+\frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{2\cdot 4!}x^4-\cdots \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(n+2)!\,x^n}{2(\mathrm{n}!)}=\sum_{n=0}^{\infty}\frac{(-1)^n(n+2)(n+1)x^n}{2}\end{aligned}$$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+3)(n+2)x^{n+1}}{2}\cdot\frac{2}{(n+2)(n+1)x^n}\right|=|x|\lim_{n\to\infty}\frac{n+3}{n+1}=|x|<1$ for convergence, so $R=1$ (Ratio Test).

6.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1+x)$	0
1	$(1+x)^{-1}$	1
2	$-(1+x)^{-2}$	-1
3	$2(1+x)^{-3}$	2
4	$-6(1+x)^{-4}$	-6
5	$24(1+x)^{-5}$	24
⋮	⋮	⋮

$$\begin{aligned}\ln(1+x) &= f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3 \\ &\quad+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5+\cdots \\ &= x-\frac{1}{2}x^2+\frac{2}{6}x^3-\frac{6}{24}x^4+\frac{24}{120}x^5-\cdots \\ &= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n\end{aligned}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{1+1/n} = |x| < 1 \text{ for convergence, so } R=1 .$$

7.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{5x}	1
1	$5e^{5x}$	5
2	5^2e^{5x}	25
3	5^3e^{5x}	125
4	5^4e^{5x}	625
. . .	. . .	. . .

$$e^{5x} = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{5^n}{n!}x^n .$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{5^{n+1}|x|^{n+1}}{(n+1)!}\cdot\frac{n!}{5^n|x|^n}\right]$$

$$= \lim_{n\to\infty}\frac{5|x|}{n+1} = 0 < 1 \text{ for all } x, \text{ so } R=\infty .$$

8.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^x	0
1	$(x+1)e^x$	1
2	$(x+2)e^x$	2
3	$(x+3)e^x$	3
. . .	. . .	. . .

$$xe^x = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{n}{n!}x^n = \sum_{n=1}^{\infty}\frac{n}{n!}x^n = \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!} .$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left[\frac{|x|^{n+1}}{n!}\cdot\frac{(n-1)!}{|x|^n}\right]=\lim_{n\to\infty}\frac{|x|}{n}=0<1 \text{ for all } x \text{ , so } R=\infty \text{ .}$$

9.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	sinh x	0
1	cosh x	1
2	sinh x	0
3	cosh x	1
4	sinh x	0
. . .	. . .	. . .

$$f^{(n)}(0)=\begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd}\end{cases} \quad \text{so } \sinh x=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} \text{ .}$$

Use the Ratio Test to find R . If $a_n=\dfrac{x^{2n+1}}{(2n+1)!}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$$

$$= x^2\cdot\lim_{n\to\infty}\frac{1}{(2n+3)(2n+2)}=0<1$$

for all x , so $R=\infty$.

10.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	cosh x	1
1	sinh x	0
2	cosh x	1
3	sinh x	0
. . .	. . .	. . .

$$f^{(n)}(0)=\begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd}\end{cases} \quad \text{so } \cosh x=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} \text{ .}$$

Use the Ratio Test to find R . If

$a_n = \dfrac{x^{2n}}{(2n)!}$, then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right|$$

$$= x^2 \cdot \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0 < 1$$

for all x , so $R=\infty$.

11.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
1	$1+2x$	5
2	2	2
3	0	0
4	0	0
. . .	. . .	. . .

$$f(x) = 7+5(x-2)+\frac{2}{2!}(x-2)^2+\sum_{n=3}^{\infty}\frac{0}{n!}(x-2)^n$$

$$= 7+5(x-2)+(x-2)^2$$

Since $a_n=0$ for large n , $R=\infty$.

12.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	x^3	-1
1	$3x^2$	3
2	$6x$	-6
3	6	6
4	0	0
5	0	0
. . .	. . .	. . .

$$f(x) = -1+3(x+1) - \frac{6}{2!}(x+1)^2 + \frac{6}{3!}(x+1)^3$$
$$= -1+3(x+1)-3(x+1)^2+(x+1)^3$$

Since $a_n = 0$ for large n , $R = \infty$.

13. Clearly, $f^{(n)}(x) = e^x$, so $f^{(n)}(3) = e^3$ and $e^x = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$. If $a_n = \frac{e^3}{n!}(x-3)^n$, then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{e^3 (x-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{e^3 (x-3)^n} \right| = \lim_{n\to\infty} \frac{|x-3|}{n+1} = 0 < 1$$ for all x , so $R = \infty$.

14.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln x$	$\ln 2$
1	x^{-1}	$\frac{1}{2}$
2	$-x^{-2}$	$-\frac{1}{4}$
3	$2x^{-3}$	$\frac{2}{8}$
4	$-3\cdot 2x^{-4}$	$-\frac{3\cdot 2}{16}$
⋮	⋮	⋮

$f^{(n)}(2) = \frac{(-1)^{n-1}(n-1)!}{2^n}$ for $n \geq 1$, so $\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-2)^n}{n\cdot 2^n}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x-2|}{2} \lim_{n\to\infty} \frac{n}{n+1} = \frac{|x-2|}{2} < 1$$ for convergence, so $|x-2| < 2 \Rightarrow R = 2$.

15.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos x$	-1

1	$-\sin x$	0
2	$-\cos x$	1
3	$\sin x$	0
4	$\cos x$	-1
. . .	. . .	. . .

$$\cos x=\sum_{k=0}^{\infty}\frac{f^{(k)}(\pi)}{k!}(x-\pi)^k=-1+\frac{(x-\pi)^2}{2!}-\frac{(x-\pi)^4}{4!}+\frac{(x-\pi)^6}{6!}-\cdots=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\pi)^{2n}}{(2n)!}\ .$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left[\frac{|x-\pi|^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{|x-\pi|^{2n}}\right]=\lim_{n\to\infty}\frac{|x-\pi|^2}{(2n+2)(2n+1)}=0<1 \text{ for all } x\text{ , so } R=\infty\ .$$

16.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin x$	1
1	$\cos x$	0
2	$-\sin x$	-1
3	$-\cos x$	0
4	$\sin x$	1
. . .	. . .	. . .

$$\sin x=\sum_{k=0}^{\infty}\frac{f^{(k)}(\pi/2)}{k!}\left(x-\frac{\pi}{2}\right)^k$$

$$=1-\frac{(x-\pi/2)^2}{2!}+\frac{(x-\pi/2)^4}{4!}-\frac{(x-\pi/2)^6}{6!}+\cdots$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{(x-\pi/2)^{2n}}{(2n)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left[\frac{|x-\pi/2|^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{|x-\pi/2|^{2n}}\right]=\lim_{n\to\infty}\frac{|x-\pi/2|^2}{(2n+2)(2n+1)}=0<1 \text{ for all } x\text{ , so } R=\infty\ .$$

17.

n	$f^{(n)}(x)$	$f^{(n)}(0)$

0	$x^{-1/2}$	$\frac{1}{3}$
1	$-\frac{1}{2}x^{-3/2}$	$-\frac{1}{2}\cdot\frac{1}{3^3}$
2	$\frac{3}{4}x^{-5/2}$	$-\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\cdot\frac{1}{3^5}$
3	$-\frac{15}{8}x^{-7/2}$	$-\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\frac{1}{3^7}$
⋮	⋮	⋮

$$\frac{1}{\sqrt{x}}=\frac{1}{3}-\frac{1}{2\cdot 3^3}(x-9)+\frac{3}{2^2\cdot 3^5}\frac{(x-9)^2}{2!}-\frac{3\cdot 5}{2^3\cdot 3^7}\frac{(x-9)^3}{3!}+\cdots$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\ \cdots\ \cdot(2n-1)}{2^n\cdot 3^{2n+1}\cdot n!}(x-9)^n .$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left[\frac{1\cdot 3\cdot 5\cdot\ \cdots\ \cdot(2n-1)[2(n+1)-1]|x-9|^{n+1}}{2^{n+1}\cdot 3^{[2(n+1)+1]}\cdot(n+1)!}\cdot\frac{2^n\cdot 3^{2n+1}\square!}{1\cdot 3\cdot 5\cdot\ \cdots\ \cdot(2n-1)|x-9|^n}\right]$$

$$=\lim_{n\to\infty}\left[\frac{(2n+1)|x-9|}{2\cdot 3^2(n+1)}\right]=\frac{1}{9}|x-9|<1$$

for convergence, so $|x-9|<9$ and $R=9$.

18.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	x^{-2}	1
1	$-2x^{-3}$	-2
2	$6x^{-4}$	6

3	$-24x^{-5}$	-24
4	$120x^{-6}$	120
. . .	. . .	. . .

$$x^{-2}=1-2(x-1)+6\cdot\frac{(x-1)^2}{2!}-24\cdot\frac{(x-1)^3}{3!}+120\cdot\frac{(x-1)^4}{4!}-\cdots$$
$$=1-2(x-1)+3(x-1)^2-4(x-1)^3+5(x-1)^4-\cdots$$
$$=\sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n .$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{(n+2)|x-1|^{n+1}}{(n+1)|x-1|^n}=\lim_{n\to\infty}\left[\frac{n+2}{n+1}\cdot|x-1|\right]=|x-1|<1 \text{ for convergence, so } R=1 .$$

19. If $f(x)=\cos x$, then $f^{(n+1)}(x)=\pm\sin x$ or $\pm\cos x$. In each case, $\left|f^{(n+1)}(x)\right|\leq 1$, so by Formula 9 with $a=0$ and $M=1$, $\left|R_n(x)\right|\leq\frac{1}{(n+1)!}|x|^{n+1}$. Thus, $\left|R_n(x)\right|\to 0$ as $n\to\infty$ by Equation 10. So $\lim_{n\to\infty} R_n(x)=0$ and, by Theorem 8, the series in Exercise 3 represents $\cos x$ for all x .

20. If $f(x)=\sin x$, then $f^{(n+1)}(x)=\pm\sin x$ or $\pm\cos x$. In each case, $\left|f^{(n+1)}(x)\right|\leq 1$, so by Formula 9 with $a=0$ and $M=1$, $\left|R_n(x)\right|\leq\frac{1}{(n+1)!}\left|x-\frac{\pi}{2}\right|^{n+1}$. Thus, $\left|R_n(x)\right|\to 0$ as $n\to\infty$ by Equation 10. So $\lim_{n\to\infty} R_n(x)=0$ and, by Theorem 8, the series in Exercise 16 represents $\sin x$ for all x .

21. If $f(x)=\sinh x$, then for all n , $f^{(n+1)}(x)=\cosh x$ or $\sinh x$. Since $|\sinh x|<|\cosh x|=\cosh x$ for all x , we have $\left|f^{(n+1)}(x)\right|\leq\cosh x$ for all n . If d is any positive number and $|x|\leq d$, then $\left|f^{(n+1)}(x)\right|\leq\cosh x\leq\cosh d$, so by Formula 9 with $a=0$ and $M=\cosh d$, we have $\left|R_n(x)\right|\leq\frac{\cosh d}{(n+1)!}|x|^{n+1}$. It follows that $\left|R_n(x)\right|\to 0$ as $n\to\infty$ for $|x|\leq d$ (by Equation 10). But d was an arbitrary positive number. So by Theorem 8, the series represents $\sinh x$ for all x .

22. If $f(x)=\cosh x$, then for all n , $f^{(n+1)}(x)=\cosh x$ or $\sinh x$. Since $|\sinh x|<|\cosh x|=\cosh x$ for all x , we have $\left|f^{(n+1)}(x)\right|\leq\cosh x$ for all n . If d is any positive number and $|x|\leq d$, then

$\left|f^{(n+1)}(x)\right| \leq \cosh x \leq \cosh d$, so by Formula 9 with $a=0$ and $M=\cosh d$, we have $\left|R_n(x)\right| \leq \dfrac{\cosh d}{(n+1)!}\,|x|^{n+1}$. It follows that $\left|R_n(x)\right| \to 0$ as $n\to\infty$ for $|x| \leq d$ (by Equation 10). But d was an arbitrary positive number. So by Theorem 8, the series represents $\cosh x$ for all x .

23. $\cos x=\sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n}}{(2n)!} \Rightarrow f(x)=\cos(\pi x)=\sum_{n=0}^{\infty} \dfrac{(-1)^n(\pi x)^{2n}}{(2n)!}=\sum_{n=0}^{\infty} \dfrac{(-1)^n \pi^{2n} x^{2n}}{(2n)!}$, $R=\infty$

24. $e^x=\sum_{n=0}^{\infty} \dfrac{x^n}{n!} \Rightarrow f(x)=e^{-x/2}=\sum_{n=0}^{\infty} \dfrac{(-x/2)^n}{n!}=\sum_{n=0}^{\infty} \dfrac{(-1)^n}{2^n n!}\, x^n$, $R=\infty$

25. $\tan^{-1} x=\sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n+1}}{2n+1} \Rightarrow f(x)=x\tan^{-1} x=x\sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n+2}}{2n+1}$, $R=1$

26. $\sin x=\sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n+1}}{(2n+1)!} \Rightarrow f(x)=\sin\left(x^4\right)=\sum_{n=0}^{\infty}(-1)^n \dfrac{\left(x^4\right)^{2n+1}}{(2n+1)!}=\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!}\, x^{8n+4}$, $R=\infty$

27. $e^x=\sum_{n=0}^{\infty} \dfrac{x^n}{n!} \Rightarrow f(x)=x^2 e^{-x}=x^2\sum_{n=0}^{\infty} \dfrac{(-x)^n}{n!}=\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{n+2}}{n!}$, $R=\infty$

28. $\cos x=\sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n}}{(2n)!} \Rightarrow \cos 2x=\sum_{n=0}^{\infty}(-1)^n \dfrac{(2x)^{2n}}{(2n)!}=\sum_{n=0}^{\infty} \dfrac{(-1)^n 2^{2n}}{(2n)!}\, x^{2n} \Rightarrow$
$f(x)=x\cos 2x=\sum_{n=0}^{\infty} \dfrac{(-1)^n 2^{2n}}{(2n)!}\, x^{2n+1}$, $R=\infty$

29.

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)=\frac{1}{2}\left[1-\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!}\right]=\frac{1}{2}\left[1-1-\sum_{n=1}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!}\right]$$
$$=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^{2n-1}x^{2n}}{(2n)!}\text{ , } R=\infty$$

30.

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)=\frac{1}{2}\left[1+\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!}\right]=\frac{1}{2}\left[1+1+\sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n}x^{2n}}{(2n)!}\right]$$

$$=1+\sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n-1}x^{2n}}{(2n)!}\ ,\ R=\infty$$

Another method: Use $\cos^2 x=1-\sin^2 x$ and Exercise 29.

31. $\frac{\sin x}{x}=\frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}$ and this series also gives the required value at $x=0$ (namely 1); $R=\infty$.

32.
$$\frac{x-\sin x}{x^3}=\frac{1}{x^3}\left[x-\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right]=\frac{1}{x^3}\left[x-x-\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right]=\frac{1}{x^3}\left[-\sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}\right]$$

$=\frac{1}{x^3}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+3}}{(2n+3)!}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+3)!}$ and this series also gives the required value at $x=0$ (namely 1/6); $R=\infty$.

33.

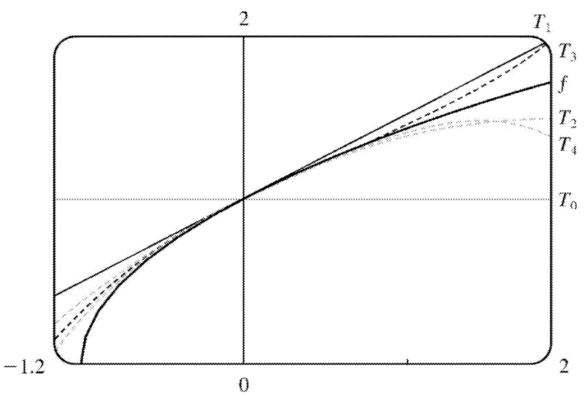
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{1/2}$	1
1	$\frac{1}{2}(1+x)^{-1/2}$	$\frac{1}{2}$
2	$-\frac{1}{4}(1+x)^{-3/2}$	$-\frac{1}{4}$
3	$\frac{3}{8}(1+x)^{-5/2}$	$\frac{3}{8}$
4	$-\frac{15}{16}(1+x)^{-7/2}$	$-\frac{15}{16}$
⋮	⋮	⋮

So $f^{(n)}(0)=\frac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n}$ for $n\geq 2$, and

$\sqrt{1+x}=1+\frac{x}{2}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n n!}x^n$. If $a_n=\frac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n n!}x^n$,

then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)(2n-1)x^{n+1}}{2^{n+1}(n+1)!}\cdot\frac{2^n n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)x^n}\right|$

$$=\frac{|x|}{2}\lim_{n\to\infty}\frac{2n-1}{n+1}=\frac{|x|}{2}\cdot 2=|x|<1 \text{ for convergence, so } R=1 .$$

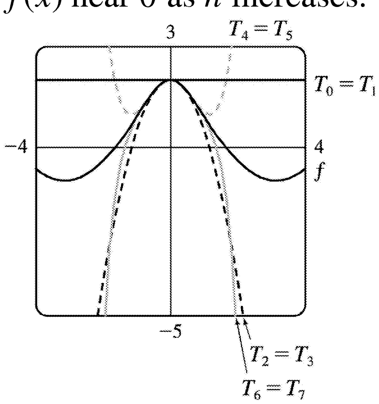


Notice that, as n increases, $T_n(x)$ becomes a better approximation to $f(x)$ for $-1<x<1$.

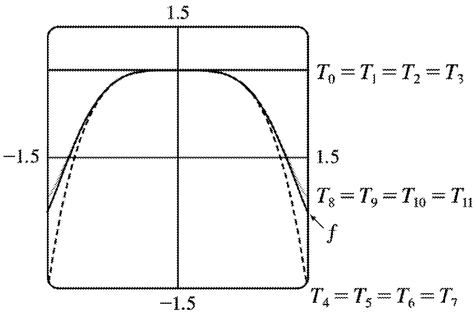
34. $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$, so $e^{-x^2}=\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!}$. Also, $\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$, so

$f(x)=e^{-x^2}+\cos x=\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{n!}+\frac{1}{(2n)!}\right)x^{2n}=2-\frac{3}{2}x^2+\frac{13}{24}x^4-\frac{121}{720}x^6+\cdots$.

The series for e^x and $\cos x$ converge for all x , so the same is true of the series for $f(x)$; that is, $R=\infty$. From the graphs of f and the first few Taylor polynomials, we see that $T_n(x)$ provides a closer fit to $f(x)$ near 0 as n increases.



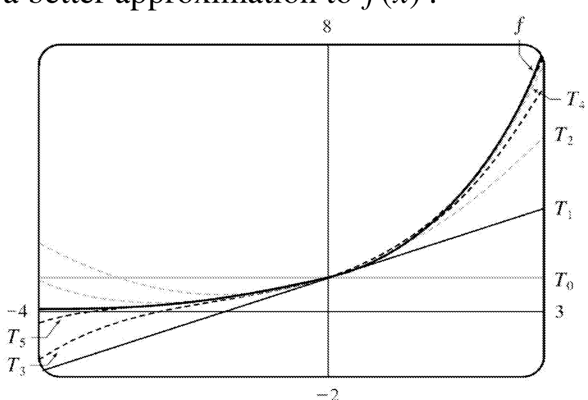
35. $\cos x=\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{(2n)!}\Rightarrow f(x)=\cos\left(x^2\right)=\sum_{n=0}^{\infty}\dfrac{(-1)^n\left(x^2\right)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{4n}}{(2n)!}$, $R=\infty$



Notice that, as n increases, $T_n(x)$ becomes a better approximation to $f(x)$.

36. $2^x=\left(e^{\ln 2}\right)^x=e^{x\ln 2}=\sum_{n=0}^{\infty}\dfrac{(x\ln 2)^n}{n!}=\sum_{n=0}^{\infty}\dfrac{(\ln 2)^n x^n}{n!}$, $R=\infty$.

Notice that, as n increases, $T_n(x)$ becomes
a better approximation to $f(x)$.



37. $e^x=\sum_{n=0}^{\infty}\dfrac{x^n}{n!}$, so

$e^{-0.2}=\sum_{n=0}^{\infty}\dfrac{(-0.2)^n}{n!}=1-0.2+\dfrac{1}{2!}(0.2)^2-\dfrac{1}{3!}(0.2)^3+\dfrac{1}{4!}(0.2)^4-\dfrac{1}{5!}(0.2)^5+\dfrac{1}{6!}(0.2)^6-\cdots$. But

$\dfrac{1}{6!}(0.2)^6=8.\overline{8}\times 10^{-8}$, so by the Alternating Series Estimation Theorem, $e^{-0.2}\approx\sum_{n=0}^{5}\dfrac{(-0.2)^n}{n!}\approx 0.81873$
, correct to five decimal places.

38. $3^\circ=\dfrac{\pi}{60}$ radians and $\sin x=\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$, so

$\sin \frac{\pi}{60} = \frac{\pi}{60} - \frac{\left(\frac{\pi}{60}\right)^3}{3!} + \frac{\left(\frac{\pi}{60}\right)^5}{5!} - \cdots = \frac{\pi}{60} - \frac{\pi^3}{1{,}296{,}000} + \frac{\pi^5}{93{,}312{,}000{,}000} - \cdots$. But

$\frac{\pi^5}{93{,}312{,}000{,}000} < 10^{-8}$, so by the Alternating Series Estimation Theorem,

$\sin \frac{\pi}{60} \approx \frac{\pi}{60} - \frac{\pi^3}{1{,}296{,}000} \approx 0.05234$.

39. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow \cos (x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!} \Rightarrow$

$x\cos (x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(2n)!} \Rightarrow \int x\cos (x^3)\, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}$, with $R=\infty$.

40. $\frac{\sin x}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$, so

$\int \frac{\sin x}{x}\, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$

41. Using the series from Exercise 33 and substituting x^3 for x , we get

$$\int \sqrt{x^3+1}\, dx = \int \left[1 + \frac{x^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1\cdot 3\cdot 5\cdot \cdots \cdot (2n-3)}{2^n \mathrm{n}!} x^{3n} \right] dx$$

$$= C + x + \frac{x^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1\cdot 3\cdot 5\cdot \cdots \cdot (2n-3)}{2^n \mathrm{n}!(3n+1)} x^{3n+1}$$

42. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{\mathrm{n}!} \Rightarrow e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{\mathrm{n}!} \Rightarrow \frac{e^x-1}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{\mathrm{n}!} \Rightarrow$

$\int \frac{e^x-1}{x}\, dx = C + \sum_{n=1}^{\infty} \frac{x^n}{n\cdot \mathrm{n}!}$, with $R=\infty$.

43. By Exercise 39, $\int x\cos (x^3)\, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}$, so $\int_0^1 x\cos (x^3)\, dx$

$= \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!} \right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+2)(2n)!} = \frac{1}{2} - \frac{1}{8\cdot 2!} + \frac{1}{14\cdot 4!} - \frac{1}{20\cdot 6!} + \cdots$, but

$\frac{1}{20\cdot 6!} = \frac{1}{14{,}400} \approx 0.000069$, so

$\int_0^1 x\cos(x^3)\,dx \approx \frac{1}{2} - \frac{1}{16} + \frac{1}{336} \approx 0.440$ (correct to three decimal places) by the Alternating Series Estimation Theorem.

44. From the table of Maclaurin series in Section .10, we see that

$\tan^{-1}x = \Sigma_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ for x in $[-1,1]$ and $\sin x = \Sigma_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all real numbers x , so

$\tan^{-1}(x^3) + \sin(x^3) = \Sigma_{n=0}^{\infty}(-1)^n \frac{x^{6n+3}}{2n+1} + \Sigma_{n=0}^{\infty}(-1)^n \frac{x^{6n+3}}{(2n+1)!}$ for x^3 in $[-1,1] \Leftrightarrow x$ in $[-1,1]$. Thus,

$$I = \int_0^{0.2} dx = \int_0^{0.2} \Sigma_{n=0}^{\infty}(-1)^n x^{6n+3}\left(\frac{1}{2n+1} + \frac{1}{(2n+1)!}\right)dx$$

$$= \left[\Sigma_{n=0}^{\infty}(-1)^n \frac{x^{6n+4}}{6n+4}\left(\frac{1}{2n+1} + \frac{1}{(2n+1)!}\right)\right]_0^{0.2}$$

$$= \Sigma_{n=0}^{\infty}(-1)^n \frac{(0.2)^{6n+4}}{6n+4}\left(\frac{1}{2n+1} + \frac{1}{(2n+1)!}\right) = \frac{(0.2)^4}{4}(1+1) - \frac{(0.2)^{10}}{10}\left(\frac{1}{3} + \frac{1}{3!}\right) + \cdots .$$

But $\frac{(0.2)^{10}}{10}\left(\frac{1}{3} + \frac{1}{3!}\right) = \frac{(0.2)^{10}}{20} = 5.12 \times 10^{-9}$, so by the Alternating Series Estimation Theorem,

$I \approx \frac{(0.2)^4}{2} = 0.000\,80$ (correct to five decimal places).

45. We first find a series representation for $f(x) = (1+x)^{-1/2}$, and then substitute.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-1/2}$	1
1	$-\frac{1}{2}(1+x)^{-3/2}$	$-\frac{1}{2}$
2	$\frac{3}{4}(1+x)^{-5/2}$	$\frac{3}{4}$
3	$-\frac{15}{8}(1+x)^{-7/2}$	$-\frac{15}{8}$
⋮	⋮	⋮

$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3}{4}\left(\frac{x^2}{2!}\right) - \frac{15}{8}\left(\frac{x^3}{3!}\right) + \cdots \Rightarrow \frac{1}{\sqrt{1+x^3}} = 1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9 + \cdots \Rightarrow$

$\int_0^{0.1} \frac{dx}{\sqrt{1+x^3}} = \left[x - \frac{1}{8}x^4 + \frac{3}{56}x^7 - \frac{1}{32}x^{10} + \cdots\right]_0^{0.1} \approx (0.1) - \frac{1}{8}(0.1)^4$, by the Alternating Series Estimation Theorem, since $\frac{3}{56}(0.1)^7 \approx 0.0000000054 < 10^{-8}$, which is the maximum desired error.

Therefore, $\int_0^{0.1} \frac{dx}{\sqrt{1+x^3}} \approx 0.09998750$.

46. $\int_0^{0.5} x^2 e^{-x^2} dx = \int_0^{0.5} \Sigma_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!} dx = \Sigma_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n+3}}{n!(2n+3)}\right]_0^{1/2} = \Sigma_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+3)2^{2n+3}}$ and since the term with $n=2$ is $\frac{1}{1792} < 0.001$, we use $\Sigma_{n=0}^{1} \frac{(-1)^n}{n!(2n+3)2^{2n+3}} = \frac{1}{24} - \frac{1}{160} \approx 0.0354$.

47.

$$\lim_{x\to 0} \frac{x - \tan^{-1} x}{x^3} = \lim_{x\to 0} \frac{x - \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots\right)}{x^3} = \lim_{x\to 0} \frac{\frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{1}{7}x^7 - \cdots}{x^3}$$

$$= \lim_{x\to 0}\left(\frac{1}{3} - \frac{1}{5}x^2 + \frac{1}{7}x^4 - \cdots\right) = \frac{1}{3}$$

since power series are continuous functions.

48.

$$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x} = \lim_{x\to 0} \frac{1 - \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right)}{1+x - \left(1+x+\frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots\right)}$$

$$=\lim_{x\to 0}\frac{\frac{1}{2!}x^2-\frac{1}{4!}x^4+\frac{1}{6!}x^6-\cdots}{-\frac{1}{2!}x^2-\frac{1}{3!}x^3-\frac{1}{4!}x^4-\frac{1}{5!}x^5-\frac{1}{6!}x^6-\cdots}$$

$$=\lim_{x\to 0}\frac{\frac{1}{2!}-\frac{1}{4!}x^2+\frac{1}{6!}x^4-\cdots}{-\frac{1}{2!}-\frac{1}{3!}x-\frac{1}{4!}x^2-\frac{1}{5!}x^3-\frac{1}{6!}x^4-\cdots}=\frac{\frac{1}{2}-0}{-\frac{1}{2}-0}=-1$$

since power series are continuous functions.

49.

$$\lim_{x\to 0}\frac{\sin x-x+\frac{1}{6}x^3}{x^5}=\lim_{x\to 0}\frac{\left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots\right)-x+\frac{1}{6}x^3}{x^5}$$

$$=\lim_{x\to 0}\frac{\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots}{x^5}=\lim_{x\to 0}\left(\frac{1}{5!}-\frac{x^2}{7!}+\frac{x^4}{9!}-\cdots\right)=\frac{1}{5!}=\frac{1}{120}$$

since power series are continuous functions.

50.

$$\lim_{x\to 0}\frac{\tan x-x}{x^3}=\lim_{x\to 0}\frac{\left(x+\frac{1}{3}x^3+\frac{2}{15}x^5+\cdots\right)-x}{x^3}=\lim_{x\to 0}\frac{\frac{1}{3}x^3+\frac{2}{15}x^5+\cdots}{x^3}=\lim_{x\to 0}\left(\frac{1}{3}+\frac{2}{15}x^2+\cdots\right)=\frac{1}{3}$$

since power series are continuous functions.

51. As in Example 8(a), we have $e^{-x^2}=1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots$ and we know that $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$ from Equation 16. Therefore, $e^{-x^2}\cos x=\left(1-x^2+\frac{1}{2}x^4-\cdots\right)\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots\right)$. Writing only the terms with degree ≤ 4, we get

$$e^{-x^2}\cos x=1-\frac{1}{2}x^2+\frac{1}{24}x^4-x^2+\frac{1}{2}x^4+\frac{1}{2}x^4+\cdots=1-\frac{3}{2}x^2+\frac{25}{24}x^4+\cdots.$$

52.

$$
\begin{array}{r|l}
 & 1+\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots \\
\hline
1-\frac{1}{2}x^2+\frac{5}{24}x^4-\cdots & 1 \\
 & 1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots \\
\hline
 & \frac{1}{2}x^2-\frac{1}{24}x^4+\cdots \\
 & \frac{1}{2}x^2-\frac{1}{4}x^4+\cdots \\
\hline
 & \frac{5}{24}x^4+\cdots \\
 & \frac{5}{24}x^4+\cdots \\
\hline
 & \cdots
\end{array}
$$

$\sec x=\dfrac{1}{\cos x}=\dfrac{1}{1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots}$. From the long division above, $\sec x=1+\frac{1}{2}x^2+\frac{5}{24}x^4+\cdots$.

53.

$$
\begin{array}{r|l}
 & 1+\frac{1}{6}x^2+\frac{7}{360}x^4+\cdots \\
\hline
x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots & x \\
 & x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots \\
\hline
 & \frac{1}{6}x^3-\frac{1}{120}x^5+\cdots \\
 & \frac{1}{6}x^3-\frac{1}{36}x^5+\cdots \\
\hline
 & \frac{7}{360}x^5+\cdots \\
 & \frac{7}{360}x^5+\cdots \\
\hline
 & \cdots
\end{array}
$$

$\dfrac{x}{\sin x} = \dfrac{x}{x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots}$. From the long division above, $\dfrac{x}{\sin x} = 1+\frac{1}{6}x^2+\frac{7}{360}x^4+\cdots$.

54. From Example 6 in Section .9, we have $\ln(1-x) = -x-\frac{1}{2}x^2-\frac{1}{3}x^3-\cdots$, $|x|<1$. Therefore,

$$\begin{aligned} e^x \ln(1-x) &= \left(1+x+\frac{1}{2}x^2+\cdots\right)\left(-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\cdots\right) \\ &= -x-\frac{1}{2}x^2-\frac{1}{3}x^3-x^2-\frac{1}{2}x^3-\frac{1}{2}x^3-\cdots \\ &= -x-\frac{3}{2}x^2-\frac{4}{3}x^3-\cdots, |x|<1 \end{aligned}$$

55. $\sum_{n=0}^{\infty}(-1)^n \dfrac{x^{4n}}{n!} = \sum_{n=0}^{\infty} \dfrac{\left(-x^4\right)^n}{n!} = e^{-x^4}$, by (11).

56. $\sum_{n=0}^{\infty} \dfrac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty}(-1)^n \dfrac{\left(\frac{\pi}{6}\right)^{2n}}{(2n)!} = \cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$, by (16).

57. $\sum_{n=0}^{\infty} \dfrac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty} \dfrac{(-1)^n \left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} = \sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$, by (15).

58. $\sum_{n=0}^{\infty} \dfrac{3^n}{5^n n!} = \sum_{n=0}^{\infty} \dfrac{(3/5)^n}{n!} = e^{3/5}$, by (11).

59. $3+\dfrac{9}{2!}+\dfrac{27}{3!}+\dfrac{81}{4!}+\cdots = \dfrac{3^1}{1!}+\dfrac{3^2}{2!}+\dfrac{3^3}{3!}+\dfrac{3^4}{4!}+\cdots = \sum_{n=1}^{\infty}\dfrac{3^n}{n!} = \sum_{n=0}^{\infty}\dfrac{3^n}{n!}-1 = e^3-1$, by (11).

60. $1-\ln 2+\dfrac{(\ln 2)^2}{2!}-\dfrac{(\ln 2)^3}{3!}+\cdots = \sum_{n=0}^{\infty}\dfrac{(-\ln 2)^n}{n!} = e^{-\ln 2} = \left(e^{\ln 2}\right)^{-1} = 2^{-1} = \dfrac{1}{2}$, by (11).

61. Assume that $\left|f'''(x)\right| \le M$, so $f'''(x) \le M$ for $a \le x \le a+d$. Now $\int_a^x f'''(t)\,dt \le \int_a^x M\,dt \Rightarrow$ $f''(x)-f''(a) \le M(x-a) \Rightarrow f''(x) \le f''(a)+M(x-a)$. Thus, $\int_a^x f''(t)\,dt \le \int_a^x \left[f''(a)+M(t-a)\right]dt$ $\Rightarrow$

$f'(x)-f'(a)\le f''(a)(x-a)+\frac{1}{2}M(x-a)^2 \Rightarrow f'(x)\le f'(a)+f''(a)(x-a)+\frac{1}{2}M(x-a)^2 \Rightarrow$

$\int_a^x f'(t)\,dt \le \int_a^x \left[f'(a)+f''(a)(t-a)+\frac{1}{2}M(t-a)^2 \right] dt \Rightarrow$

$f(x)-f(a)\le f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2+\frac{1}{6}M(x-a)^3$. So

$f(x)-f(a)-f'(a)(x-a)-\frac{1}{2}f''(a)(x-a)^2\le \frac{1}{6}M(x-a)^3$. But

$R_2(x)=f(x)-T_2(x)=f(x)-f(a)-f'(a)(x-a)-\frac{1}{2}f''(a)(x-a)^2$, so $R_2(x)\le \frac{1}{6}M(x-a)^3$.

A similar argument using $f'''(x)\ge -M$ shows that $R_2(x)\ge -\frac{1}{6}M(x-a)^3$. So

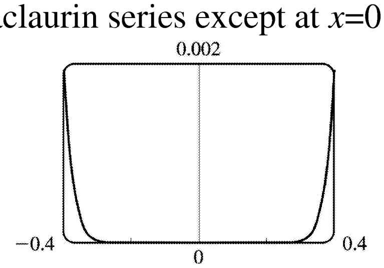
$\left|R_2\left(x_2\right)\right| \le \frac{1}{6}M|x-a|^3$.

Although we have assumed that $x>a$, a similar calculation shows that this inequality is also true if $x<a$.

62. **(a)** $f(x)=\begin{cases} e^{-1/x^2} & \text{if } x\ne 0 \\ 0 & \text{if } x=0 \end{cases}$ so

$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{e^{-1/x^2}}{x}=\lim_{x\to 0}\frac{1/x}{e^{1/x^2}}=\lim_{x\to 0}\frac{x}{2e^{1/x^2}}=0$ (using l'Hospital's Rule and simplifying in the penultimate step). Similarly, we can use the definition of the derivative and l'Hospital's Rule to show that $f''(0)=0$, $f^{(3)}(0)=0$, … , $f^{(n)}(0)=0$, so that the Maclaurin series for f consists entirely of zero terms. But since $f(x)\ne 0$ except for $x=0$, we see that f cannot equal its Maclaurin series except at $x=0$.

(b)



From the graph, it seems that the function is extremely flat at the origin. In fact, it could be said to be ‘‘infinitely flat’’ at $x=0$, since all of its derivatives are 0 there.

1. The general binomial series in (2) is

$$(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\cdots .$$

$$(1+x)^{1/2}=\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}x^n=1+\left(\frac{1}{2}\right)x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3+\cdots$$

$$=1+\frac{x}{2}-\frac{x^2}{2^2\cdot 2!}+\frac{1\cdot 3\cdot x^3}{2^3\cdot 3!}-\frac{1\cdot 3\cdot 5\cdot x^4}{2^4\cdot 4!}+\cdots$$

$$=1+\frac{x}{2}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)x^n}{2^n\cdot n!}\text{ for }|x|<1,\text{so}R=1$$

2. $\frac{1}{(1+x)^4}=(1+x)^{-4}=\sum_{n=0}^{\infty}\binom{-4}{n}x^n$. The binomial coefficient is

$$\binom{-4}{n}=\frac{(-4)(-5)(-6)\cdot\cdots\cdot(-4-n+1)}{n!}=\frac{(-4)(-5)(-6)\cdot\cdots\cdot[-(n+3)]}{n!}$$

$$=\frac{(-1)^n\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot\cdots\cdot(n+1)(n+2)(n+3)}{2\cdot 3\cdot n!}=\frac{(-1)^n(n+1)(n+2)(n+3)}{6}$$

Thus, $\frac{1}{(1+x)^4}=\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)(n+3)}{6}x^n$ for $|x|<1$, so $R=1$.

3. $\frac{1}{(2+x)^3}=\frac{1}{[2(1+x/2)]^3}=\frac{1}{8}\left(1+\frac{x}{2}\right)^{-3}=\frac{1}{8}\sum_{n=0}^{\infty}\binom{-3}{n}\left(\frac{x}{2}\right)^n$. The binomial coefficient is

$$\binom{-3}{n}=\frac{(-3)(-4)(-5)\cdot\cdots\cdot(-3-n+1)}{n!}=\frac{(-3)(-4)(-5)\cdot\cdots\cdot[-(n+2)]}{n!}$$

$$=\frac{(-1)^n\cdot 2\cdot 3\cdot 4\cdot 5\cdot\cdots\cdot(n+1)(n+2)}{2\cdot n!}=\frac{(-1)^n(n+1)(n+2)}{2}$$

Thus, $\frac{1}{(2+x)^3}=\frac{1}{8}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)}{2}\frac{x^n}{2^n}=\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)x^n}{2^{n+4}}$ for $\left|\frac{x}{2}\right|<1\Leftrightarrow$ $|x|<2$, so $R=2$.

4.

$$(1-x)^{2/3}=\sum_{n=0}^{\infty}\binom{\frac{2}{3}}{n}(-x)^n$$

$$=1+\frac{2}{3}(-x)+\frac{\frac{2}{3}\left(-\frac{1}{3}\right)}{2!}(-x)^2+\frac{\frac{2}{3}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{3!}(-x)^3+\cdot\cdot\cdot$$

$$=1-\frac{2}{3}x+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(-1)^n\cdot 2\cdot[1\cdot 4\cdot 7\cdot\cdot\cdot\cdot\cdot(3n-5)]}{3^n\cdot n!}x^n$$

$$=1-\frac{2}{3}x-2\sum_{n=2}^{\infty}\frac{1\cdot 4\cdot 7\cdot\cdot\cdot\cdot\cdot(3n-5)}{3^n\cdot n!}x^n$$

and $|-x|<1\Leftrightarrow|x|<1$, so $R=1$.

5.

$$\sqrt[4]{1-8x}=(1-8x)^{1/4}=\sum_{n=0}^{\infty}\binom{\frac{1}{4}}{n}(-8x)^n$$

$$=1+\frac{1}{4}(-8x)+\frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2!}(-8x)^2+\frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{3!}(-8x)^3+\cdot\cdot\cdot$$

$$=1-2x+\sum_{n=2}^{\infty}\frac{(-1)^n(-1)^{n-1}\cdot 3\cdot 7\cdot\cdot\cdot\cdot\cdot(4n-5)8^n}{4^n\cdot n!}x^n$$

$$=1-2x-\sum_{n=2}^{\infty}\frac{3\cdot 7\cdot\cdot\cdot\cdot\cdot(4n-5)2^n}{n!}x^n$$

and $|-8x|<1\Leftrightarrow|x|<\frac{1}{8}$, so $R=\frac{1}{8}$.

6.

$$\frac{1}{\sqrt[5]{32-x}}=\frac{1}{2\sqrt[5]{1-x/32}}=\frac{1}{2}\left(1-\frac{x}{32}\right)^{-1/5}=\frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{5}}{n}\left(-\frac{x}{32}\right)^n=\frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{5}}{n}\frac{(-1)^nx^n}{2^{5n}}$$

$$=\frac{1}{2}\left[1+\left(-\frac{1}{5}\right)\left(-\frac{x}{2^5}\right)+\frac{\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)}{2!}\frac{x^2}{2^{10}}+\frac{\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)\left(-\frac{11}{5}\right)}{3!}\left(-\frac{x^3}{2^{15}}\right)+\cdot\cdot\cdot\right.$$

$$= \frac{1}{2} + \frac{1}{5\cdot 2^6}x + \frac{1\cdot 6}{5^2\cdot 2!\cdot 2^{11}}x^2 + \frac{1\cdot 6\cdot 11}{5^3\cdot 3!\cdot 2^{16}}x^3 + \cdots = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1\cdot 6\cdot \cdots \cdot (5n-4)}{5^n 2^{5n+1} n!}x^n$$

The radius of convergence is 32 .

7. We must write the binomial in the form (1+ expression), so we'll factor out a 4 .

$$\frac{x}{\sqrt{4+x^2}} = \frac{x}{\sqrt{4\left(1+x^2/4\right)}} = \frac{x}{2\sqrt{1+x^2/4}} = \frac{x}{2}\left(1+\frac{x^2}{4}\right)^{-1/2} = \frac{x}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x^2}{4}\right)^n$$

$$= \frac{x}{2}\left[1+\left(-\frac{1}{2}\right)\frac{x^2}{4} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x^2}{4}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x^2}{4}\right)^3 + \cdots\right]$$

$$= \frac{x}{2} + \frac{x}{2}\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{2^n\cdot 4^n\cdot n!}x^{2n}$$

$$= \frac{x}{2} + \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{n!\,2^{3n+1}}x^{2n+1} \text{ and } \frac{x^2}{4} < 1 \Leftrightarrow \frac{|x|}{2} < 1 \Leftrightarrow$$

$|x|<2$, so $R=2$.

8.

$$\frac{x^2}{\sqrt{2+x}} = \frac{x^2}{\sqrt{2(1+x/2)}} = \frac{x^2}{\sqrt{2}}\left(1+\frac{x}{2}\right)^{-1/2} = \frac{x^2}{\sqrt{2}}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x}{2}\right)^n$$

$$= \frac{x^2}{\sqrt{2}}\left[1+\left(-\frac{1}{2}\right)\left(\frac{x}{2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x}{2}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x}{2}\right)^3 + \cdots\right]$$

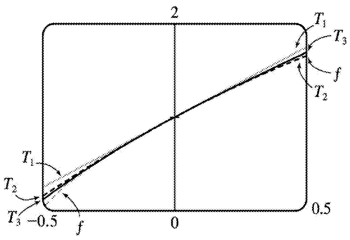
$$= \frac{x^2}{\sqrt{2}} + \frac{x^2}{\sqrt{2}}\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{n!\,2^{2n}}x^n$$

$$= \frac{x^2}{\sqrt{2}} + \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{n!\,2^{2n+1/2}}x^{n+2} \text{ and } \left|\frac{x}{2}\right| < 1 \Leftrightarrow |x| < 2 \text{ , so } R=2 \text{ .}$$

9.

$$(1+2x)^{3/4} = 1+\frac{3}{4}(2x)+\frac{\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)}{2!}(2x)^2+\frac{\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{3!}(2x)^3+\cdots$$
$$=1+\frac{3}{2}x+3\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1\cdot 5\cdot 9\cdot\cdots\cdot(4n-7)}{4^n\cdot n!}\cdot 2^n x^n$$
$$=1+\frac{3}{2}x+3\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1\cdot 5\cdot 9\cdot\cdots\cdot(4n-7)}{2^n\cdot n!}x^n \text{ and } |2x|<1 \Leftrightarrow |x|<\frac{1}{2}\text{ , so } R=\frac{1}{2}\text{ .}$$

The three Taylor polynomials are $T_1(x)=1+\frac{3}{2}x$, $T_2(x)=1+\frac{3}{2}x-\frac{3}{8}x^2$, and $T_3(x)=1+\frac{3}{2}x-\frac{3}{8}x^2+\frac{5}{16}x^3$.

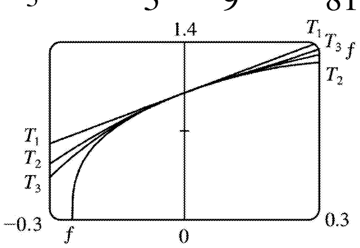


10.

$$\sqrt[3]{1+4x} = (1+4x)^{1/3}$$
$$=1+\frac{1}{3}(4x)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}(4x)^2+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}(4x)^3+\cdots$$
$$=1+\frac{4}{3}x+\sum_{n=2}^{\infty}(-1)^{n+1}\frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-4)}{3^n\cdot n!}(4x)^n \text{ and } |4x|<1 \Leftrightarrow |x|<\frac{1}{4}\text{ , so } R=\frac{1}{4}\text{ .}$$

The three Taylor polynomials are $T_1(x)=1+\frac{4}{3}x$, $T_2(x)=1+\frac{4}{3}x-\frac{16}{9}x^2$, and

$T_3(x)=1+\frac{4}{3}x-\frac{16}{9}x^2+\frac{320}{81}x^3$.



11. **(a)**

$$1/\sqrt{1-x^2}=\left[1+\left(-x^2\right)\right]^{-1/2}$$
$$=1+\left(-\frac{1}{2}\right)\left(-x^2\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-x^2\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-x^2\right)^3+\cdots$$
$$=1+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n\cdot n!}x^{2n}$$

(b)

$$\sin^{-1}x=\int\frac{1}{\sqrt{1-x^2}}\,dx=C+x+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(2n+1)2^n\cdot n!}x^{2n+1}$$
$$=x+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(2n+1)2^n\cdot n!}x^{2n+1}\text{ since }0=\sin^{-1}0=C\,.$$

12. **(a)** $\left(1+x^2\right)^{-1/2}=\sum_{n=0}^{\infty}\binom{-1/2}{n}x^{2n}=1+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)x^{2n}}{2^n\cdot n!}$

(b) $\sinh^{-1}x=\int\frac{dx}{\sqrt{1+x^2}}=C+x+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)x^{2n+1}}{2^n\cdot n!\,(2n+1)}$, but $C=0$ since $\sinh^{-1}0=0$, so

$\sinh^{-1}x=x+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)x^{2n+1}}{2^n\cdot n!\,(2n+1)}$, $R=1$.

13. **(a)**

$$\sqrt[3]{1+x}=(1+x)^{1/3}=\sum_{n=0}^{\infty}\binom{\frac{1}{3}}{n}x^n$$
$$=1+\frac{1}{3}x+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}x^2+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}x^3+\cdots$$
$$=1+\frac{x}{3}+\sum_{n=2}^{\infty}(-1)^{n+1}\frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-4)}{3^n\cdot n!}x^n$$

(b) $\sqrt[3]{1+x}=1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\cdots$. $\sqrt[3]{1.01}=\sqrt[3]{1+0.01}$, so let $x=0.01$. The sum of the first two terms is then

$1+\frac{1}{3}(0.01)\approx 1.0033$. The third term is $\frac{1}{9}(0.01)^2\approx 0.00001$, which does not affect the fourth decimal place of the sum, so we have $\sqrt[3]{1.01}\approx 1.0033$.

14. **(a)**

$$1/\sqrt[4]{1+x} =(1+x)^{-1/4}=\sum_{n=0}^{\infty}\binom{-\frac{1}{4}}{n}x^n$$
$$=1-\frac{1}{4}x+\frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2!}x^2+\frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)}{3!}x^3+\cdots$$
$$=1-\frac{1}{4}x+\sum_{n=2}^{\infty}(-1)^n\frac{1\cdot 5\cdot 9\cdot\cdots\cdot(4n-3)}{4^n\cdot n!}x^n$$

(b) $1/\sqrt[4]{1+x}=1-\frac{1}{4}x+\frac{5}{32}x^2-\frac{15}{128}x^3+\frac{195}{2048}x^4-\cdots$. $1/\sqrt[4]{1.1}=1/\sqrt[4]{1+0.1}$, so let $x=0.1$. The sum of the first four terms is then $1-\frac{1}{4}(0.1)+\frac{5}{32}(0.1)^2-\frac{15}{128}(0.1)^3\approx 0.976$. The fifth term is $\frac{195}{2048}(0.1)^4\approx 0.0000095$, which does not affect the third decimal place of the sum, so we have $1/\sqrt[4]{1.1}\approx 0.976$. (Note that the third decimal place of the sum of the first three terms is affected by the fourth term, so we need to use more than three terms for the sum.)

15. **(a)**

$$[1+(-x)]^{-2} =1+(-2)(-x)+\frac{(-2)(-3)}{2!}(-x)^2+\frac{(-2)(-3)(-4)}{3!}(-x)^3+\cdots$$
$$=1+2x+3x^2+4x^3+\cdots=\sum_{n=0}^{\infty}(n+1)x^n ,$$

so $\frac{x}{(1-x)^2}=x\sum_{n=0}^{\infty}(n+1)x^n=\sum_{n=0}^{\infty}(n+1)x^{n+1}=\sum_{n=1}^{\infty}nx^n$.

(b) With $x=\frac{1}{2}$ in part (a), we have $\sum_{n=1}^{\infty}n\left(\frac{1}{2}\right)^n=\sum_{n=1}^{\infty}\frac{n}{2^n}=\frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2}=\frac{\frac{1}{2}}{\frac{1}{4}}=2$.

16. **(a)**

$$[1+(-x)]^{-3} = \sum_{n=0}^{\infty} \binom{-3}{n} (-x)^n$$
$$= 1+(-3)(-x)+ \frac{(-3)(-4)}{2!}(-x)^2 + \frac{(-3)(-4)(-5)}{3!}(-x)^3 + \cdots$$
$$= 1+\sum_{n=1}^{\infty} \frac{3\cdot 4\cdot 5\cdot \cdots \cdot (n+2)}{n!} x^n = \sum_{n=0}^{\infty} \frac{2\cdot 3\cdot 4\cdot 5\cdot \cdots \cdot (n+2)}{2\cdot n!} x^n$$
$$= \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n \Rightarrow$$

$$\left(x+x^2\right)[1+(-x)]^{-3} = x[1+(-x)]^{-3} + x^2[1+(-x)]^{-3}$$
$$= \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^{n+1} + \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^{n+2}$$
$$= \sum_{n=1}^{\infty} \frac{n(n+1)}{2} x^n + \sum_{n=1}^{\infty} \frac{n(n+1)}{2} x^{n+1}$$
$$= x+\sum_{n=2}^{\infty} \frac{n(n+1)}{2} x^n + \sum_{n=2}^{\infty} \frac{(n-1)n}{2} x^n = x+\sum_{n=2}^{\infty} \left[\frac{n(n+1)}{2} + \frac{(n-1)n}{2} \right] x^n$$
$$= x+\sum_{n=2}^{\infty} n^2 x^n = \sum_{n=1}^{\infty} n^2 x^n, \ -1<x<1$$

(b) Setting $x=\frac{1}{2}$ in the last series above gives the required series, so $\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{\frac{1}{2}+\left(\frac{1}{2}\right)^2}{\left(1-\frac{1}{2}\right)^3} = \frac{\frac{3}{4}}{\frac{1}{8}} = 6$

.

17. **(a)**

$$\left(1+x^2\right)^{1/2} = 1+\left(\frac{1}{2}\right)x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(x^2\right)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(x^2\right)^3 + \cdots$$
$$= 1+\frac{x^2}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1\cdot 3\cdot 5\cdot \cdots \cdot (2n-3)}{2^n \cdot n!} x^{2n}$$

(b) The coefficient of x^{10} (corresponding to $n=5$) in the above Maclaurin series is $\frac{f^{(10)}(0)}{10!}$, so

$$\frac{f^{(10)}(0)}{10!}=\frac{(-1)^4\cdot 1\cdot 3\cdot 5\cdot 7}{2^5\cdot 5!}\Rightarrow f^{(10)}(0)=10!\left(\frac{1\cdot 3\cdot 5\cdot 7}{2^5\cdot 5!}\right)=99\ ,225\ .$$

18. **(a)**

$$\left(1+x^3\right)^{-1/2}=\Sigma_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(x^3\right)^n$$

$$=1+\left(-\frac{1}{2}\right)\left(x^3\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(x^3\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(x^3\right)^3+\cdots$$

$$=1+\Sigma_{n=1}^{\infty}\frac{(-1)^n 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)x^{3n}}{2^n\cdot n!}$$

(b) The coefficient of x^9 (corresponding to $n=3$) in the preceding series is

$\dfrac{f^{(9)}(0)}{9!}$, so $\dfrac{f^{(9)}(0)}{9!}=\dfrac{(-1)^3 1\cdot 3\cdot 5}{2^3\cdot 3!}\Rightarrow f^{(9)}(0)=-\dfrac{9!\cdot 5}{8\cdot 2}=-113\ ,400$.

19. **(a)** $g(x)=\Sigma_{n=0}^{\infty}\binom{k}{n}x^n\Rightarrow g'(x)=\Sigma_{n=1}^{\infty}\binom{k}{n}nx^{n-1}$, so

$$(1+x)g'(x)=(1+x)\Sigma_{n=1}^{\infty}\binom{k}{n}nx^{n-1}=\Sigma_{n=1}^{\infty}\binom{k}{n}nx^{n-1}+\Sigma_{n=1}^{\infty}\binom{k}{n}nx^n$$

$$=\Sigma_{n=0}^{\infty}\binom{k}{n}(n+1)x^n+\Sigma_{n=0}^{\infty}\binom{k}{n+1}nx^n\left[\begin{array}{c}\text{Replace } n \text{ with } n+1\\ \text{in the first series}\end{array}\right]$$

$$=\Sigma_{n=0}^{\infty}(n+1)\frac{k(k-1)(k-2)\cdots(k-n+1)(k-n)}{(n+1)!}x^n$$

$$+\Sigma_{n=0}^{\infty}\left[(n)\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}\right]x^n$$

$$=\Sigma_{n=0}^{\infty}\frac{(n+1)k(k-1)(k-2)\cdots(k-n+1)}{(n+1)!}[(k-n)+n]x^n$$

$$=k\Sigma_{n=0}^{\infty}\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n=k\Sigma_{n=0}^{\infty}\binom{k}{n}x^n=kg(x)$$

Thus, $g'(x)=\dfrac{kg(x)}{1+x}$.

(b)

$h(x)=(1+x)^{-k}g(x)\Rightarrow$

$h'(x)=-k(1+x)^{-k-1}g(x)+(1+x)^{-k}g'(x)$ [ProductRule

$=-k(1+x)^{-k-1}g(x)+(1+x)^{-k}\dfrac{kg(x)}{1+x}$

$=-k(1+x)^{-k-1}g(x)+k(1+x)^{-k-1}g(x)=0$

(c) From part (b) we see that $h(x)$ must be constant for $x\in(-1,1)$, so $h(x)=h(0)=1$ for $x\in(-1,1)$. Thus, $h(x)=1=(1+x)^{-k}g(x)\Leftrightarrow g(x)=(1+x)^{k}$ for $x\in(-1,1)$.

20. By Exercise .11.1, $\sqrt{1+x}=1+\dfrac{x}{2}+\sum_{n=2}^{\infty}\dfrac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)x^{n}}{2^{n}\cdot n!}$, so

$\left(1-x^{2}\right)^{1/2}=1-\dfrac{1}{2}x^{2}-\sum_{n=2}^{\infty}\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^{n}\cdot n!}x^{2n}$ and

$\sqrt{1-e^{2}\sin^{2}\theta}=1-\dfrac{1}{2}e^{2}\sin^{2}\theta-\sum_{n=2}^{\infty}\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^{n}\cdot n!}e^{2n}\sin^{2n}\theta$. Thus,

$$L=4a\int_{0}^{\pi/2}\sqrt{1-e^{2}\sin^{2}\theta}\,d\theta=4a\int_{0}^{\pi/2}\left(1-\frac{1}{2}e^{2}\sin^{2}\theta-\sum_{n=2}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^{n}\cdot n!}e^{2n}\sin^{2n}\theta\right)d\theta$$

$$=4a\left[\frac{\pi}{2}-\frac{e^{2}}{2}S_{1}-\sum_{n=2}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{n!}\left(\frac{e^{2}}{2}\right)^{n}S_{n}\right]$$

where $S_{n}=\int_{0}^{\pi/2}\sin^{2n}\theta\,d\theta=\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\dfrac{\pi}{2}$ by Exercise 44 of 8.1.

$$L=4a\left(\frac{\pi}{2}\right)\left[1-\frac{e^{2}}{2}\cdot\frac{1}{2}-\sum_{n=2}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{n!}\left(\frac{e^{2}}{2}\right)^{n}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\right]$$

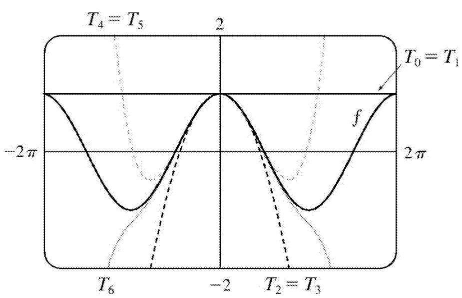
$$=2\pi a\left[1-\frac{e^{2}}{4}-\sum_{n=2}^{\infty}\frac{e^{2n}}{2^{n}}\cdot\frac{1^{2}\cdot 3^{2}\cdot 5^{2}\cdot\cdots\cdot(2n-3)^{2}(2n-1)}{n!\cdot 2^{n}\cdot n!}\right]$$

$$=2\pi a\left[1-\frac{e^{2}}{4}-\sum_{n=2}^{\infty}\frac{e^{2n}}{4^{n}}\left(\frac{1\cdot 3\cdot\cdots\cdot(2n-3)}{n!}\right)^{2}(2n-1)\right]$$

$$=2\pi a\left[1-\frac{e^{2}}{4}-\frac{3e^{4}}{64}-\frac{5e^{6}}{256}-\cdots\right]=\frac{\pi a}{128}\left(256-64e^{2}-12e^{4}-5e^{6}-\cdots\right)$$

1. **(a)**

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$T_n(x)$
0	$\cos x$	1	1
1	$-\sin x$	0	1
2	$-\cos x$	-1	$1-\frac{1}{2}x^2$
3	$\sin x$	0	$1-\frac{1}{2}x^2$
4	$\cos x$	1	$1-\frac{1}{2}x^2+\frac{1}{24}x^4$
5	$-\sin x$	0	$1-\frac{1}{2}x^2+\frac{1}{24}x^4$
6	$-\cos x$	-1	$1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6$



(b)

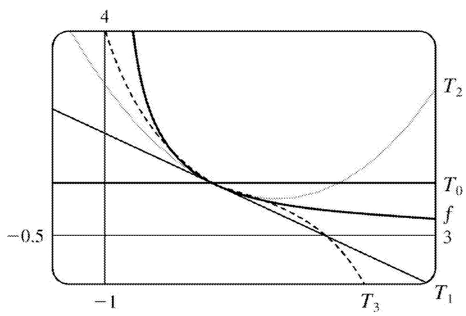
x	f	$T_0=T_1$	$T_2=T_3$	$T_4=T_5$	T_6
$\frac{\pi}{4}$	0.7071	1	0.6916	0.7074	0.7071
$\frac{\pi}{2}$	0	1	-0.2337	0.0200	-0.0009
π	-1	1	-3.9348	0.1239	-1.2114

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

2. **(a)**

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$T_n(x)$

0	x^{-1}	1	1
1	$-x^{-2}$	-1	$1-(x-1)=2-x$
2	$2x^{-3}$	2	$1-(x-1)+(x-1)^2=x^2-3x+3$
3	$-6x^{-4}$	-6	$1-(x-1)+(x-1)^2-(x-1)^3=-x^3+4x^2-6x+4$



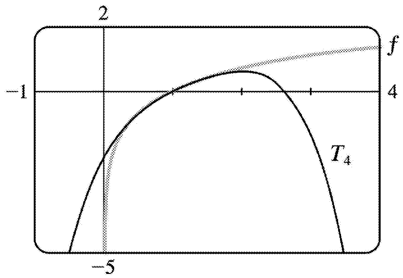
(b)

x	f	T_0	T_1	T_2	T_3
0.9	$1.\overline{1}$	1	1.1	1.11	1.111
1.3	0.7692	1	0.7	0.79	0.763

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

3.

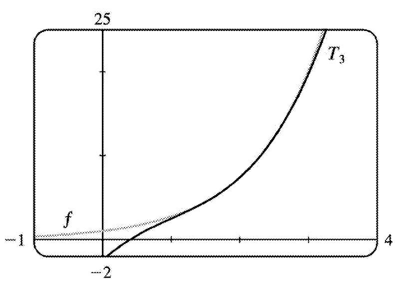
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln x$	0
1	$1/x$	1
2	$-1/x^2$	-1
3	$2/x^3$	2
4	$-6/x^4$	-6



$$T_4(x)=\sum_{n=0}^{4} \frac{f^{(n)}(1)}{n!}(x-1)^n=0+(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4$$

4.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	e^x	e^2
1	e^x	e^2
2	e^x	e^2
3	e^x	e^2

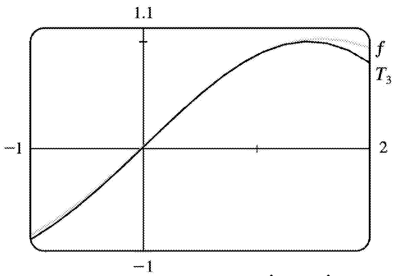


$$T_3(x)=\sum_{n=0}^{3} \frac{f^{(n)}(2)}{n!}(x-2)^n=e^2+e^2(x-2)+\frac{e^2}{2}(x-2)^2+\frac{e^2}{6}(x-2)^3$$

5.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{6}\right)$
0	$\sin x$	$\frac{1}{2}$
1	$\cos x$	$\frac{\sqrt{3}}{2}$

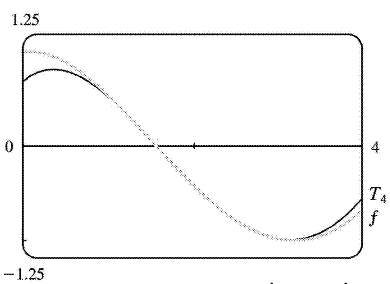
2	$-\sin x$	$-\frac{1}{2}$
3	$-\cos x$	$-\frac{\sqrt{3}}{2}$



$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}\left(\frac{\pi}{6}\right)}{n!}\left(x-\frac{\pi}{6}\right)^n=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)-\frac{1}{4}\left(x-\frac{\pi}{6}\right)^2-\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{6}\right)^3$$

6.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{2\pi}{3}\right)$
0	$\cos x$	$-\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$\frac{1}{2}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$
4	$\cos x$	$-\frac{1}{2}$

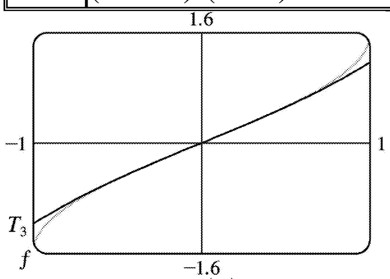


$$T_4(x)=\sum_{n=0}^{4}\frac{f^{(n)}\left(\frac{2\pi}{3}\right)}{n!}\left(x-\frac{2\pi}{3}\right)^n$$

$$=-\frac{1}{2}-\frac{\sqrt{3}}{2}\left(x-\frac{2\pi}{3}\right)+\frac{1}{4}\left(x-\frac{2\pi}{3}\right)^2+\frac{\sqrt{3}}{12}\left(x-\frac{2\pi}{3}\right)^3-\frac{1}{48}\left(x-\frac{2\pi}{3}\right)^4$$

7.

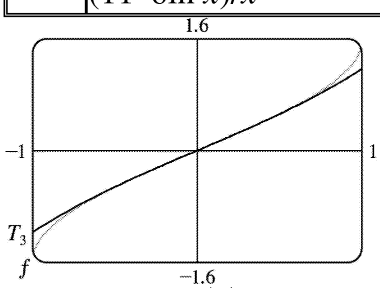
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\arcsin x$	0
1	$1/\sqrt{1-x^2}$	1
2	$x/(1-x^2)^{3/2}$	0
3	$(2x^2+1)/(1-x^2)^{5/2}$	1



$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(0)}{n!}x^n=x+\frac{x^3}{6}$$

8.

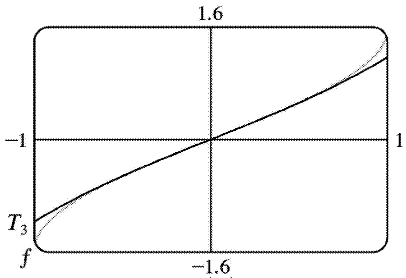
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$(\ln x)/x$	0
1	$(1-\ln x)/x^2$	1
2	$(-3+2\ln x)/x^3$	−3
3	$(11-6\ln x)/x^4$	11



$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(1)}{n!}(x-1)^n=(x-1)-\frac{3}{2}(x-1)^2+\frac{11}{6}(x-1)^3$$

9.

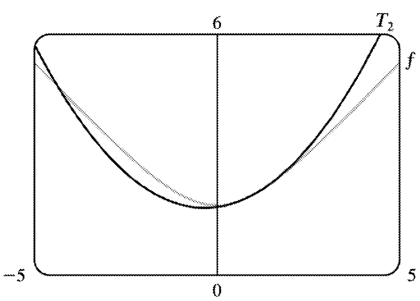
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^{-2x}	0
1	$(1-2x)e^{-2x}$	1
2	$4(x-1)e^{-2x}$	-4
3	$4(3-2x)e^{-2x}$	12



$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(0)}{n!}x^n=\frac{0}{1}\cdot 1+\frac{1}{1}x^1+\frac{-4}{2}x^2+\frac{12}{6}x^3=x-2x^2+2x^3$$

10.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\left(3+x^2\right)^{1/2}$	2
1	$x\left(3+x^2\right)^{-1/2}$	$\frac{1}{2}$
2	$3\left(3+x^2\right)^{-3/2}$	$\frac{3}{8}$

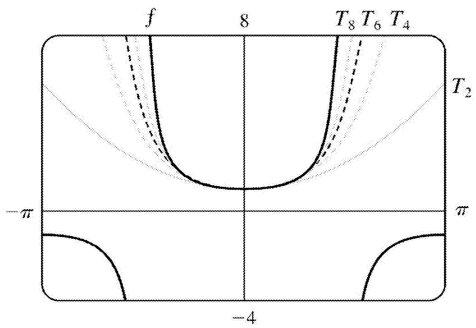


$$T_2(x)=\sum_{n=0}^{2}\frac{f^{(n)}(1)}{n!}(x-1)^n=2+\frac{1}{2}(x-1)+\frac{3/8}{2}(x-1)^2=2+\frac{1}{2}(x-1)+\frac{3}{16}(x-1)^2$$

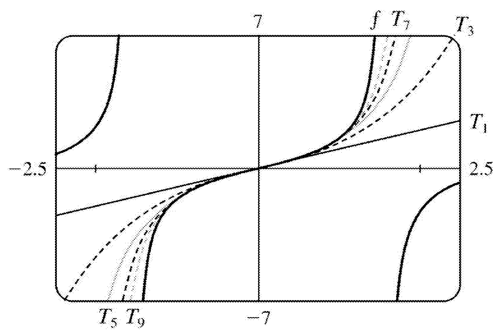
11. In Maple, we can find the Taylor polynomials by the following method: first define `f:=sec(x);` and then set

T2:=convert(taylor(f,x=0,3),polynom);, T4:=convert(taylor(f,x=0,5),polynom);, etc. (The third argument in the taylor function is one more than the degree of the desired polynomial). We must convert to the type polynom because the output of the taylor function contains an error term which we do not want. In Mathematica, we use Tn:=Normal[Series[f,{x,0,n}]] , with n=2,4, etc. Note that in Mathematica, the "degree" argument is the same as the degree of the desired polynomial. In Derive, author sec x , then enter Calculus ,Taylor,8,0; and then simplify the expression. The eighth Taylor polynomial is

$T_8(x)=1+\frac{1}{2}x^2+\frac{5}{24}x^4+\frac{61}{720}x^6+\frac{277}{8064}x^8$.



12. See Exercise 11 for the CAS commands used to generate the Taylor polynomials. The ninth Taylor polynomial for tan x is $T_9(x)=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\frac{62}{2835}x^9$.



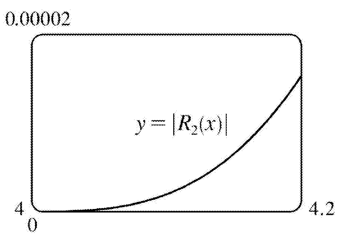
13.

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$\sqrt{x}$	2
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{4}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{32}$
3	$\frac{3}{8}x^{-5/2}$	

(a) $f(x)=\sqrt{x}\approx T_2(x)=2+\frac{1}{4}(x-4)-\frac{1/32}{2!}(x-4)^2=2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2$

(b) $\left|R_2(x)\right|\le\frac{M}{3!}|x-4|^3$, where $\left|f'''(x)\right|\le M$. Now $4\le x\le 4.2\Rightarrow|x-4|\le 0.2\Rightarrow|x-4|^3\le 0.008$.
Since $f'''(x)$ is decreasing on $[4,4.2]$, we can take $M=\left|f'''(4)\right|=\frac{3}{8}4^{-5/2}=\frac{3}{256}$, so
$\left|R_2(x)\right|\le\frac{3/256}{6}(0.008)=\frac{0.008}{512}=0.000015625$.

(c) From the graph of $\left|R_2(x)\right|=\left|\sqrt{x}-T_2(x)\right|$, it seems that the error is less than 1.52×10^{-5} on $[4,4.2]$
.



14.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-2}	1
1	$-2x^{-3}$	-2
2	$6x^{-4}$	6
3	$-24x^{-5}$	

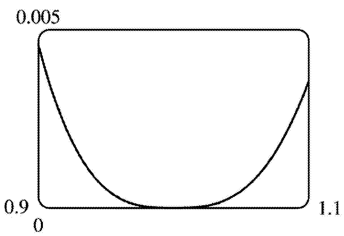
(a)

$$f(x)=x^{-2}\approx T_2(x)$$
$$=1-2(x-1)+\frac{6}{2!}(x-1)^2$$
$$=1-2(x-1)+3(x-1)^2$$

(b) $\left|R_2(x)\right|\le\frac{M}{3!}|x-1|^3$, where $\left|f'''(x)\right|\le M$. Now $0.9\le x\le 1.1\Rightarrow|x-1|\le 0.1\Rightarrow|x-1|^3\le 0.001$.
Since $f'''(x)$ is decreasing on $[0.9,1.1]$, we can take

$M=\left|f'''(0.9)\right|=\dfrac{24}{(0.9)^5}$, so

$$\left|R_2(x)\right| \le \frac{24/(0.9)^5}{6}(0.001)=\frac{0.004}{0.59049} \approx 0.00677404$$

(c) 

From the graph of $\left|R_2(x)\right|=\left|x^{-2}-T_2(x)\right|$, it seems that the error is less than 0.0046 on $[0.9,1.1]$.

15.

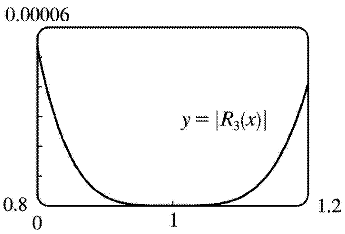
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{2/3}$	1
1	$\frac{2}{3}x^{-1/3}$	$\frac{2}{3}$
2	$-\frac{2}{9}x^{-4/3}$	$-\frac{2}{9}$
3	$\frac{8}{27}x^{-7/3}$	$\frac{8}{27}$
4	$-\frac{56}{81}x^{-10/3}$	

(a) $f(x)=x^{2/3}\approx T_3(x)=1+\frac{2}{3}(x-1)-\frac{2/9}{2!}(x-1)^2+\frac{8/27}{3!}(x-1)^3=1+\frac{2}{3}(x-1)-\frac{1}{9}(x-1)^2+\frac{4}{81}(x-1)^3$

(b) $\left|R_3(x)\right|\le\frac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right|\le M$. Now $0.8\le x\le 1.2\Rightarrow |x-1|\le 0.2\Rightarrow |x-1|^4\le 0.0016$.
Since $\left|f^{(4)}(x)\right|$ is decreasing on $[0.8,1.2]$, we can take $M=\left|f^{(4)}(0.8)\right|=\frac{56}{81}(0.8)^{-10/3}$, so

$$\left|R_3(x)\right|\le\frac{\frac{56}{81}(0.8)^{-10/3}}{24}(0.0016)\approx 0.00009697 .$$

(c) From the graph of $\left|R_3(x)\right| = \left|x^{2/3} - T_3(x)\right|$, it seems that the error is less than 0.0000533 on $[0.8, 1.2]$.



16.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{3}\right)$
0	$\cos x$	$\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$-\frac{1}{2}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$
4	$\cos x$	$\frac{1}{2}$
5	$-\sin x$	

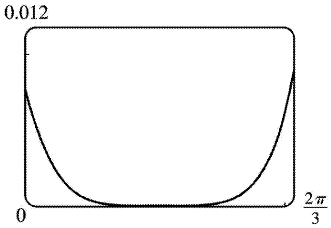
(a)

$f(x) = \cos x \approx T_4(x)$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{48}\left(x - \frac{\pi}{3}\right)^4$$

(b) $\left|R_4(x)\right| \le \frac{M}{5!}\left|x - \frac{\pi}{3}\right|^5$, where $\left|f^{(5)}(x)\right| \le M$. Now $0 \le x \le \frac{2\pi}{3} \Rightarrow \left(x - \frac{\pi}{3}\right)^5 \le \left(\frac{\pi}{3}\right)^5$, and letting $x = \frac{\pi}{2}$ gives $M = 1$, so $\left|R_4(x)\right| \le \frac{1}{5!}\left(\frac{\pi}{3}\right)^5 \approx 0.0105$.

(c)



From the graph of $\left|R_4(x)\right|=\left|\cos x - T_4(x)\right|$, it seems that the error is less than 0.01 on $\left[0, \frac{2\pi}{3}\right]$.

17.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tan x$	0
1	$\sec^2 x$	1
2	$2\sec^2 x \tan x$	0
3	$4\sec^2 x \tan^2 x + 2\sec^4 x$	2
4	$8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$	

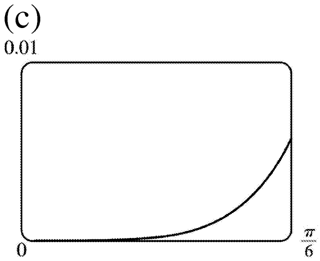
(a) $f(x)=\tan x \approx T_3(x)=x+\frac{1}{3}x^3$

(b) $\left|R_3(x)\right| \le \frac{M}{4!}\left|x\right|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now $0 \le x \le \frac{\pi}{6} \Rightarrow x^4 \le \left(\frac{\pi}{6}\right)^4$, and letting $x=\frac{\pi}{6}$
$\left[\text{since} f^{(4)} \text{is increasing on} \left(0, \frac{\pi}{6}\right)\right]$ gives

$$\left|R_3(x)\right| \le \frac{8\left(\frac{2}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right)^3+16\left(\frac{2}{\sqrt{3}}\right)^4\left(\frac{1}{\sqrt{3}}\right)}{4!}\left(\frac{\pi}{6}\right)^4$$

$$= \frac{4\sqrt{3}}{9}\left(\frac{\pi}{6}\right)^4 \approx 0.057859$$

(c)



From the graph of

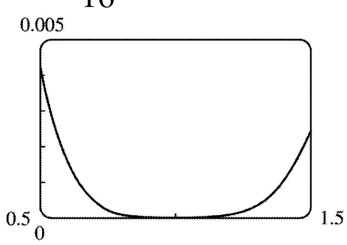
$\left|R_3(x)\right| = \left|\tan x - T_3(3)\right|$, it seems that the error is less than 0.006 on $[0, \pi/6]$.

18.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln(1+2x)$	$\ln 3$
1	$2/(1+2x)$	$\frac{2}{3}$
2	$-4/(1+2x)^2$	$-\frac{4}{9}$
3	$16/(1+2x)^3$	$\frac{16}{27}$
4	$-96/(1+2x)^4$	

(a) $f(x) = \ln(1+2x) \approx T_3(x) = \ln 3 + \frac{2}{3}(x-1) - \frac{4/9}{2!}(x-1)^2 + \frac{16/27}{3!}(x-1)^3$

(b) $\left|R_3(x)\right| \le \frac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now $0.5 \le x \le 1.5 \Rightarrow -0.5 \le x-1 \le 0.5 \Rightarrow |x-1| \le 0.5 \Rightarrow$ $|x-1|^4 \le \frac{1}{16}$, and letting $x=0.5$ gives $M=6$, so $\left|R_3(x)\right| \le \frac{6}{4!} \cdot \frac{1}{16} = \frac{1}{64} = 0.015625$.



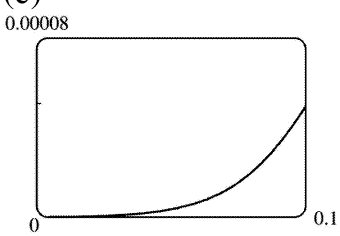
(c)
From the graph of $\left|R_3(x)\right| = \left|\ln(1+2x) - T_3(x)\right|$, it seems that the error is less than 0.005 on $[0.5, 1.5]$.

19.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{x^2}	1
1	$e^{x^2}(2x)$	0
2	$e^{x^2}(2+4x^2)$	2
3	$e^{x^2}(12x+8x^3)$	0
4	$e^{x^2}(12+48x^2+16x^4)$	

(a) $f(x)=e^{x^2}\approx T_3(x)=1+\frac{2}{2!}x^2=1+x^2$

(b) $\left|R_3(x)\right|\le\frac{M}{4!}|x|^4$, where $\left|f^{(4)}(x)\right|\le M$.
Now $0\le x\le 0.1\Rightarrow x^4\le(0.1)^4$, and
letting $x=0.1$ gives
$\left|R_3(x)\right|\le\frac{e^{0.01}(12+0.48+0.0016)}{24}(0.1)^4\approx 0.00006$.

(c)


From the graph of $\left|R_3(x)\right|=\left|e^{x^2}-\left(1+x^2\right)\right|$, it appears that the error is less than 0.000051 on $[0,0.1]$.

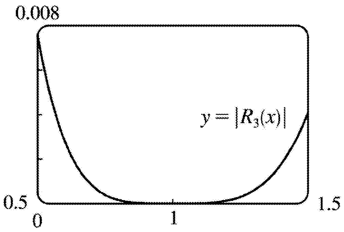
20.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x\ln x$	0
1	$\ln x+1$	1
2	$1/x$	1
3	$-1/x^2$	-1
4	$2/x^3$	

(a) $f(x)=x\ln x\approx T_3(x)=(x-1)+\frac{1}{2}(x-1)^2-\frac{1}{6}(x-1)^3$

(b) $\left|R_3(x)\right|\le\frac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right|\le M$. Now $0.5\le x\le 1.5\Rightarrow|x-1|\le\frac{1}{2}\Rightarrow|x-1|^4\le\frac{1}{16}$.
Since $\left|f^{(4)}(x)\right|$ is decreasing on $[0.5,1.5]$, we can take $M=\left|f^{(4)}(0.5)\right|=2/(0.5)^3=16$, so
$\left|R_3(x)\right|\le\frac{16}{24}(1/16)=\frac{1}{24}=0.041\overline{6}$.

(c) From the graph of $\left|R_3(x)\right|=\left|x\ln x-T_3(x)\right|$, it seems that the error is less than 0.0076 on $[0.5,1.5]$
.



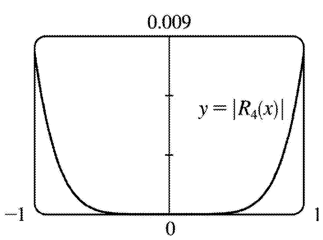
21.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$x\sin x$	0
1	$\sin x+x\cos x$	0
2	$2\cos x-x\sin x$	2
3	$-3\sin x-x\cos x$	0
4	$-4\cos x+x\sin x$	-4
5	$5\sin x+x\cos x$	

(a) $f(x)=x\sin x\approx T_4(x)=\frac{2}{2!}(x-0)^2+\frac{-4}{4!}(x-0)^4=x^2-\frac{1}{6}x^4$

(b) $\left|R_4(x)\right|\leq\frac{M}{5!}|x|^5$, where $\left|f^{(5)}(x)\right|\leq M$. Now $-1\leq x\leq 1\Rightarrow|x|\leq 1$, and a graph of $f^{(5)}(x)$ shows that $\left|f^{(5)}(x)\right|\leq 5$ for $-1\leq x\leq 1$. Thus, we can take $M=5$ and get $\left|R_4(x)\right|\leq\frac{5}{5!}\cdot 1^5=\frac{1}{24}=0.041\overline{6}$.

(c) From the graph of $\left|R_4(x)\right|=\left|x\sin x-T_4(x)\right|$, it seems that the error is less than 0.0082 on $[-1,1]$.



22.

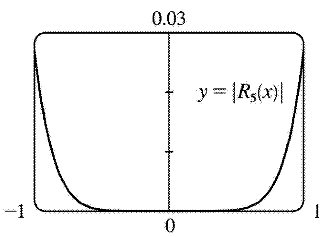
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sinh 2x$	0
1	$2\cosh 2x$	2
2	$4\sinh 2x$	0
3	$8\cosh 2x$	8
4	$16\sinh 2x$	0
5	$32\cosh 2x$	32

6	$64\sinh 2x$	

(a) $f(x)=\sinh 2x \approx T_5(x)=2x+\frac{8}{3!}x^3+\frac{32}{5!}x^5=2x+\frac{4}{3}x^3+\frac{4}{15}x^5$

(b) $\left|R_5(x)\right| \le \frac{M}{6!}|x|^6$, where $\left|f^{(6)}(x)\right| \le M$. For x in $[-1,1]$, we have $|x| \le 1$. Since $f^{(6)}(x)$ is an increasing odd function on $[-1,1]$, we see that $\left|f^{(6)}(x)\right| \le f^{(6)}(1)=64\sinh 2=32(e^2-e^{-2}) \approx 232.119$, so we can take $M=232.12$ and get $\left|R_5(x)\right| \le \frac{232.12}{720}\cdot 1^6 \approx 0.3224$.

(c) From the graph of $\left|R_5(x)\right|=\left|\sinh 2x-T_5(x)\right|$, it seems that the error is less than 0.027 on $[-1,1]$.



23. From Exercise 5, $\sin x=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)-\frac{1}{4}\left(x-\frac{\pi}{6}\right)^2-\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{6}\right)^3+R_3(x)$, where $\left|R_3(x)\right| \le \frac{M}{4!}\left|x-\frac{\pi}{6}\right|^4$ with $\left|f^{(4)}(x)\right|=|\sin x| \le M=1$. Now $x=35^\circ=(30^\circ+5^\circ)=\left(\frac{\pi}{6}+\frac{\pi}{36}\right)$ radians, so the error is

$\left|R_3\left(\frac{\pi}{36}\right)\right| \le \frac{\left(\frac{\pi}{36}\right)^4}{4!}<0.000003$. Therefore, to five decimal places,

$\sin 35^\circ \approx \frac{1}{2}+\frac{\sqrt{3}}{2}\left(\frac{\pi}{36}\right)-\frac{1}{4}\left(\frac{\pi}{36}\right)^2-\frac{\sqrt{3}}{12}\left(\frac{\pi}{36}\right)^3 \approx 0.57358$.

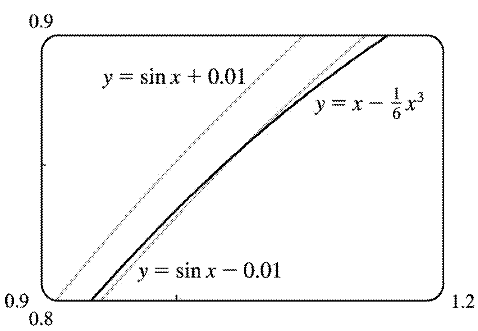
24. From Exercise 16,

$\cos x=\frac{1}{2}-\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)-\frac{1}{4}\left(x-\frac{\pi}{3}\right)^2+\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{3}\right)^3+\frac{1}{48}\left(x-\frac{\pi}{3}\right)^4+R_4(x)$. Now since $x=69^\circ=(60^\circ+9^\circ)=\left(\frac{\pi}{3}+\frac{\pi}{20}\right)$ radians, the error is $\left|R_4(x)\right| \le \frac{\left(\frac{\pi}{20}\right)^5}{5!}<8\times 10^{-7}$. Therefore, to five decimal places, $\cos 69^\circ \approx \frac{1}{2}-\frac{\sqrt{3}}{2}\left(\frac{\pi}{20}\right)-\frac{1}{4}\left(\frac{\pi}{20}\right)^2+\frac{\sqrt{3}}{12}\left(\frac{\pi}{20}\right)^3+\frac{1}{48}\left(\frac{\pi}{20}\right)^4 \approx 0.35837$.

25. All derivatives of e^x are e^x , so $\left|R_n(x)\right| \le \frac{e^x}{(n+1)!}\,|x|^{n+1}$, where $0<x<0.1$. Letting $x=0.1$, $R_n(0.1) \le \frac{e^{0.1}}{(n+1)!}(0.1)^{n+1} < 0.00001$, and by trial and error we find that $n=3$ satisfies this inequality since $R_3(0.1)<0.0000046$. Thus, by adding the four terms of the Maclaurin series for e^x corresponding to $n=0$, 1 , 2 , and 3 , we can estimate $e^{0.1}$ to within 0.00001 . (In fact, this sum is $1.1051\overline{6}$ and $e^{0.1} \approx 1.10517$.)

26. Example 6 in Section .9 gives the Maclaurin series for $\ln(1-x)$ as $-\sum_{n=1}^{\infty} \frac{x^n}{n}$ for $|x|<1$. Thus, $\ln 1.4 = \ln[1-(-0.4)] = -\sum_{n=1}^{\infty} \frac{(-0.4)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.4)^n}{n}$. Since this is an alternating series, the error is less than the first neglected term by the Alternating Series Estimation Theorem, and we find that $\left|a_6\right| = (0.4)^6/6 \approx 0.0007 < 0.001$. So we need the first five (non–zero) terms of the Maclaurin series for the desired accuracy. (In fact, this sum is approximately 0.33698 and $\ln 1.4 \approx 0.33647$.)

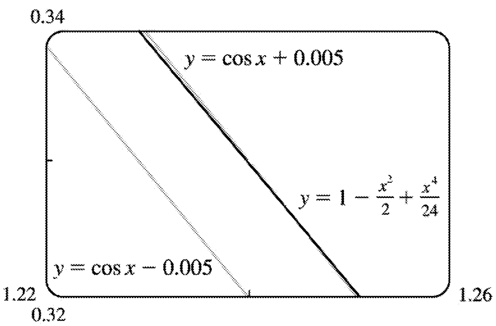
27. $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$. By the Alternating Series Estimation Theorem, the error in the approximation $\sin x = x - \frac{1}{3!}x^3$ is less than $\left|\frac{1}{5!}x^5\right| < 0.01 \Leftrightarrow \left|x^5\right| < 120(0.01) \Leftrightarrow |x| < (1.2)^{1/5} \approx 1.037$. The curves $y = x - \frac{1}{6}x^3$ and $y = \sin x - 0.01$ intersect at $x \approx 1.043$, so the graph confirms our estimate. Since both the sine function and



the given approximation are odd functions, we need to check the estimate only for $x>0$. Thus, the desired range of values for x is $-1.037<x<1.037$.

28.

$\cos x=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots$. By the Alternating Series Estimation Theorem, the error is less than $\left|-\frac{1}{6!}x^6\right|<0.005\Leftrightarrow x^6<720(0.005)\Leftrightarrow |x|<(3.6)^{1/6}\approx 1.238$. The curves $y=1-\frac{1}{2}x^2+\frac{1}{24}x^4$ and $y=\cos x+0.005$ intersect at $x\approx 1.244$, so the graph confirms our estimate. Since both the cosine function and the given approximation



are even functions, we need to check the estimate only for $x>0$. Thus, the desired range of values for x is $-1.238<x<1.238$.

29. Let $s(t)$ be the position function of the car, and for convenience set $s(0)=0$. The velocity of the car is
$v(t)=s'(t)$ and the acceleration is $a(t)=s''(t)$, so the second degree Taylor polynomial is $T_2(t)=s(0)+v(0)t+\frac{a(0)}{2}t^2=20t+t^2$. We estimate the distance travelled during the next second to be $s(1)\approx T_2(1)=20+1=21$ m. The function $T_2(t)$ would not be accurate over a full minute, since the car could not possibly maintain an acceleration of 2 m / s^2 for that long (if it did, its final speed would be 140 m / s $\approx$313 mi / h!)

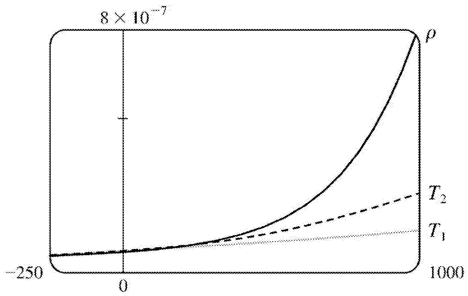
30. **(a)**

n	$\rho^{(n)}(t)$	$\rho^{(n)}(20)$
0	$\rho_{20}e^{\alpha(t-20)}$	ρ_{20}
1	$\alpha\rho_{20}e^{\alpha(t-20)}$	$\alpha\rho_{20}$
2	$\alpha^2\rho_{20}e^{\alpha(t-20)}$	$\alpha^2\rho_{20}$

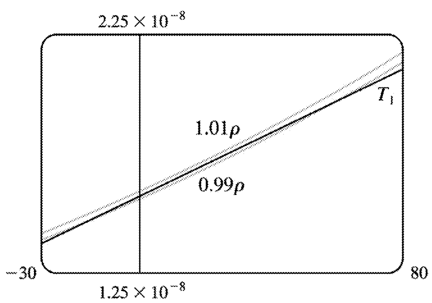
The linear approximation is $T_1(t)=\rho(20)+\rho'(20)(t-20)=\rho_{20}[1+\alpha(t-20)]$. The quadratic approximation is

$$T_2(t)=\rho(20)+\rho'(20)(t-20)+\frac{\rho''(20)}{2}(t-20)^2=\rho_{20}\left[1+\alpha(t-20)+\frac{1}{2}\alpha^2(t-20)^2\right]$$

(b)



(c)



From the graph, it seems that $T_1(t)$ is within 1% of $\rho(t)$, that is, $0.99\rho(t)\le T_1(t)\le 1.01\rho(t)$, for -14° C $\le t\le 58^\circ$ C.

31. $E=\frac{q}{D^2}-\frac{q}{(D+d)^2}=\frac{q}{D^2}-\frac{q}{D^2(1+d/D)^2}=\frac{q}{D^2}\left[1-\left(1+\frac{d}{D}\right)^{-2}\right]$.

We use the Binomial Series to expand $(1+d/D)^{-2}$:

$$E=\frac{q}{D^2}\left[1-\left(1-2\left(\frac{d}{D}\right)+\frac{2\cdot 3}{2!}\left(\frac{d}{D}\right)^2-\frac{2\cdot 3\cdot 4}{3!}\left(\frac{d}{D}\right)^3+\cdots\right)\right]$$

$$=\frac{q}{D^2}\left[2\left(\frac{d}{D}\right)-3\left(\frac{d}{D}\right)^2+4\left(\frac{d}{D}\right)^3-\cdots\right]\approx\frac{q}{D^2}\cdot 2\left(\frac{d}{D}\right)=2qd\cdot\frac{1}{D^3}$$

when D is much larger than d; that is, when P is far away from the dipole.

32. **(a)** $\frac{n_1}{\ell_o}+\frac{n_2}{\ell_i}=\frac{1}{R}\left(\frac{n_2 s_i}{\ell_i}-\frac{n_1 s_o}{\ell_o}\right)$ (Equation 1) where

$$\ell_o=\sqrt{R^2+\left(s_o+R\right)^2-2R\left(s_o+R\right)\cos\phi}\ \text{ and } \ell_i=\sqrt{R^2+\left(s_i-R\right)^2+2R\left(s_i-R\right)\cos\phi}\quad (2)$$

Using $\cos\phi\approx 1$ gives

$$\ell_o=\sqrt{R^2+\left(s_o+R\right)^2-2R\left(s_o+R\right)}=\sqrt{R^2+s_o^2+2Rs_o+R^2-2Rs_o-2R^2}=\sqrt{s_o^2}=s_o$$

and similarly, $\ell_i=s_i$. Thus, Equation 1 becomes

$$\frac{n_1}{s_o}+\frac{n_2}{s_i}=\frac{1}{R}\left(\frac{n_2 s_i}{s_i}-\frac{n_1 s_o}{s_o}\right)\Rightarrow\frac{n_1}{s_o}+\frac{n_2}{s_i}=\frac{n_2-n_1}{R}$$

(b) Using $\cos\phi\approx 1-\frac{1}{2}\phi^2$ in (2) gives us

$$\begin{aligned}\ell_o &= \sqrt{R^2+\left(s_o+R\right)^2-2R\left(s_o+R\right)\left(1-\frac{1}{2}\phi^2\right)}\\ &= \sqrt{R^2+s_o^2+2Rs_o+R^2-2Rs_o+Rs_o\phi^2-2R^2+R^2\phi^2}=\sqrt{s_o^2+Rs_o\phi^2+R^2\phi^2}\end{aligned}$$

Anticipating that we will use the binomial series expansion $(1+x)^k\approx 1+kx$, we can write the last expression for ℓ_o as $s_o\sqrt{1+\phi^2\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)}$ and similarly, $\ell_i=s_i\sqrt{1-\phi^2\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)}$. Thus, from Equation 1, $\frac{n_1}{\ell_o}+\frac{n_2}{\ell_i}=\frac{1}{R}\left(\frac{n_2 s_i}{\ell_i}-\frac{n_1 s_o}{\ell_o}\right)\Leftrightarrow n_1\ell_o^{-1}+n_2\ell_i^{-1}=\frac{n_2}{R}\cdot\frac{s_i}{\ell_i}-\frac{n_1}{R}\cdot\frac{s_o}{\ell_o}\Leftrightarrow$

$$\frac{n_1}{s_o}\left[1+\phi^2\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\right]^{-1/2}+\frac{n_2}{s_i}\left[1-\phi^2\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\right]^{-1/2}$$
$$=\frac{n_2}{R}\left[1-\phi^2\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\right]^{-1/2}-\frac{n_1}{R}\left[1+\phi^2\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\right]^{-1/2}$$

Approximating the expressions for ℓ_o^{-1} and ℓ_i^{-1} by the first two terms in their binomial series, we get

$$\frac{n_1}{s_o}\left[1-\frac{1}{2}\phi^2\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\right]+\frac{n_2}{s_i}\left[1+\frac{1}{2}\phi^2\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\right]$$
$$=\frac{n_2}{R}\left[1+\frac{1}{2}\phi^2\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\right]-\frac{n_1}{R}\left[1-\frac{1}{2}\phi^2\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\right]\Leftrightarrow$$
$$\frac{n_1}{s_o}-\frac{n_1\phi^2}{2s_o}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)+\frac{n_2}{s_i}+\frac{n_2\phi^2}{2s_i}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)$$
$$=\frac{n_2}{R}+\frac{n_2\phi^2}{2R}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)-\frac{n_1}{R}+\frac{n_1\phi^2}{2R}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\Leftrightarrow$$
$$\frac{n_1}{s_o}+\frac{n_2}{s_i}$$
$$=\frac{n_2}{R}-\frac{n_1}{R}+\frac{n_1\phi^2}{2s_o}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)+\frac{n_1\phi^2}{2R}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)+\frac{n_2\phi^2}{2R}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)-\frac{n_2\phi^2}{2s_i}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)$$
$$=\frac{n_2-n_1}{R}+\frac{n_1\phi^2}{2}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\left(\frac{1}{s_o}+\frac{1}{R}\right)+\frac{n_2\phi^2}{2}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\left(\frac{1}{R}-\frac{1}{s_i}\right)$$
$$=\frac{n_2-n_1}{R}+\frac{n_1\phi^2R^2}{2s_o}\left(\frac{1}{R}+\frac{1}{s_o}\right)\left(\frac{1}{R}+\frac{1}{s_o}\right)+\frac{n_2\phi^2R^2}{2s_i}\left(\frac{1}{R}-\frac{1}{s_i}\right)\left(\frac{1}{R}-\frac{1}{s_i}\right)$$
$$=\frac{n_2-n_1}{R}+\phi^2R^2\left[\frac{n_1}{2s_o}\left(\frac{1}{R}+\frac{1}{s_o}\right)^2+\frac{n_2}{2s_i}\left(\frac{1}{R}-\frac{1}{s_i}\right)^2\right]$$

From Figure 8, we see that $\sin\phi=h/R$. So if we approximate $\sin\phi$ with ϕ , we get $h=R\phi$ and $h^2=\phi^2R^2$ and hence, Equation 4, as desired.

33. **(a)** If the water is deep, then $2\pi d/L$ is large, and we know that $\tanh x\to 1$ as $x\to\infty$. So we can approximate $\tanh(2\pi d/L)\approx 1$, and so $v^2\approx gL/(2\pi)\Leftrightarrow v\approx\sqrt{gL/(2\pi)}$.
(b) From the table, the first term in the Maclaurin series of $\tanh x$ is x , so if the water is shallow, we can approximate

$\tanh \frac{2\pi d}{L} \approx \frac{2\pi d}{L}$, and so $v^2 \approx \frac{gL}{2\pi} \cdot \frac{2\pi d}{L} \Leftrightarrow v \approx \sqrt{gd}$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tanh x$	0
1	$\operatorname{sech}^2 x$	1
2	$-2\operatorname{sech}^2 x \tanh x$	0
3	$2\operatorname{sech}^2 x(3\tanh^2 x - 1)$	-2

(c) Since tanh x is an odd function, its Maclaurin series is alternating, so the error in the approximation $\tanh \frac{2\pi d}{L} \approx \frac{2\pi d}{L}$ is less than the first neglected term, which is $\frac{\left|f'''(0)\right|}{3!}\left(\frac{2\pi d}{L}\right)^3 = \frac{1}{3}\left(\frac{2\pi d}{L}\right)^3$.

If $L > 10d$, then $\frac{1}{3}\left(\frac{2\pi d}{L}\right)^3 < \frac{1}{3}\left(2\pi \cdot \frac{1}{10}\right)^3 = \frac{\pi^3}{375}$, so the error in the approximation $v^2 = gd$ is less than $\frac{gL}{2\pi} \cdot \frac{\pi^3}{375} \approx 0.0132gL$.

34. **(a)** $4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}} = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1+\left(-k^2\sin^2 x\right)\right]^{-1/2}dx$

$$=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1-\frac{1}{2}\left(-k^2\sin^2 x\right)+\frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\left(-k^2\sin^2 x\right)^2-\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}\left(-k^2\sin^2 x\right)^3+\cdots\right]dx$$

$$=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1+\left(\frac{1}{2}\right)k^2\sin^2 x+\left(\frac{1\cdot 3}{2\cdot 4}\right)k^4\sin^4 x+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)k^6\sin^6 x+\cdots\right]dx$$

$$=4\sqrt{\frac{L}{g}}\left[\frac{\pi}{2}+\left(\frac{1}{2}\right)\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)\left(\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2}\right)k^4\right.$$
$$\left.+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2}\right)k^6+\cdots\right]$$

[split up the integral and use the result from Exercise 8.1.44]

$$=2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1^2}{2^2}k^2+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}k^4+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}k^6+\cdots\right]$$

(b) The first of the two inequalities is true because all of the terms in the series are positive. For the second,

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1^2}{2^2}k^2+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}k^4+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}k^6+\frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2}{2^2\cdot 4^2\cdot 6^2\cdot 8^2}k^8+\cdots\right]$$

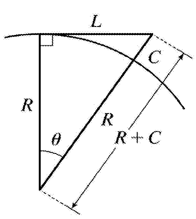
$$\leq 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1}{4}k^2+\frac{1}{4}k^4+\frac{1}{4}k^6+\frac{1}{4}k^8+\cdots\right]$$

The terms in brackets (after the first) form a geometric series with $a=\frac{1}{4}k^2$ and $r=k^2=\sin^2\left(\frac{1}{2}\theta_0\right)<1$. So $T\leq 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{k^2/4}{1-k^2}\right]=2\pi\sqrt{\frac{L}{g}}\,\frac{4-3k^2}{4-4k^2}$.

(c) We substitute $L=1$, $g=9.8$, and $k=\sin\left(10^\circ/2\right)\approx 0.08716$, and the inequality from part (b) becomes $2.01090\leq T\leq 2.01093$, so $T\approx 2.0109$. The estimate $T\approx 2\pi\sqrt{L/g}\approx 2.0071$ differs by about 0.2% .

If $\theta_0=42^\circ$, then $k\approx 0.35837$ and the inequality becomes $2.07153\leq T\leq 2.08103$, so $T\approx 2.0763$. The one–term estimate is the same, and the discrepancy between the two estimates increases to about 3.4% .

35. **(a)** L is the length of the arc subtended by the angle θ , so $L=R\theta\Rightarrow\theta=L/R$. Now $\sec\theta=(R+C)/R$ $\Rightarrow R\sec\theta=R+C\Rightarrow C=R\sec\theta-R=R\sec(L/R)-R$.



(b) From Exercise 11, $\sec x\approx T_4(x)=1+\frac{1}{2}x^2+\frac{5}{24}x^4$. By part (a),

$$C\approx R\left[1+\frac{1}{2}\left(\frac{L}{R}\right)^2+\frac{5}{24}\left(\frac{L}{R}\right)^4\right]-R=R+\frac{1}{2}R\cdot\frac{L^2}{R^2}+\frac{5}{24}R\cdot\frac{L^4}{R^4}-R=\frac{L^2}{2R}+\frac{5L^4}{24R^3}\ .$$

(c) Taking $L=100$ km and $R=6370$ km, the formula in part (a) says that $C=R\sec(L/R)-R=6370\sec(100/6370)-6370\approx 0.785\,009\,965\,44$ km

The formula in part (b) says that $C\approx\frac{L^2}{2R}+\frac{5L^4}{24R^3}=\frac{100^2}{2\cdot 6370}+\frac{5\cdot 100^4}{24\cdot 6370^3}\approx 0.785\,009\,957\,36$ km.

The difference between these two results is only 0.00000000808 km, or 0.00000808 m!

36. $T_n(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$. Let $0\le m\le n$. Then

$$T_n^{(m)}(x)=m!\,\frac{f^{(m)}(a)}{m!}(x-a)^0+(m+1)(m)\cdots(2)\,\frac{f^{(m+1)}(a)}{(m+1)!}(x-a)^1+\cdots$$
$$+n(n-1)\cdots(n-m+1)\,\frac{f^{(n)}(a)}{n!}(x-a)^{n-m}$$

For $x=a$, all terms in this sum except the first one are 0 , so $T_n^{(m)}(a)=\frac{m!\,f^{(m)}(a)}{m!}=f^{(m)}(a)$.

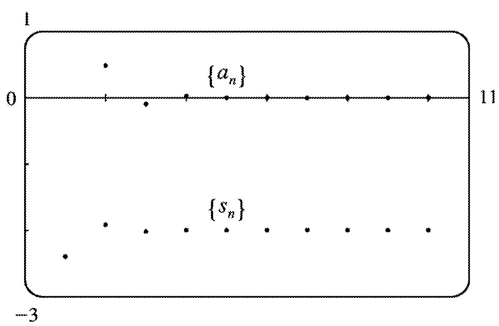
37. Using $f(x)=T_n(x)+R_n(x)$ with $n=1$ and $x=r$, we have $f(r)=T_1(r)+R_1(r)$, where T_1 is the first–degree Taylor polynomial of f at a . Because $a=x_n$, $f(r)=f\left(x_n\right)+f'\left(x_n\right)\left(r-x_n\right)+R_1(r)$. But r is a root of f , so $f(r)=0$ and we have $0=f\left(x_n\right)+f'\left(x_n\right)\left(r-x_n\right)+R_1(r)$. Taking the first two terms to the left side gives us $f'\left(x_n\right)\left(x_n-r\right)-f\left(x_n\right)=R_1(r)$. Dividing by $f'\left(x_n\right)$, we get $x_n-r-\frac{f\left(x_n\right)}{f'\left(x_n\right)}=\frac{R_1(r)}{f'\left(x_n\right)}$. By the formula for Newton's method, the left side of the preceding equation is $x_{n+1}-r$, so $\left|x_{n+1}-r\right|=\left|\frac{R_1(r)}{f'\left(x_n\right)}\right|$. Taylor's Inequality gives us $\left|R_1(r)\right|\le\frac{\left|f''(r)\right|}{2!}\left|r-x_n\right|^2$. Combining this inequality with the facts $\left|f''(x)\right|\le M$ and $\left|f'(x)\right|\ge K$ gives us $\left|x_{n+1}-r\right|\le\frac{M}{2K}\left|x_n-r\right|^2$.

1. **(a)** A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.
(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

2. $\Sigma_{n=1}^{\infty} a_n = 5$ means that by adding sufficiently many terms of the series we can get as close as we like to the number 5 .
In other words, it means that ${}_{n\to\infty} s_n = 5$, where s_n is the n th partial sum, that is, $\Sigma_{i=1}^{n} a_i$.

3.

n	s_n
1	–2.40000
2	–1.92000
3	–2.01600
4	–1.99680
5	–2.00064
6	–1.99987
7	–2.00003
8	–1.99999
9	–2.00000
10	–2.00000

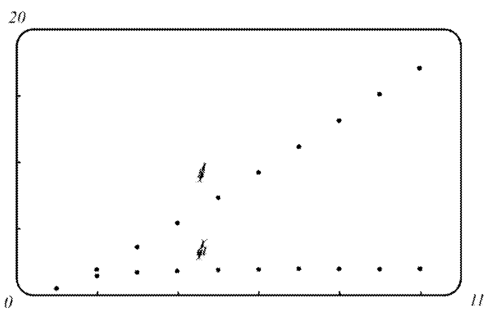


From the graph and the table, it seems that the series converges to –2 . In fact, it is a geometric series with $a = -2.4$ and $r = -\frac{1}{5}$, so its sum is $\Sigma_{n=1}^{\infty} \frac{12}{(-5)^n} = \frac{-2.4}{1-\left(-\frac{1}{5}\right)} = \frac{-2.4}{1.2} = -2$. Note that the dot corresponding to $n=1$ is part of both $\left\{ a_n \right\}$ and $\left\{ s_n \right\}$.
TI–86 Note: To graph $\left\{ a_n \right\}$ and $\left\{ s_n \right\}$, set your calculator to Param mode and DrawDot mode. (DrawDot is under GRAPH, MORE, FORMT (F3).) Now under E(t)= make the assignments: xt1=t, yt1=12/(–5)^t, xt2=t, yt2=sum seq(yt1,t,1,t,1). (sum and seq are under

LIST, OPS (F5), MORE.) Under WIND use 1,10,1,0,10,1,–3,1,1 to obtain a graph similar to the one above. Then use TRACE\;(F4) to see the values.

4.

n	s_n
1	0.50000
2	1.90000
3	3.60000
4	5.42353
5	7.30814
6	9.22706
7	11.16706
8	13.12091
9	15.08432
10	17.05462

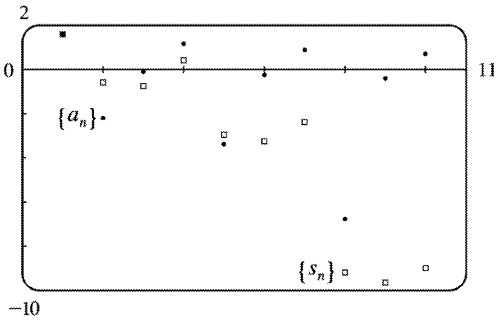


The series $\sum_{n=1}^{\infty} \frac{2n^2-1}{n^2+1}$ diverges, since its terms do not approach 0.

5.

n	s_n
1	1.55741
2	–0.62763
3	–0.77018
4	0.38764
5	–2.99287
6	–3.28388
7	–2.41243
8	–9.21214
9	–9.66446

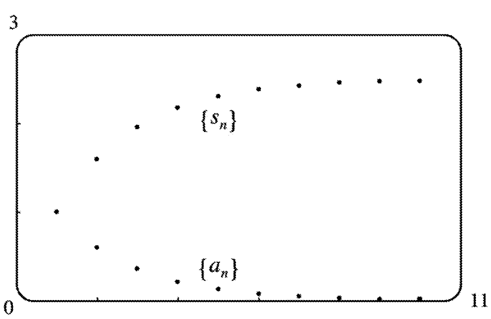
10	–9.01610



The series $\sum_{n=1}^{\infty} \tan n$ diverges, since its terms do not approach 0.

6.

n	s_n
1	1.00000
2	1.60000
3	1.96000
4	2.17600
5	2.30560
6	2.38336
7	2.43002
8	2.45801
9	2.47481
10	2.48488

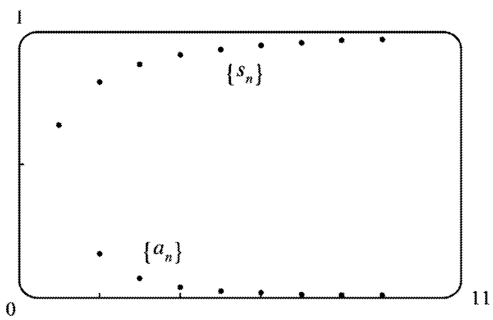


From the graph and the table, it seems that the series converges to 2.5.

In fact, it is a geometric series with $a=1$ and $r=0.6$, so its sum is $\sum_{n=1}^{\infty}(0.6)^{n-1}=\frac{1}{1-0.6}=\frac{1}{2/5}=2.5$.

7.

n	s_n
1	0.64645
2	0.80755
3	0.87500
4	0.91056
5	0.93196
6	0.94601
7	0.95581
8	0.96296
9	0.96838
10	0.97259



From the graph, it seems that the series converges to 1 . To find the sum, we write

$$s_n = \sum_{i=1}^{n} \left(\frac{1}{i^{1.5}} - \frac{1}{(i+1)^{1.5}} \right)$$

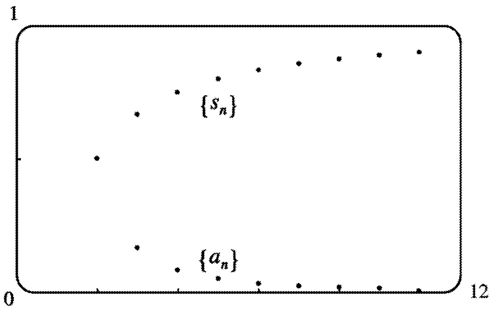
$$= \left(1 - \frac{1}{2^{1.5}}\right) + \left(\frac{1}{2^{1.5}} - \frac{1}{3^{1.5}}\right) + \left(\frac{1}{3^{1.5}} - \frac{1}{4^{1.5}}\right) + \cdots + \left(\frac{1}{n^{1.5}} - \frac{1}{(n+1)^{1.5}}\right) = 1 - \frac{1}{(n+1)^{1.5}}$$

So the sum is $\lim_{n\to\infty} s_n = 1 - 0 = 1$.

8.

n	s_n
2	0.50000
3	0.66667
4	0.75000
5	0.80000
6	0.83333

7	0.85714
8	0.87500
9	0.88889
10	0.90000
11	0.90909
100	0.99000



From the graph and the table, it seems that the series converges to 1 . To find the sum, we write

$$s_n = \sum_{i=2}^{n} \frac{1}{i(i-1)} = \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{1}{i} \right) \quad \text{[partial fractions]}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 - \frac{1}{n},$$

and so the sum is $\lim_{n\to\infty} s_n = 1 - 0 = 1$.

9. **(a)** $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3}$, so the *sequence* $\{a_n\}$ is convergent by (.1.1).

(b) Since $\lim_{n\to\infty} a_n = \frac{2}{3} \neq 0$, the *series* $\sum_{n=1}^{\infty} a_n$ is divergent by the Test for Divergence (7).

10. **(a)** Both $\sum_{i=1}^{n} a_i$ and $\sum_{j=1}^{n} a_j$ represent the sum of the first n terms of the sequence $\{a_n\}$, that is, the n th partial sum.

(b) $\sum_{i=1}^{n} a_j = \underbrace{a_j + a_j + \ldots + a_j}_{n \text{ terms}} = na_j$, which, in general, is not the same as $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.

11. $3 + 2 + \frac{4}{3} + \frac{8}{9} + \cdots$ is a geometric series with first term $a=3$ and common ratio $r = \frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series converges to $\frac{a}{1-r} = \frac{3}{1-2/3} = \frac{3}{1/3} = 9$.

12. $\frac{1}{8}-\frac{1}{4}+\frac{1}{2}-1+\cdots$ is a geometric series with $r=-2$. Since $|r|=2>1$, the series diverges.

13. $-2+\frac{5}{2}-\frac{25}{8}+\frac{125}{32}-\cdots$ is a geometric series with $a=-2$ and $r=\frac{5/2}{-2}=-\frac{5}{4}$. Since $|r|=\frac{5}{4}>1$, the series diverges by (4).

14. $1+0.4+0.16+0.064+\cdots$ is a geometric series with ratio 0.4 . The series converges to $\frac{a}{1-r}=\frac{1}{1-2/5}=\frac{5}{3}$ since $|r|=\frac{2}{5}<1$.

15. $\Sigma_{n=1}^{\infty}5\left(\frac{2}{3}\right)^{n-1}$ is a geometric series with $a=5$ and $r=\frac{2}{3}$. Since $|r|=\frac{2}{3}<1$, the series converges to $\frac{a}{1-r}=\frac{5}{1-2/3}=\frac{5}{1/3}=15$.

16. $\Sigma_{n=1}^{\infty}\frac{(-6)^{n-1}}{5^{n-1}}$ is a geometric series with $a=1$ and $r=-\frac{6}{5}$. The series diverges since $|r|=\frac{6}{5}>1$.

17. $\Sigma_{n=1}^{\infty}\frac{(-3)^{n-1}}{4^{n}}=\frac{1}{4}\Sigma_{n=1}^{\infty}\left(-\frac{3}{4}\right)^{n-1}$. The latter series is geometric with $a=1$ and $r=-\frac{3}{4}$. Since $|r|=\frac{3}{4}<1$, it converges to $\frac{1}{1-(-3/4)}=\frac{4}{7}$. Thus, the given series converges to $\left(\frac{1}{4}\right)\left(\frac{4}{7}\right)=\frac{1}{7}$.

18. $\Sigma_{n=0}^{\infty}\frac{1}{(\sqrt{2})^{n}}$ is a geometric series with ratio $r=\frac{1}{\sqrt{2}}$. Since $|r|=\frac{1}{\sqrt{2}}<1$, the series converges. Its sum is $\frac{1}{1-1/\sqrt{2}}=\frac{\sqrt{2}}{\sqrt{2}-1}=\frac{\sqrt{2}}{\sqrt{2}-1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}+1}=\sqrt{2}(\sqrt{2}+1)=2+\sqrt{2}$.

19. $\Sigma_{n=0}^{\infty}\frac{\pi^{n}}{3^{n+1}}=\frac{1}{3}\Sigma_{n=0}^{\infty}\left(\frac{\pi}{3}\right)^{n}$ is a geometric series with ratio $r=\frac{\pi}{3}$. Since $|r|>1$, the series diverges.

20. $\Sigma_{n=1}^{\infty}\frac{e^{n}}{3^{n-1}}=3\Sigma_{n=1}^{\infty}\left(\frac{e}{3}\right)^{n}$ is a geometric series with first term $3(e/3)=e$ and ratio $r=\frac{e}{3}$. Since $|r|<1$, the series converges. Its sum is $\frac{e}{1-e/3}=\frac{3e}{3-e}$.

21. $\sum_{n=1}^{\infty} \frac{n}{n+5}$ diverges since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+5} = 1 \neq 0$.

22. $\sum_{n=1}^{\infty} \frac{3}{n} = 3\sum_{n=1}^{\infty} \frac{1}{n}$ diverges since each of its partial sums is 3 times the corresponding partial sum of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. In general, constant multiples of divergent series are divergent.

23. Using partial fractions, the partial sums are

$$s_n = \sum_{i=2}^{n} \frac{2}{(i-1)(i+1)} = \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{1}{i+1} \right)$$

$$= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n-3}-\frac{1}{n-1}\right)+\left(\frac{1}{n-2}-\frac{1}{n}\right)$$

This sum is a telescoping series and $s_n = 1+\frac{1}{2}-\frac{1}{n-1}-\frac{1}{n}$.

Thus, $\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{n\to\infty} \left(1+\frac{1}{2}-\frac{1}{n-1}-\frac{1}{n}\right) = \frac{3}{2}$.

24. $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$ diverges by (7), the Test for Divergence, since

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n} = \lim_{n\to\infty} \left(1+\frac{1}{n^2+2n}\right) = 1 \neq 0$.

25. $\sum_{k=2}^{\infty} \frac{k^2}{k^2-1}$ diverges by the Test for Divergence since $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^2}{k^2-1} = 1 \neq 0$.

26. Converges. $s_n = \sum_{i=1}^{n} \frac{2}{i^2+4i+3} = \sum_{i=1}^{n} \left(\frac{1}{i+1}-\frac{1}{i+3}\right)$ (using partial fractions). The latter sum is

$$\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+2}\right)+\left(\frac{1}{n+1}-\frac{1}{n+3}\right) = \frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}$$

(telescoping series). Thus, $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \lim_{n\to\infty} \left(\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}\right) = \frac{1}{2}+\frac{1}{3} = \frac{5}{6}$.

27. Converges.

$\sum_{n=1}^{\infty} \frac{3^n+2^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{3^n}{6^n} + \frac{2^n}{6^n} \right) = \sum_{n=1}^{\infty} \left[\left(\frac{1}{2} \right)^n + \left(\frac{1}{3} \right)^n \right] = \frac{1/2}{1-1/2} + \frac{1/3}{1-1/3} = 1 + \frac{1}{2} = \frac{3}{2}$

28. $\sum_{n=1}^{\infty} \left[(0.8)^{n-1} - (0.3)^n \right] = \sum_{n=1}^{\infty} (0.8)^{n-1} - \sum_{n=1}^{\infty} (0.3)^n = \frac{1}{1-0.8} - \frac{0.3}{1-0.3} = 5 - \frac{3}{7} = \frac{32}{7}$.

29. $\sum_{n=1}^{\infty} \sqrt[n]{2} = 2 + \sqrt{2} + \sqrt[3]{2} + \sqrt[4]{2} + \cdots$ diverges by the Test for Divergence since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{2} = \lim_{n\to\infty} 2^{1/n} = 2^0 = 1 \neq 0$.

30. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln \left(\frac{n}{2n+5} \right) = \lim_{n\to\infty} \ln \left(\frac{1}{2+5/n} \right) = \ln \frac{1}{2} \neq 0$, so the series diverges by the Test for Divergence.

31. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$, so the series diverges by the Test for Divergence.

32. $\sum_{k=1}^{\infty} (\cos 1)^k$ is a geometric series with ratio $r = \cos 1 \approx 0.540302$. It converges because $|r| < 1$. Its sum is $\frac{\cos 1}{1-\cos 1} \approx 1.175343$.

33. The first series is a telescoping sum:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{3}{n(n+3)} &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3} \right) = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+3} \right) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{n+2} - \frac{1}{n+3} \right) \\
&= 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}
\end{aligned}$$

The second series is geometric with first term $\frac{5}{4}$ and ratio $\frac{1}{4}$: $\sum_{n=1}^{\infty} \frac{5}{4^n} = \frac{5/4}{1-1/4} = \frac{5}{3}$. Thus,

$$\sum_{n=1}^{\infty} \left(\frac{3}{n(n+3)} + \frac{5}{4^n} \right) = \sum_{n=1}^{\infty} \frac{3}{n(n+3)} + \sum_{n=1}^{\infty} \frac{5}{4^n} = \frac{11}{6} + \frac{5}{3} = \frac{7}{2} .$$

34. $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n} \right)$ diverges because $\sum_{n=1}^{\infty} \frac{2}{n} = 2\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (If it converged, then

$\frac{1}{2}\cdot 2\sum_{n=1}^{\infty}\frac{1}{n}$ would also converge by Theorem 8(i), but we know from Example 7 that the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.) If the given series converges, then the difference $\sum_{n=1}^{\infty}\left(\frac{3}{5^n}+\frac{2}{n}\right)-\sum_{n=1}^{\infty}\frac{3}{5^n}$ must converge (since $\sum_{n=1}^{\infty}\frac{3}{5^n}$ is a convergent geometric series) and equal $\sum_{n=1}^{\infty}\frac{2}{n}$, but we have just seen that $\sum_{n=1}^{\infty}\frac{2}{n}$ diverges, so the given series must also diverge.

35. $0.\overline{2}=\frac{2}{10}+\frac{2}{10^2}+\cdots$ is a geometric series with $a=\frac{2}{10}$ and $r=\frac{1}{10}$. It converges to $\frac{a}{1-r}=\frac{2/10}{1-1/10}=\frac{2}{9}$.

36. $0.\overline{73}=\frac{73}{10^2}+\frac{73}{10^4}+\cdots=\frac{73/10^2}{1-1/10^2}=\frac{73/100}{99/100}=\frac{73}{99}$

37. $3.\overline{417}=3+\frac{417}{10^3}+\frac{417}{10^6}+\cdots=3+\frac{417/10^3}{1-1/10^3}=3+\frac{417}{999}=\frac{3414}{999}=\frac{1138}{333}$

38. $6.2\overline{54}=6.2+\frac{54}{10^3}+\frac{54}{10^5}+\cdots=6.2+\frac{54/10^3}{1-1/10^2}=\frac{62}{10}+\frac{54}{990}=\frac{6192}{990}=\frac{344}{55}$

39. $0.123\overline{456}=\frac{123}{1000}+\frac{0.000456}{1-0.001}=\frac{123}{1000}+\frac{456}{999{,}000}=\frac{123{,}333}{999{,}000}=\frac{41{,}111}{333{,}000}$

40. $5.\overline{6021}=5+\frac{6021}{10^4}+\frac{6021}{10^8}+\cdots=5+\frac{6021/10^4}{1-1/10^4}=5+\frac{6021}{9999}=\frac{56{,}016}{9999}=\frac{6224}{1111}$

41. $\sum_{n=1}^{\infty}\frac{x^n}{3^n}=\sum_{n=1}^{\infty}\left(\frac{x}{3}\right)^n$ is a geometric series with $r=\frac{x}{3}$, so the series converges $\Leftrightarrow |r|<1 \Leftrightarrow \frac{|x|}{3}<1 \Leftrightarrow |x|<3$; that is, $-3<x<3$. In that case, the sum of the series is $\frac{a}{1-r}=\frac{x/3}{1-x/3}=\frac{x/3}{1-x/3}\cdot\frac{3}{3}=\frac{x}{3-x}$.

42. $\sum_{n=1}^{\infty}(x-4)^n$ is a geometric series with $r=x-4$, so the series converges $\Leftrightarrow |r|<1 \Leftrightarrow |x-4|<1 \Leftrightarrow$ $3<x<5$. In that case, the sum of the series is $\dfrac{x-4}{1-(x-4)}=\dfrac{x-4}{5-x}$.

43. $\sum_{n=0}^{\infty}4^n x^n=\sum_{n=0}^{\infty}(4x)^n$ is a geometric series with $r=4x$, so the series converges $\Leftrightarrow |r|<1 \Leftrightarrow 4|x|<1$ $\Leftrightarrow |x|<\dfrac{1}{4}$. In that case, the sum of the series is $\dfrac{1}{1-4x}$.

44. $\sum_{n=0}^{\infty}\dfrac{(x+3)^n}{2^n}$ is a geometric series with $r=\dfrac{x+3}{2}$, so the series converges $\Leftrightarrow |r|<1 \Leftrightarrow \dfrac{|x+3|}{2}<1 \Leftrightarrow$ $|x+3|<2 \Leftrightarrow -5<x<-1$. For these values of x , the sum of the series is $\dfrac{1}{1-(x+3)/2}=\dfrac{2}{2-(x+3)}=-\dfrac{2}{x+1}$.

45. $\sum_{n=0}^{\infty}\dfrac{\cos^n x}{2^n}$ is a geometric series with first term 1 and ratio $r=\dfrac{\cos x}{2}$, so it converges $\Leftrightarrow |r|<1$. But $|r|=\dfrac{|\cos x|}{2}\le\dfrac{1}{2}$ for all x . Thus, the series converges for all real values of x and the sum of the series is $\dfrac{1}{1-(\cos x)/2}=\dfrac{2}{2-\cos x}$.

46. Because $\dfrac{1}{n}\to 0$ and ln is continuous, we have $\lim_{n\to\infty}\ln\left(1+\dfrac{1}{n}\right)=\ln 1=0$. We now show that the series $\sum_{n=1}^{\infty}\ln\left(1+\dfrac{1}{n}\right)=\sum_{n=1}^{\infty}\ln\left(\dfrac{n+1}{n}\right)=\sum_{n=1}^{\infty}[\ln(n+1)-\ln n]$ diverges.
$s_n=(\ln 2-\ln 1)+(\ln 3-\ln 2)+\cdots+(\ln(n+1)-\ln n)=\ln(n+1)-\ln 1=\ln(n+1)$. As $n\to\infty$, $s_n=\ln(n+1)\to\infty$, so the series diverges.

47. After defining f , We use `convert(f,parfrac);` in Maple, `Apart` in Mathematica, or `Expand Rational` and `Simplify` in Derive to find that the general term is $\dfrac{1}{(4n+1)(4n-3)}=-\dfrac{1/4}{4n+1}+\dfrac{1/4}{4n-3}$. So the n th partial sum is

$$s_n=\sum_{k=1}^{n}\left(-\frac{1/4}{4k+1}+\frac{1/4}{4k-3}\right)=\frac{1}{4}\sum_{k=1}^{n}\left(\frac{1}{4k-3}-\frac{1}{4k+1}\right)$$
$$=\frac{1}{4}\left[\left(1-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{13}\right)+\cdots+\left(\frac{1}{4n-3}-\frac{1}{4n+1}\right)\right]=\frac{1}{4}\left(1-\frac{1}{4n+1}\right)$$

The series converges to $\lim_{n\to\infty} s_n = \frac{1}{4}$. This can be confirmed by directly computing the sum using sum(f,1..infinity); (in Maple), Sum[f,{n,1,Infinity}] (in Mathematica), or Calculus Sum (from 1 to ∞) and Simplify (in Derive).

48. See Exercise 47 for specific CAS commands. $\frac{n^2+3n+1}{\left(n^2+n\right)^2} = \frac{1}{n^2} + \frac{1}{n} - \frac{1}{(n+1)^2} - \frac{1}{n+1}$. So the n th partial sum is

$$s_n = \sum_{k=1}^{n} \left(\frac{1}{k^2} + \frac{1}{k} - \frac{1}{(k+1)^2} - \frac{1}{k+1} \right)$$

$$= \left(1+1-\frac{1}{2^2} - \frac{1}{2} \right) + \left(\frac{1}{2^2} + \frac{1}{2} - \frac{1}{3^2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{n^2} + \frac{1}{n} - \frac{1}{(n+1)^2} - \frac{1}{n+1} \right)$$

$$= 1+1-\frac{1}{(n+1)^2} - \frac{1}{n+1}$$

The series converges to $\lim_{n\to\infty} s_n = 2$.

49. For $n=1$, $a_1=0$ since $s_1=0$. For $n>1$,

$$a_n = s_n - s_{n-1} = \frac{n-1}{n+1} - \frac{(n-1)-1}{(n-1)+1} = \frac{(n-1)n-(n+1)(n-2)}{(n+1)n} = \frac{2}{n(n+1)}$$

Also, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-1/n}{1+1/n} = 1$.

50. $a_1 = s_1 = 3 - \frac{1}{2} = \frac{5}{2}$. For $n \neq 1$,

$$a_n = s_n - s_{n-1} = \left(3-n2^{-n}\right) - \left[3-(n-1)2^{-(n-1)}\right] = -\frac{n}{2^n} + \frac{n-1}{2^{n-1}} \cdot \frac{2}{2} = \frac{2(n-1)}{2^n} - \frac{n}{2^n} = \frac{n-2}{2^n}$$

Also, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(3 - \frac{n}{2^n} \right) = 3$ because $\lim_{x\to\infty} \frac{x}{2^x} = \lim_{x\to\infty} \frac{1}{2^x \ln 2} = 0$.

51. **(a)** The first step in the chain occurs when the local government spends D dollars. The people who receive it spend a fraction c of those D dollars, that is, Dc dollars. Those who receive the Dc

dollars spend a fraction c of it, that is, Dc^2 dollars. Continuing in this way, we see that the total spending after n transactions is $S_n = D + Dc + Dc^2 + \cdots + Dc^{n-1} = \dfrac{D\left(1-c^n\right)}{1-c}$ by (3).

(b)

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{D\left(1-c^n\right)}{1-c} = \frac{D}{1-c} \lim_{n\to\infty} \left(1-c^n\right) = \frac{D}{1-c} \quad (\text{since } 0<c<1 \Rightarrow \lim_{n\to\infty} c^n = 0)$$
$$= \frac{D}{s} \quad (\text{since } c+s=1) = kD \quad (\text{since } k=1/s)$$

If $c=0.8$, then $s=1-c=0.2$ and the multiplier is $k=1/s=5$.

52. **(a)** Initially, the ball falls a distance H, then rebounds a distance rH, falls rH, rebounds r^2H, falls r^2H, etc. The total distance it travels is

$$\mathrm{H} + 2\mathrm{rH} + 2\mathrm{r}^2\mathrm{H} + 2\mathrm{r}^3\mathrm{H} + \cdots = \mathrm{H}\left(1 + 2r + 2r^2 + 2r^3 + \cdots\right)$$
$$= \mathrm{H}\left[1 + 2r\left(1 + r + r^2 + \cdots\right)\right] = \mathrm{H}\left[1 + 2r\left(\frac{1}{1-r}\right)\right] = \mathrm{H}\left(\frac{1+r}{1-r}\right) \text{meters}$$

(b) From Example 3 in Section 2.1, we know that a ball falls $\frac{1}{2}gt^2$ meters in t seconds, where g is the gravitational acceleration. Thus, a ball falls h meters in $t=\sqrt{2h/g}$ seconds. The total travel time in seconds is

$$\sqrt{\frac{2H}{g}} + 2\sqrt{\frac{2H}{g}r} + 2\sqrt{\frac{2H}{g}r^2} + 2\sqrt{\frac{2H}{g}r^3} + \cdots = \sqrt{\frac{2H}{g}}\left[1 + 2\sqrt{r} + 2\sqrt{r}^2 + 2\sqrt{r}^3 + \cdots\right]$$
$$= \sqrt{\frac{2H}{g}}\left(1 + 2\sqrt{r}\left[1 + \sqrt{r} + \sqrt{r}^2 + \cdots\right]\right) = \sqrt{\frac{2H}{g}}\left[1 + 2\sqrt{r}\left(\frac{1}{1-\sqrt{r}}\right)\right] = \sqrt{\frac{2H}{g}}\,\frac{1+\sqrt{r}}{1-\sqrt{r}}$$

(c) It will help to make a chart of the time for each descent and each rebound of the ball, together with the velocity just before and just after each bounce. Recall that the time in seconds needed to fall h meters is $\sqrt{2h/g}$. The ball hits the ground with velocity $-g\sqrt{2h/g} = -\sqrt{2hg}$ (taking the upward direction to be positive) and rebounds with velocity $kg\sqrt{2h/g} = k\sqrt{2hg}$, taking time $k\sqrt{2h/g}$ to reach the top of its bounce, where its velocity is 0. At that point, its height is k^2h. All these results follow from the formulas for vertical motion with gravitational acceleration $-g$:

$\frac{d^2y}{dt^2}=-g \Rightarrow v=\frac{dy}{dt}=v_0-gt \Rightarrow y=y_0+v_0t-\frac{1}{2}gt^2$.

number of descent	time of descent	speed before bounce	speed after bounce	time of ascent	peak height
1	$\sqrt{2H/g}$	$\sqrt{2Hg}$	$k\sqrt{2Hg}$	$k\sqrt{2H/g}$	k^2H
2	$\sqrt{2k^2H/g}$	$\sqrt{2k^2Hg}$	$k\sqrt{2k^2Hg}$	$k\sqrt{2k^2H/g}$	k^4H
3	$\sqrt{2k^4H/g}$	$\sqrt{2k^4Hg}$	$k\sqrt{2k^4Hg}$	$k\sqrt{2k^4H/g}$	k^6H
. . .	. . .	. . .	. . .	. . .	. . .

otal travel time in seconds is

$$\sqrt{\frac{2H}{g}}+\mathrm{k}\sqrt{\frac{2H}{g}}+\mathrm{k}\sqrt{\frac{2H}{g}}+\mathrm{k}^2\sqrt{\frac{2H}{g}}+\mathrm{k}^2\sqrt{\frac{2H}{g}}+\cdot\cdot\cdot=\sqrt{\frac{2H}{g}}\left(1+2k+2k^2+2k^3+\cdot\cdot\cdot\right)$$

$$=\sqrt{\frac{2H}{g}}\left[1+2k\left(1+k+k^2+\cdot\cdot\cdot\right)\right]=\sqrt{\frac{2H}{g}}\left[1+2k\left(\frac{1}{1-k}\right)\right]=\sqrt{\frac{2H}{g}}\frac{1+k}{1-k}$$

Another method: We could use part (b). At the top of the bounce, the height is $k^2h=rh$, so $\sqrt{r}=k$ and the result follows from part (b).

53. $\sum_{n=2}^{\infty}(1+c)^{-n}$ is a geometric series with $a=(1+c)^{-2}$ and $r=(1+c)^{-1}$, so the series converges when $\left|(1+c)^{-1}\right|<1 \Leftrightarrow |1+c|>1 \Leftrightarrow 1+c>1$ or $1+c<-1 \Leftrightarrow c>0$ or $c<-2$. We calculate the sum of the series and set it equal to 2 : $\frac{(1+c)^{-2}}{1-(1+c)^{-1}}=2 \Leftrightarrow \left(\frac{1}{1+c}\right)^2=2-2\left(\frac{1}{1+c}\right) \Leftrightarrow$

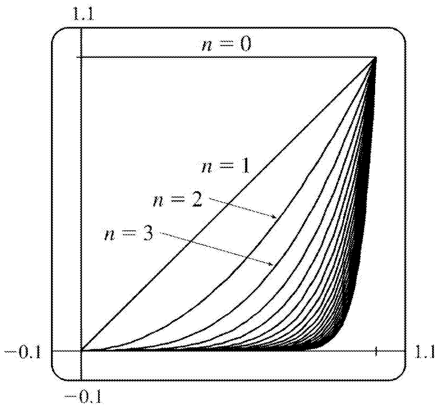
$1=2(1+c)^2-2(1+c) \Leftrightarrow 2c^2+2c-1=0 \Leftrightarrow c=\frac{-2\pm\sqrt{12}}{4}=\frac{\pm\sqrt{3}-1}{2}$. However, the negative root is inadmissible because $-2<\frac{-\sqrt{3}-1}{2}<0$. So $c=\frac{\sqrt{3}-1}{2}$.

54. The area between $y=x^{n-1}$ and $y=x^n$ for $0\leq x\leq 1$ is

$$\int_0^1\left(x^{n-1}-x^n\right)dx=\left[\frac{x^n}{n}-\frac{x^{n+1}}{n+1}\right]_0^1=\frac{1}{n}-\frac{1}{n+1}$$

$$=\frac{(n+1)-n}{n(n+1)}=\frac{1}{n(n+1)}$$

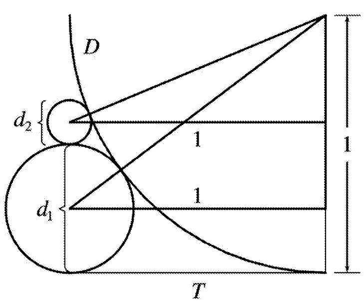
We can see from the diagram that as $n\to\infty$, the sum of the areas between the successive curves approaches the area of the unit square, that is, 1 . So $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.



55. Let d_n be the diameter of C_n . We draw lines from the centers of the C_i to the center of D (or C), and using the Pythagorean Theorem, we can write $1^2+\left(1-\frac{1}{2}d_1\right)^2=\left(1+\frac{1}{2}d_1\right)^2 \Leftrightarrow$

$1=\left(1+\frac{1}{2}d_1\right)^2-\left(1-\frac{1}{2}d_1\right)^2=2d_1$ (difference of squares) $\Rightarrow d_1=\frac{1}{2}$. Similarly,

$$1 = \left(1+\frac{1}{2}d_2\right)^2-\left(1-d_1-\frac{1}{2}d_2\right)^2=2d_2+2d_1-d_1^2-d_1d_2$$
$$= (2-d_1)(d_1+d_2) \Leftrightarrow$$



$d_2=\frac{1}{2-d_1}-d_1=\frac{\left(1-d_1\right)^2}{2-d_1}$, $1=\left(1+\frac{1}{2}d_3\right)^2-\left(1-d_1-d_2-\frac{1}{2}d_3\right)^2 \Leftrightarrow d_3=\frac{\left[1-\left(d_1+d_2\right)\right]^2}{2-\left(d_1+d_2\right)}$, and in general, $d_{n+1}=\frac{\left(1-\sum_{i=1}^{n}d_i\right)^2}{2-\sum_{i=1}^{n}d_i}$. If we actually calculate d_2 and d_3 from the formulas above, we find that they are $\frac{1}{6}=\frac{1}{2\cdot 3}$ and $\frac{1}{12}=\frac{1}{3\cdot 4}$ respectively, so we suspect that in general, $d_n=\frac{1}{n(n+1)}$. To prove this, we use induction: Assume that for all $k\le n$, $d_k=\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$. Then $\sum_{i=1}^{n} d_i=1-\frac{1}{n+1}=\frac{n}{n+1}$ (telescoping sum). Substituting this into our formula for d_{n+1} , we get

$$d_{n+1}=\frac{\left[1-\frac{n}{n+1}\right]^2}{2-\left(\frac{n}{n+1}\right)}=\frac{\frac{1}{(n+1)^2}}{\frac{n+2}{n+1}}=\frac{1}{(n+1)(n+2)}$$, and the induction is complete.

Now, we observe that the partial sums $\sum_{i=1}^{n} d_i$ of the diameters of the circles approach 1 as $n\to\infty$; that is, $\sum_{n=1}^{\infty} a_n=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1$, which is what we wanted to prove.

56. $|CD|=b\sin\theta$, $|DE|=|CD|\sin\theta=b\sin^2\theta$, $|EF|=|DE|\sin\theta=b\sin^3\theta$, … . Therefore, $|CD|+|DE|+|EF|+|FG|+\cdot\cdot\cdot=b\sum_{n=1}^{\infty}\sin^n\theta=b\left(\frac{\sin\theta}{1-\sin\theta}\right)$ since this is a geometric series with $r=\sin\theta$ and $|\sin\theta|<1$ (because $0<\theta<\frac{\pi}{2}$).

57. The series $1-1+1-1+1-1+\cdot\cdot\cdot$ diverges (geometric series with $r=-1$) so we cannot say that $0=1-1+1-1+1-1+\cdot\cdot\cdot$.

58. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n=0$ by Theorem 6, so $\lim_{n\to\infty}\frac{1}{a_n}\neq 0$, and so $\sum_{n=1}^{\infty}\frac{1}{a_n}$ is divergent by the Test for Divergence.

59. $\sum_{n=1}^{\infty} ca_n=\lim_{n\to\infty}\sum_{i=1}^{n} ca_i=\lim_{n\to\infty} c\sum_{i=1}^{n} a_i=c\lim_{n\to\infty}\sum_{i=1}^{n} a_i=c\sum_{n=1}^{\infty} a_n$, which exists by hypothesis.

60. If $\sum ca_n$ were convergent, then $\sum(1/c)(ca_n)=\sum a_n$ would be also, by Theorem 8. But this is not the case, so $\sum ca_n$ must diverge.

61. Suppose on the contrary that $\sum\left(a_n+b_n\right)$ converges. Then $\sum\left(a_n+b_n\right)$ and $\sum a_n$ are convergent series. So by Theorem 8, $\sum\left[\left(a_n+b_n\right)-a_n\right]$ would also be convergent. But $\sum\left[\left(a_n+b_n\right)-a_n\right]=\sum b_n$, a contradiction, since $\sum b_n$ is given to be divergent.

62. No. For example, take

$\Sigma a_n = \Sigma n$ and $\Sigma b_n = \Sigma(-n)$, which both diverge, yet $\Sigma\left(a_n + b_n\right) = \Sigma 0$, which converges with sum 0 .

63. The partial sums $\left\{s_n\right\}$ form an increasing sequence, since $s_n - s_{n-1} = a_n > 0$ for all n . Also, the sequence $\left\{s_n\right\}$ is bounded since $s_n \le 1000$ for all n . So by Theorem .1.11, the sequence of partial sums converges, that is, the series Σa_n is convergent.

64. **(a)**

$$\text{RHS} = \frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}} = \frac{f_n f_{n+1} - f_n f_{n-1}}{f_n^2 f_{n-1} f_{n+1}} = \frac{f_{n+1} - f_{n-1}}{f_n f_{n-1} f_{n+1}} = \frac{\left(f_{n-1} + f_n\right) - f_{n-1}}{f_n f_{n-1} f_{n+1}}$$

$$= \frac{1}{f_{n-1} f_{n+1}} = \text{LHS}$$

(b) $\Sigma_{n=2}^{\infty} \dfrac{1}{f_{n-1}f_{n+1}} = \Sigma_{n=2}^{\infty}\left(\dfrac{1}{f_{n-1}f_n} - \dfrac{1}{f_n f_{n+1}}\right)$

$$= \lim_{n\to\infty}\left[\left(\frac{1}{f_1 f_2} - \frac{1}{f_2 f_3}\right) + \left(\frac{1}{f_2 f_3} - \frac{1}{f_3 f_4}\right) + \left(\frac{1}{f_3 f_4} - \frac{1}{f_4 f_5}\right) + \cdots + \left(\frac{1}{f_{n-1} f_n} - \frac{1}{f_n f_{n+1}}\right)\right.$$

$$= \lim_{n\to\infty}\left(\frac{1}{f_1 f_2} - \frac{1}{f_n f_{n+1}}\right) = \frac{1}{f_1 f_2} - 0 = \frac{1}{1\cdot 1} = 1 \text{ because } f_n \to \infty \text{ as } n \to \infty .$$

(c) $\Sigma_{n=2}^{\infty} \dfrac{f_n}{f_{n-1}f_{n+1}} = \Sigma_{n=2}^{\infty}\left(\dfrac{f_n}{f_{n-1}f_n} - \dfrac{f_n}{f_n f_{n+1}}\right)$ (as above)

$$= \Sigma_{n=2}^{\infty}\left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}\right)$$

$$= \lim_{n\to\infty}\left[\left(\frac{1}{f_1} - \frac{1}{f_3}\right) + \left(\frac{1}{f_2} - \frac{1}{f_4}\right) + \left(\frac{1}{f_3} - \frac{1}{f_5}\right) + \left(\frac{1}{f_4} - \frac{1}{f_6}\right) + \cdots + \left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}\right)\right.$$

$$= \lim_{n\to\infty}\left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{1}{f_n} - \frac{1}{f_{n+1}}\right) = 1 + 1 - 0 - 0 = 2 \text{ because } f_n \to \infty \text{ as } n \to \infty .$$

65. **(a)** At the first step, only the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ (length $\frac{1}{3}$) is removed. At the second step, we remove the intervals $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$, which have a total length of $2\cdot\left(\frac{1}{3}\right)^2$. At the third

step, we remove 2^2 intervals, each of length $\left(\frac{1}{3}\right)^3$. In general, at the n th step we remove 2^{n-1} intervals, each of length $\left(\frac{1}{3}\right)^n$, for a length of $2^{n-1}\cdot\left(\frac{1}{3}\right)^n=\frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$. Thus, the total length of all removed intervals is $\sum_{n=1}^{\infty}\frac{1}{3}\left(\frac{2}{3}\right)^{n-1}=\frac{1/3}{1-2/3}=1$ (geometric series with $a=\frac{1}{3}$ and $r=\frac{2}{3}$). Notice that at the n th step, the leftmost interval that is removed is $\left(\left(\frac{1}{3}\right)^n,\left(\frac{2}{3}\right)^n\right)$, so we never remove 0 , and 0 is in the Cantor set. Also, the rightmost interval removed is $\left(1-\left(\frac{2}{3}\right)^n,1-\left(\frac{1}{3}\right)^n\right)$, so 1 is never removed. Some other numbers in the Cantor set are $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{9}$, $\frac{2}{9}$, $\frac{7}{9}$, and $\frac{8}{9}$.

(b) The area removed at the first step is $\frac{1}{9}$; at the second step, $8\cdot\left(\frac{1}{9}\right)^2$; at the third step, $(8)^2\cdot\left(\frac{1}{9}\right)^3$. In general, the area removed at the n th step is $(8)^{n-1}\left(\frac{1}{9}\right)^n=\frac{1}{9}\left(\frac{8}{9}\right)^{n-1}$, so the total area of all removed squares is $\sum_{n=1}^{\infty}\frac{1}{9}\left(\frac{8}{9}\right)^{n-1}=\frac{1/9}{1-8/9}=1$.

66. **(a)**

a_1	1	2	4	1	1	1000
a_2	2	3	1	4	1000	1
a_3	1.5	2.5	2.5	2.5	500.5	500.5
a_4	1.75	2.75	1.75	3.25	750.25	250.75
a_5	1.625	2.625	2.125	2.875	625.375	375.625
a_6	1.6875	2.6875	1.9375	3.0625	687.813	313.188
a_7	1.65625	2.65625	2.03125	2.96875	656.594	344.406
a_8	1.67188	2.67188	1.98438	3.01563	672.203	328.797
a_9	1.66406	2.66406	2.00781	2.99219	664.398	336.602
a_{10}	1.66797	2.66797	1.99609	3.00391	668.301	332.699
a_{11}	1.66602	2.66602	2.00195	2.99805	666.350	334.650

a_{12}	1.66699	2.66699	1.99902	3.00098	667.325	333.675

The limits seem to be $\frac{5}{3}$, $\frac{8}{3}$, 2 , 3 , 667 , and 334 . Note that the limits appear to be "weighted'' more toward a_2 . In general, we guess that the limit is $\frac{a_1+2a_2}{3}$.

(b)

$$a_{n+1}-a_n = \frac{1}{2}\left(a_n+a_{n-1}\right)-a_n = -\frac{1}{2}\left(a_n-a_{n-1}\right) = -\frac{1}{2}\left[\frac{1}{2}\left(a_{n-1}+a_{n-2}\right)-a_{n-1}\right]$$
$$= -\frac{1}{2}\left[-\frac{1}{2}\left(a_{n-1}-a_{n-2}\right)\right] = \cdots = \left(-\frac{1}{2}\right)^{n-1}\left(a_2-a_1\right)$$

Note that we have used the formula $a_k=\frac{1}{2}\left(a_{k-1}+a_{k-2}\right)$ a total of $n-1$ times in this calculation, once for each k between 3 and $n+1$. Now we can write

$$a_n = a_1+\left(a_2-a_1\right)+\left(a_3-a_2\right)+\cdots+\left(a_{n-1}-a_{n-2}\right)+\left(a_n-a_{n-1}\right)$$
$$= a_1+\sum_{k=1}^{n-1}\left(a_{k+1}-a_k\right) = a_1+\sum_{k=1}^{n-1}\left(-\frac{1}{2}\right)^{k-1}\left(a_2-a_1\right)$$

and so

$$\lim_{n\to\infty} a_n = a_1+\left(a_2-a_1\right)\sum_{k=1}^{\infty}\left(-\frac{1}{2}\right)^{k-1} = a_1+\left(a_2-a_1\right)\left[\frac{1}{1-(-1/2)}\right]$$
$$= a_1+\frac{2}{3}\left(a_2-a_1\right) = \frac{a_1+2a_2}{3}$$

67. **(a)** For $\sum_{n=1}^{\infty}\frac{n}{(n+1)!}$, $s_1=\frac{1}{1\cdot 2}=\frac{1}{2}$, $s_2=\frac{1}{2}+\frac{2}{1\cdot 2\cdot 3}=\frac{5}{6}$, $s_3=\frac{5}{6}+\frac{3}{1\cdot 2\cdot 3\cdot 4}=\frac{23}{24}$, $s_4=\frac{23}{24}+\frac{4}{1\cdot 2\cdot 3\cdot 4\cdot 5}=\frac{119}{120}$. The denominators are $(n+1)!$, so a guess would be $s_n=\frac{(n+1)!-1}{(n+1)!}$.

(b) For $n=1$, $s_1=\frac{1}{2}=\frac{2!-1}{2!}$, so the formula holds for $n=1$. Assume $s_k=\frac{(k+1)!-1}{(k+1)!}$. Then

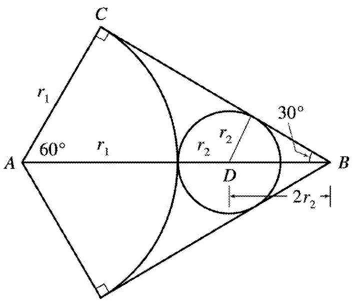
$$s_{k+1} = \frac{(k+1)!-1}{(k+1)!}+\frac{k+1}{(k+2)!} = \frac{(k+1)!-1}{(k+1)!}+\frac{k+1}{(k+1)!(k+2)}$$
$$= \frac{(k+2)!-(k+2)+k+1}{(k+2)!} = \frac{(k+2)!-1}{(k+2)!}$$

Thus, the formula is true for $n=k+1$. So by induction, the guess is correct.

(c)

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{(n+1)!-1}{(n+1)!} = \lim_{n\to\infty} \left[1 - \frac{1}{(n+1)!} \right] = 1$ and so $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$.

68.

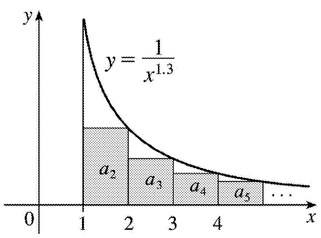


Let r_1= radius of the large circle, r_2= radius of next circle, and so on. From the figure we have $\angle BAC = 60^\circ$ and $\cos 60^\circ = r_1 / |AB|$, so $|AB| = 2r_1$ and $|DB| = 2r_2$. Therefore, $2r_1 = r_1 + r_2 + 2r_2 = r_1 + 3r_2 \Rightarrow r_1 = 3r_2$. In general, we have $r_{n+1} = \frac{1}{3} r_n$, so the total area is

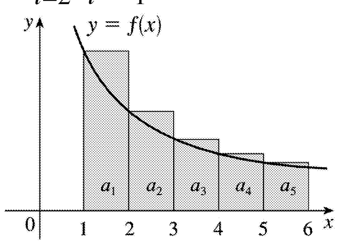
$$\begin{aligned} A &= \pi r_1^2 + 3\pi r_2^2 + 3\pi r_3^2 + \cdots \\ &= \pi r_1^2 + 3\pi r_2^2 \left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \cdots \right) \\ &= \pi r_1^2 + 3\pi r_2^2 \cdot \frac{1}{1-1/9} = \pi r_1^2 + \frac{27}{8} \pi r_2^2 \end{aligned}$$

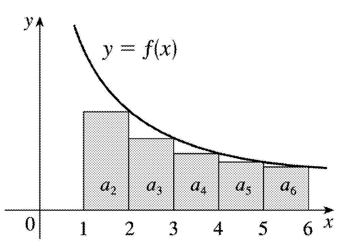
Since the sides of the triangle have length 1 , $|BC| = \frac{1}{2}$ and $\tan 30^\circ = \frac{r_1}{1/2}$. Thus, $r_1 = \frac{\tan 30^\circ}{2} = \frac{1}{2\sqrt{3}}$ $\Rightarrow r_2 = \frac{1}{6\sqrt{3}}$, so $A = \pi \left(\frac{1}{2\sqrt{3}} \right)^2 + \frac{27\pi}{8} \left(\frac{1}{6\sqrt{3}} \right)^2 = \frac{\pi}{12} + \frac{\pi}{32} = \frac{11\pi}{96}$. The area of the triangle is $\frac{\sqrt{3}}{4}$, so the circles occupy about 83.1% of the area of the triangle.

1. The picture shows that $a_2 = \frac{1}{2^{1.3}} < \int_1^2 \frac{1}{x^{1.3}}\,dx$, $a_3 = \frac{1}{3^{1.3}} < \int_2^3 \frac{1}{x^{1.3}}\,dx$, and so on, so $\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^{\infty} \frac{1}{x^{1.3}}\,dx$. The integral converges by (.2) with $p=1.3>1$, so the series converges.



2. From the first figure, we see that $\int_1^6 f(x)dx < \sum_{i=1}^{5} a_i$. From the second figure, we see that $\sum_{i=2}^{6} a_i < \int_1^6 f(x)dx$. Thus, we have $\sum_{i=2}^{6} a_i < \int_1^6 f(x)dx < \sum_{i=1}^{5} a_i$.





3. The function $f(x)=1/x^4$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies. $\int_1^{\infty} \frac{1}{x^4}\,dx = \lim_{t\to\infty} \int_1^t x^{-4}\,dx = \lim_{t\to\infty} \left[\frac{x^{-3}}{-3} \right]_1^t = \lim_{t\to\infty} \left(-\frac{1}{3t^3} + \frac{1}{3} \right) = \frac{1}{3}$. Since this improper integral is convergent, the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is also convergent by the Integral Test.

4. The function $f(x)=1/\sqrt[4]{x}=x^{-1/4}$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral

Test applies. $\int_1^{\infty} x^{-1/4} dx = \lim_{t\to\infty} \int_1^{t} x^{-1/4} dx = \lim_{t\to\infty} \left[\frac{4}{3} x^{3/4} \right]_1^t = \lim_{t\to\infty} \left(\frac{4}{3} t^{3/4} - \frac{4}{3} \right) = \infty$, so $\Sigma_{n=1}^{\infty} 1/\sqrt[4]{n}$ diverges.

5. The function $f(x)=1/(3x+1)$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies.

$$\int_1^{\infty} \frac{}{dx}\, 3x+1 = \lim_{b\to\infty} \int_1^{b} \frac{}{dx}\, 3x+1 = \lim_{b\to\infty} \left[\frac{1}{3} \ln(3x+1) \right]_1^b = \lim_{b\to\infty} \left[\frac{1}{3} \ln(3b+1) - \frac{1}{3} \ln 4 \right] = \infty$$

so the improper integral diverges, and so does the series $\Sigma_{n=1}^{\infty} 1/(3n+1)$.

6. The function $f(x)=e^{-x}$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies. $\int_1^{\infty} e^{-x} dx = \lim_{b\to\infty} \int_1^{b} e^{-x} dx = \lim_{b\to\infty} \left[-e^{-x} \right]_1^b = \lim_{b\to\infty} \left(-e^{-b} + e^{-1} \right) = e^{-1}$, so $\Sigma_{n=1}^{\infty} e^{-n}$ converges. *Note:* This is a geometric series, with first term $a=e^{-1}$ and ratio $r=e^{-1}$. Since $|r|<1$, the series converges to $e^{-1}/(1-e^{-1}) = 1/(e-1)$.

7. $f(x)=xe^{-x}$ is continuous and positive on $[1,\infty)$. $f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x) < 0$ for $x>1$, so f is decreasing on $[1,\infty)$. Thus, the Integral Test applies.

$$\int_1^{\infty} xe^{-x} dx = \lim_{b\to\infty} \int_1^{b} xe^{-x} dx = \lim_{b\to\infty} \left[-xe^{-x} - e^{-x} \right]_1^b \text{(byparts)}$$

$$= \lim_{b\to\infty} \left[-be^{-b} - e^{-b} + e^{-1} + e^{-1} \right] = 2/e$$

since $\lim_{b\to\infty} be^{-b} = \lim_{b\to\infty} \left(b/e^{b} \right) = \lim_{b\to\infty} \left(1/e^{b} \right) = 0$ and $\lim_{b\to\infty} e^{-b} = 0$. Thus, $\Sigma_{n=1}^{\infty} ne^{-n}$ converges.

8. The function $f(x) = \frac{x+2}{x+1} = 1 + \frac{1}{x+1}$ is continuous, positive, and decreasing on $[1,\infty)]$, so the Integral Test applies. $\int_1^{\infty} f(x)dx = \lim_{t\to\infty} \int_1^{t} \left(1 + \frac{1}{x+1} \right) dx = \lim_{t\to\infty} [x+\ln(x+1)]_1^t = \lim_{t\to\infty} (t+\ln(t+1)-1-\ln 2) = \infty$, so $\int_1^{\infty} \frac{x+2}{x+1} dx$ is divergent and the series $\Sigma_{n=1}^{\infty} \frac{n+2}{n+1}$ is divergent. NOTE: $\lim_{n\to\infty} \frac{n+2}{n+1} = 1$, so the given series diverges by the Test for Divergence.

9. The series $\sum_{n=1}^{\infty} \frac{1}{n^{0.85}}$ is a p –series with $p=0.85\leq 1$, so it diverges by (1). Therefore, the series $\sum_{n=1}^{\infty} \frac{2}{n^{0.85}}$ must also diverge, for if it converged, then $\sum_{n=1}^{\infty} \frac{1}{n^{0.85}}$ would have to converge (by Theorem 8(i) in Section 11.2).

10. $\sum_{n=1}^{\infty} n^{-1.4}$ and $\sum_{n=1}^{\infty} n^{-1.2}$ are p –series with $p>1$, so they converge by (1). Thus, $\sum_{n=1}^{\infty} 3n^{-1.2}$ converges by Theorem 8(i) in Section 11.2. It follows from Theorem 8(ii) that the given series $\sum_{n=1}^{\infty}(n^{-1.4}+3n^{-1.2})$ also converges.

11. $1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\frac{1}{125}+\cdots=\sum_{n=1}^{\infty} \frac{1}{n^{3}}$. This is a p –series with $p=3>1$, so it converges by (1).

12. $1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+\frac{1}{5\sqrt{5}}+\cdots=\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}=\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. This is a p –series with $p=\frac{3}{2}>1$, so it converges by (1).

13. $\sum_{n=1}^{\infty} \frac{5-2\sqrt{n}}{n^{3}}=5\sum_{n=1}^{\infty} \frac{1}{n^{3}}-2\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ by Theorem .2.8, since $\sum_{n=1}^{\infty} \frac{1}{n^{3}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ both converge by (1) (with $p=3>1$ and $p=\frac{5}{2}>1$). Thus, $\sum_{n=1}^{\infty} \frac{5-2\sqrt{n}}{n^{3}}$ converges.

14. The function $f(x)=\frac{5}{x-2}$ is continuous, positive, and decreasing on $[3,\infty)]$, so we can apply the Integral Test. $\int_{3}^{\infty} \frac{5}{x-2}\, dx=\lim_{t\to\infty} \int_{3}^{t} \frac{5}{x-2}\, dx=\lim_{t\to\infty} [5\ln (x-2)]_{3}^{t}=\lim_{t\to\infty} [5\ln (t-2)-0]=\infty$, so the series $\sum_{n=3}^{\infty} \frac{5}{n-2}$ diverges.

15. The function $f(x)=\frac{1}{x^{2}+4}$ is continuous, positive, and decreasing on $[1,\infty)]$, so we can apply the Integral Test.

$$\int_{1}^{\infty} \frac{1}{x^{2}+4}\, dx = \lim_{t\to\infty} \int_{1}^{t} \frac{1}{x^{2}+4}\, dx=\lim_{t\to\infty} \left[\frac{1}{2} \tan^{-1} \frac{x}{2} \right]_{1}^{t}=\frac{1}{2} \lim_{t\to\infty} \left[\tan^{-1} \left(\frac{t}{2} \right)-\tan^{-1} \left(\frac{1}{2} \right) \right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)\right]$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ converges.

16. The function $f(x) = \frac{3x+2}{x(x+1)} = \frac{2}{x} + \frac{1}{x+1}$ is continuous, positive, and decreasing on $[1,\infty)]$ since it is the sum of two such functions. Thus, we can apply the Integral Test.

$$\int_1^{\infty} \frac{3x+2}{x(x+1)}\,dx = \lim_{t\to\infty}\int_1^t \left[\frac{2}{x} + \frac{1}{x+1}\right]dx = \lim_{t\to\infty}\,[2\ln x + \ln(x+1)]_1^t$$

$$= \lim_{t\to\infty}\,[2\ln t + \ln(t+1) - \ln 2] = \infty$$

Thus, the series $\sum_{n=1}^{\infty} \frac{3n+2}{n(n+1)}$ diverges.

17. $f(x) = \frac{x}{x^2+1}$ is continuous and positive on $[1,\infty)$, and since $f'(x) = \frac{1-x^2}{\left(x^2+1\right)^2} < 0$ for $x>1$, f is also decreasing. Using the Integral Test,

$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{t\to\infty}\int_1^t \frac{x}{x^2+1}\,dx = \lim_{t\to\infty}\left[\frac{\ln(x^2+1)}{2}\right]_1^t = \frac{1}{2}\lim_{t\to\infty}\,[\ln(t^2+1) - \ln 2] = \infty$, so the series diverges.

18. The function $f(x) = \frac{1}{x^2-4x+5} = \frac{1}{(x-2)^2+1}$ is continuous, positive, and decreasing on $[2,\infty)]$, so the Integral Test applies.

$$\int_2^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_2^t f(x)\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{(x-2)^2+1}\,dx = \lim_{t\to\infty}\,[\tan^{-1}(x-2)]_2^t = \lim_{t\to\infty}\,[\tan^{-1}(t-2) - \tan^{-1}0] = \frac{\pi}{2} - 0 = \frac{\pi}{2}\ ,$$

so the series $\sum_{n=2}^{\infty} \frac{1}{n^2-4n+5}$ converges. Of course this means that $\sum_{n=1}^{\infty} \frac{1}{n^2-4n+5}$ converges too.

19. $f(x) = xe^{-x^2}$ is continuous and positive on $[1,\infty)$, and since $f'(x) = e^{-x^2}\left(1-2x^2\right) < 0$ for $x>1$, f is decreasing as well. Thus, we can use the Integral Test.

$\int_1^{\infty} xe^{-x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_1^t = 0 - \left(-\frac{1}{2}e^{-1}\right) = 1/(2e)$. Since the integral converges, the series converges.

20. $f(x)=\frac{\ln x}{x^2}$ is continuous and positive for $x\geq 2$, and $f'(x)=\frac{1-2\ln x}{x^3}<0$ for $x\geq 2$, so f is decreasing. $\int_2^{\infty} \frac{\ln x}{x^2}\,dx=\lim_{t\to\infty}\left[-\frac{\ln x}{x}-\frac{1}{x}\right]_2^t=1$. Thus, $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}=\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$ converges by the Integral Test.

21. $f(x)=\frac{1}{x\ln x}$ is continuous and positive on $[2,\infty)$, and also decreasing since $f'(x)=-\frac{1+\ln x}{x^2(\ln x)^2}<0$ for $x>2$, so we can use the Integral Test. $\int_2^{\infty}\frac{1}{x\ln x}\,dx=\lim_{t\to\infty}[\ln(\ln x)]_2^t=\lim_{t\to\infty}[\ln(\ln t)-\ln(\ln 2)]=\infty$, so the series diverges.

22. The function $f(x)=\frac{x}{x^4+1}$ is positive, continuous, and decreasing on $[1,\infty)]$. Thus, we can apply the Integral Test.

$$\int_1^{\infty}\frac{x}{x^4+1}\,dx=\lim_{t\to\infty}\int_1^t\frac{\frac{1}{2}(2x)}{1+(x^2)^2}\,dx=\lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}(x^2)\right]_1^t=\frac{1}{2}\lim_{t\to\infty}[\tan^{-1}(t^2)-\tan^{-1}1]$$
$$=\frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)=\frac{\pi}{8}$$

so the series $\sum_{n=1}^{\infty}\frac{n}{n^4+1}$ converges.

23. The function $f(x)=\frac{1}{x^3+x}$ is continuous, positive, and decreasing on $[1,\infty)]$, so the Integral Test applies. We use partial fractions to evaluate the integral:

$$\int_1^{\infty}\frac{1}{x^3+x}\,dx=\lim_{t\to\infty}\int_1^t\left[\frac{1}{x}-\frac{x}{1+x^2}\right]dx=\lim_{t\to\infty}\left[\ln x-\frac{1}{2}\ln(1+x^2)\right]_1^t$$
$$=\lim_{t\to\infty}\left[\ln\frac{x}{\sqrt{1+x^2}}\right]_1^t=\lim_{t\to\infty}\left(\ln\frac{t}{\sqrt{1+t^2}}-\ln\frac{1}{\sqrt{2}}\right)$$

$$= \lim_{t\to\infty}\left(\ln \frac{1}{\sqrt{1+1/t^2}} + \frac{1}{2}\ln 2 \right) = \frac{1}{2}\ln 2$$

so the series $\sum_{n=1}^{\infty} \frac{1}{n^3+n}$ converges.

24. $f(x)=\frac{1}{x\ln x\ln(\ln x)}$ is positive and continuous on $[3,\infty)$, and is decreasing since x , $\ln x$, and $\ln(\ln x)$ are all increasing; so we can apply the Integral Test.
$\int_3^{\infty} \frac{dx}{x\ln x\ln(\ln x)} = \lim_{t\to\infty} [\ln(\ln(\ln x))]_3^t = \infty$. The integral diverges, so $\sum_{n=3}^{\infty} \frac{1}{n\ln n\ln(\ln n)}$ diverges.

25. We have already shown (in Exercise 21) that when $p=1$ the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ diverges, so assume that $p\neq 1$. $f(x)=\frac{1}{x(\ln x)^p}$ is continuous and positive on $[2,\infty)$, and $f'(x)=-\frac{p+\ln x}{x^2(\ln x)^{p+1}}<0$ if $x>e^{-p}$, so that f is eventually decreasing and we can use the Integral Test.

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^{1-p}}{1-p} \right]_2^t (\text{for } p\neq 1) = \lim_{t\to\infty}\left[\frac{(\ln t)^{1-p}}{1-p} \right] - \frac{(\ln 2)^{1-p}}{1-p}$$

This limit exists whenever $1-p<0 \Leftrightarrow p>1$, so the series converges for $p>1$.

26. As in Exercise 24, we can apply the Integral Test. $\int_3^{\infty} \frac{dx}{x\ln x(\ln\ln x)^p} = \lim_{t\to\infty}\left[\frac{(\ln\ln x)^{-p+1}}{-p+1} \right]_3^t$ (for $p\neq 1$; if $p=1$ see Exercise 24) and $\lim_{t\to\infty} \frac{(\ln\ln t)^{-p+1}}{-p+1}$ exists whenever $-p+1<0 \Leftrightarrow p>1$, so the series converges for $p>1$.

27. Clearly the series cannot converge if $p\geq -\frac{1}{2}$, because then $\lim_{n\to\infty} n(1+n^2)^p \neq 0$. Also, if $p=-1$ the series diverges (see Exercise 17). So assume $p<-\frac{1}{2}$, $p\neq -1$. Then $f(x)=x(1+x^2)^p$ is continuous,

positive, and eventually decreasing on $[1,\infty)$, and we can use the Integral Test.
$\int_1^\infty x(1+x^2)^p dx = \lim_{t\to\infty} \left[\frac{1}{2} \cdot \frac{(1+x^2)^{p+1}}{p+1} \right]_1^t = \lim_{t\to\infty} \frac{1}{2} \cdot \frac{(1+t^2)^{p+1}}{p+1} - \frac{2^p}{p+1}$. This limit exists and is finite $\Leftrightarrow$ $p+1<0 \Leftrightarrow p<-1$, so the series converges whenever $p<-1$.

28. If $p\leq 0$, $\lim_{n\to\infty} \frac{\ln n}{n^p} = \infty$ and the series diverges, so assume $p>0$. $f(x)=\frac{\ln x}{x^p}$ is positive and continuous and $f'(x)<0$ for $x>e^{1/p}$, so f is eventually decreasing and we can use the Integral Test.
Integration by parts gives $\int_1^\infty \frac{\ln x}{x^p} dx = \lim_{t\to\infty} \left[\frac{x^{1-p}[(1-p)\ln x - 1]}{(1-p)^2} \right]_1^t$ (for $p\neq 1$)
$= \frac{1}{(1-p)^2} \left[\lim_{t\to\infty} t^{1-p}[(1-p)\ln t - 1] + 1 \right]$, which exists whenever $1-p<0 \Leftrightarrow p>1$. Since we have already done the case $p=1$ in Exercise 25 (set $p=-1$ in that exercise), $\sum_{n=1}^\infty \frac{\ln n}{n^p}$ converges $\Leftrightarrow p>1$.

29. Since this is a $p-$ series with $p=x$, $\zeta(x)$ is defined when $x>1$. Unless specified otherwise, the domain of a function f is the set of numbers x such that the expression for $f(x)$ makes sense and defines a real number. So, in the case of a series, it's the set of numbers x such that the series is convergent.

30. **(a)** $f(x)=1/x^4$ is positive and continuous and $f'(x)=-4/x^5$ is negative for $x>0$, and so the Integral Test applies. $\sum_{n=1}^\infty \frac{1}{n^4} \approx s_{10} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \frac{1}{10^4} \approx 1.082037$.

$R_{10} \leq \int_{10}^\infty \frac{1}{x^4} dx = \lim_{t\to\infty} \left[\frac{1}{-3x^3} \right]_{10}^t = \lim_{t\to\infty} \left(-\frac{1}{3t^3} + \frac{1}{3(10)^3} \right) = \frac{1}{3000}$, so the error is at most $0.000\overline{3}$.

(b) $s_{10} + \int_{11}^\infty \frac{1}{x^4} dx \leq s \leq s_{10} + \int_{10}^\infty \frac{1}{x^4} dx \Rightarrow s_{10} + \frac{1}{3(11)^3} \leq s \leq s_{10} + \frac{1}{3(10)^3} \Rightarrow$
$1.082037+0.000250=1.082287 \leq s \leq 1.082037+0.000333=1.082370$, so we get $s\approx 1.08233$ with error ≤ 0.00005 .

(c) $R_n \leq \int_n^\infty \frac{1}{x^4} dx = \frac{1}{3n^3}$. So $R_n<0.00001 \Rightarrow \frac{1}{3n^3} < \frac{1}{10^5} \Rightarrow 3n^3>10^5 \Rightarrow n>\sqrt[3]{(10)^5/3} \approx 32.2$, that is, for

$n>32$.

31. **(a)** $f(x)=\dfrac{1}{x^2}$ is positive and continuous and $f'(x)=-\dfrac{2}{x^3}$ is negative for $x>0$, and so the Integral Test applies. $\sum_{n=1}^{\infty}\dfrac{1}{n^2}\approx s_{10}=\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\cdots+\dfrac{1}{10^2}\approx 1.549768$.

$R_{10}\leq\int_{10}^{\infty}\dfrac{1}{x^2}\,dx=\lim_{t\to\infty}\left[\dfrac{-1}{x}\right]_{10}^{t}=\lim_{t\to\infty}\left(-\dfrac{1}{t}+\dfrac{1}{10}\right)=\dfrac{1}{10}$, so the error is at most 0.1 .

(b) $s_{10}+\int_{11}^{\infty}\dfrac{1}{x^2}\,dx\leq s\leq s_{10}+\int_{10}^{\infty}\dfrac{1}{x^2}\,dx\Rightarrow s_{10}+\dfrac{1}{11}\leq s\leq s_{10}+\dfrac{1}{10}\Rightarrow$

$1.549768+0.090909=1.640677\leq s\leq 1.549768+0.1=1.649768$, so we get $s\approx 1.64522$ (the average of 1.640677 and 1.649768) with error ≤ 0.005 (the maximum of $1.649768-1.64522$ and $1.64522-1.640677$, rounded up).

(c) $R_n\leq\int_{n}^{\infty}\dfrac{1}{x^2}\,dx=\dfrac{1}{n}$. So $R_n<0.001$ if $\dfrac{1}{n}<\dfrac{1}{1000}\Leftrightarrow n>1000$.

32. $f(x)=1/x^5$ is positive and continuous and $f'(x)=-5/x^6$ is negative for $x>0$, and so the Integral Test applies. Using (3), $R_n\leq\int_{n}^{\infty}x^{-5}dx=\lim_{t\to\infty}\left[\dfrac{-1}{4x^4}\right]_{n}^{t}=\dfrac{1}{4n^4}$. If we take $n=5$, then $s_5\approx 1.036662$ and $R_5\leq 0.0004$. So $s\approx s_5\approx 1.037$.

33. $f(x)=x^{-3/2}$ is positive and continuous and $f'(x)=-\dfrac{3}{2}x^{-5/2}$ is negative for $x>0$, so the Integral Test applies. From the end of Example 6, we see that the error is at most half the length of the interval. From (3), the interval is $\left(s_n+\int_{n+1}^{\infty}f(x)\mathrm{dx},s_n+\int_{n}^{\infty}f(x)dx\right)$, so its length is □ . Thus, we need n such that

$$0.01>\frac{1}{2}\int_{n}^{n+1}x^{-3/2}dx=\frac{1}{2}\left[\frac{-2}{\sqrt{x}}\right]_{n}^{n+1}=\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}$$

$\Leftrightarrow n>13.08$ (use a graphing calculator to solve $1/\sqrt{x}-1/\sqrt{x+1}<0.01$). Again from the end of Example 6, we approximate s by the midpoint of this interval. In general, the midpoint is $\dfrac{1}{2}\left[\left(s_n+\int_{n+1}^{\infty}f(x)dx\right)+\left(s_n+\int_{n}^{\infty}f(x)dx\right)\right]=s_n+\dfrac{1}{2}\left(\int_{n+1}^{\infty}f(x)dx+\int_{n}^{\infty}f(x)dx\right)$. So using $n=14$, we have $s\approx s_{14}+\frac{1}{2}\left(\int_{14}^{\infty} x^{-3/2}dx+\int_{15}^{\infty} x^{-3/2}dx\right)\approx 2.0872+\frac{1}{\sqrt{14}}+\frac{1}{\sqrt{15}}\approx 2.6127\approx 2.61$. Any larger value of n will also work. For instance, $s\approx s_{30}+\frac{1}{\sqrt{30}}+\frac{1}{\sqrt{31}}\approx 2.6124$.

34. $f(x)=\frac{1}{x(\ln x)^2}$ is positive and continuous and $f'(x)=-\frac{\ln x+2}{x^2(\ln x)^3}$ is negative for $x>1$, so the Integral Test applies. Using (2), we need $0.01>\int_n^{\infty}\frac{dx}{x(\ln x)^2}=\lim_{t\to\infty}\left[\frac{-1}{\ln x}\right]_n^t=\frac{1}{\ln n}$. This is true for $n>e^{100}$, so we would have to take this many terms, which would be problematic because $e^{100}\approx 2.7\times 10^{43}$.

35. $\sum_{n=1}^{\infty} n^{-1.001}=\sum_{n=1}^{\infty}\frac{1}{n^{1.001}}$ is a convergent p –series with $p=1.001>1$. Using (2), we get $R_n\le\int_n^{\infty} x^{-1.001}dx=\lim_{t\to\infty}\left[\frac{x^{-0.001}}{-0.001}\right]_n^t=-1000\lim_{t\to\infty}\left[\frac{1}{x^{0.001}}\right]_n^t=-1000\left(-\frac{1}{n^{0.001}}\right)=\frac{1000}{n^{0.001}}$. We want $R_n<0.000000005\Leftrightarrow\frac{1000}{n^{0.001}}<5\times 10^{-9}\Leftrightarrow n^{0.001}>\frac{1000}{5\times 10^{-9}}\Leftrightarrow$
$n>\left(2\times 10^{11}\right)^{1000}=2^{1000}\times 10^{11,000}\approx 1.07\times 10^{301}\times 10^{11,000}=1.07\times 10^{11,301}$.

36. **(a)** $f(x)=\left(\frac{\ln x}{x}\right)^2$ is continuous and positive for $x>1$, and since $f'(x)=\frac{2\ln x(1-\ln x)}{x^3}<0$ for $x>e$, we can apply the Integral Test. Using a CAS, we get $\int_1^{\infty}\left(\frac{\ln x}{x}\right)^2dx=2$, so the series also converges.

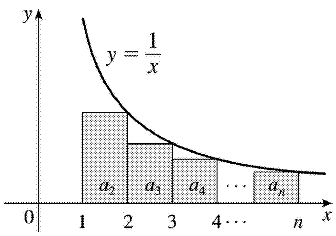
(b) Since the Integral Test applies, the error in $s\approx s_n$ is $R_n\le\int_n^{\infty}\left(\frac{\ln x}{x}\right)^2dx=\frac{(\ln n)^2+2\ln n+2}{n}$.

(c) By graphing the functions $y_1=\frac{(\ln x)^2+2\ln x+2}{x}$ and $y_2=0.05$, we see that $y_1<y_2$ for $n\ge 1373$.

(d) Using the CAS to sum the first 1373 terms, we get $s_{1373}\approx 1.94$.

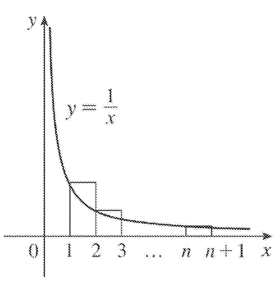
37. **(a)** From the figure,

$a_2+a_3+\cdots+a_n\le\int_1^n f(x)\,dx$, so with $f(x)=\frac{1}{x}$, $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}\le\int_1^n \frac{1}{x}\,dx=\ln n$. Thus,
$s_n=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}\le 1+\ln n$.

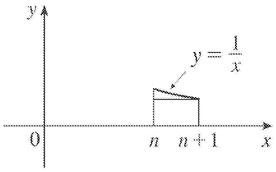


(b) By part (a), $s_{10^6}\le 1+\ln 10^6\approx 14.82<15$ and $s_{10^9}\le 1+\ln 10^9\approx 21.72<22$.

38. **(a)** The sum of the areas of the n rectangles in the graph to the right is $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$. Now $\int_1^{n+1}\frac{dx}{x}$ is less than this sum because the rectangles extend above the curve $y=1/x$, so $\int_1^{n+1}\frac{1}{x}\,dx=\ln(n+1)<1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$, and since $\ln n<\ln(n+1)$, $0<1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n=t_n$.



(b) The area under $f(x)=1/x$ between $x=n$ and $x=n+1$ is $\int_n^{n+1}\frac{dx}{x}=\ln(n+1)-\ln n$, and this is clearly greater than the area of the inscribed rectangle in the figure to the right $\left[\text{which is }\frac{1}{n+1}\right]$, so $t_n-t_{n+1}=[\ln(n+1)-\ln n]-\frac{1}{n+1}>0$, and so $t_n>t_{n+1}$, so $\{t_n\}$ is a decreasing sequence.



(c) We have shown that $\{t_n\}$ is decreasing and that $t_n>0$ for all n . Thus, $0<t_n\leq t_1=1$, so $\{t_n\}$ is a bounded monotonic sequence, and hence converges by Theorem .1.11.

39. $b^{\ln n}=\left(e^{\ln b}\right)^{\ln n}=\left(e^{\ln n}\right)^{\ln b}=n^{\ln b}=\frac{1}{n^{-\ln b}}$. This is a $p-$ series, which converges for all b such that $-\ln b>1\Leftrightarrow\ln b<-1\Leftrightarrow b<e^{-1}\Leftrightarrow b<1/e$.

1. **(a)** We cannot say anything about Σa_n . If $a_n > b_n$ for all n and Σb_n is convergent, then Σa_n could be convergent or divergent. (See the note after Example 2.)
(b) If $a_n < b_n$ for all n , then Σa_n is convergent.

2. **(a)** If $a_n > b_n$ for all n , then Σa_n is divergent.
(b) We cannot say anything about Σa_n . If $a_n < b_n$ for all n and Σb_n is divergent, then Σa_n could be convergent or divergent.

3. $\frac{1}{n^2+n+1} < \frac{1}{n^2}$ for all $n \geq 1$, so $\Sigma_{n=1}^{\infty} \frac{1}{n^2+n+1}$ converges by comparison with $\Sigma_{n=1}^{\infty} \frac{1}{n^2}$, which converges because it is a p –series with $p=2>1$.

4. $\frac{2}{n^3+4} < \frac{2}{n^3}$ for all $n \geq 1$, so $\Sigma_{n=1}^{\infty} \frac{2}{n^3+4}$ converges by comparison with $\Sigma_{n=1}^{\infty} \frac{2}{n^3} = 2\Sigma_{n=1}^{\infty} \frac{1}{n^3}$, which converges because it is a constant multiple of a convergent p –series ($p=3>1$).

5. $\frac{5}{2+3^n} < \frac{5}{3^n}$ for all $n \geq 1$, so $\Sigma_{n=1}^{\infty} \frac{5}{2+3^n}$ converges by comparison with $\Sigma_{n=1}^{\infty} \frac{5}{3^n} = 5\Sigma_{n=1}^{\infty} \frac{1}{3^n}$, which converges because $\Sigma_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series with $r=\frac{1}{3}$ ($|r|<1$).

6. $\frac{1}{n-\sqrt{n}} > \frac{1}{n}$ for all $n \geq 2$, so $\Sigma_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$ diverges by comparison with the divergent (partial) harmonic series $\Sigma_{n=2}^{\infty} \frac{1}{n}$.

7. $\frac{n+1}{n^2} > \frac{n}{n^2} = \frac{1}{n}$ for all $n \geq 1$, so $\Sigma_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges by comparison with the harmonic series $\Sigma_{n=1}^{\infty} \frac{1}{n}$.

8. $\frac{4+3^n}{2^n} > \frac{3^n}{2^n} = \left(\frac{3}{2} \right)^n$ for all $n \geq 1$, so $\Sigma_{n=1}^{\infty} \frac{4+3^n}{2^n}$ diverges by comparison with the divergent geometric series $\Sigma_{n=1}^{\infty} \left(\frac{3}{2} \right)^n$.

9. $\dfrac{\cos^2 n}{n^2+1} \le \dfrac{1}{n^2+1} < \dfrac{1}{n^2}$, so the series $\sum_{n=1}^{\infty} \dfrac{\cos^2 n}{n^2+1}$ converges by comparison with the p –series $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ $(p=2>1)$.

10. $\dfrac{n^2-1}{3n^4+1} < \dfrac{n^2}{3n^4+1} < \dfrac{n^2}{3n^4} = \dfrac{1}{3}\,\dfrac{1}{n^2}$. $\sum_{n=1}^{\infty} \dfrac{n^2-1}{3n^4+1}$ converges by comparison with $\sum_{n=1}^{\infty} \dfrac{1}{3n^2}$, which converges because it is a constant multiple of a convergent p –series $(p=2>1)$. The terms of the given series are positive for $n>1$, which is good enough.

11. If $a_n = \dfrac{n^2+1}{n^3-1}$ and $b_n = \dfrac{1}{n}$, then $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{n^3+n}{n^3-1} = \lim_{n\to\infty} \dfrac{1+1/n^2}{1-1/n^3} = 1$, so $\sum_{n=2}^{\infty} \dfrac{n^2+1}{n^3-1}$ diverges by the Limit Comparison Test with the divergent (partial) harmonic series $\sum_{n=2}^{\infty} \dfrac{1}{n}$. *Or:* Since $a_n = \dfrac{n^2+1}{n^3-1} > \dfrac{n^2+1}{n^3} > \dfrac{n^2}{n^3} = \dfrac{1}{n} = b_n$, we could use the Comparison Test.

12. $\dfrac{1+\sin n}{10^n} \le \dfrac{2}{10^n}$ and $\sum_{n=0}^{\infty} \dfrac{2}{10^n} = 2\sum_{n=0}^{\infty} \left(\dfrac{1}{10}\right)^n$, so the given series converges by comparison with a constant multiple of a convergent geometric series.

13. $\dfrac{n-1}{n4^n}$ is positive for $n>1$ and $\dfrac{n-1}{n4^n} < \dfrac{n}{n4^n} = \dfrac{1}{4^n}$, so $\sum_{n=1}^{\infty} \dfrac{n-1}{n4^n}$ converges by comparison with the convergent geometric series $\sum_{n=1}^{\infty} \left(\dfrac{1}{4}\right)^n$.

14. $\dfrac{\sqrt{n}}{n-1} > \dfrac{\sqrt{n}}{n} = \dfrac{1}{\sqrt{n}}$, so $\sum_{n=2}^{\infty} \dfrac{\sqrt{n}}{n-1}$ diverges by comparison with the divergent (partial) p –series $\sum_{n=2}^{\infty} \dfrac{1}{\sqrt{n}}$ $\left(p=\dfrac{1}{2}\le 1\right)$.

15. $\dfrac{2+(-1)^n}{n\sqrt{n}} \le \dfrac{3}{n\sqrt{n}}$, and $\sum_{n=1}^{\infty} \dfrac{3}{n\sqrt{n}}$ converges because it is a constant multiple of the convergent p –series $\sum_{n=1}^{\infty} \dfrac{1}{n\sqrt{n}}$ $\left(p=\dfrac{3}{2}>1\right)$, so the given series converges by the Comparison Test.

16. $\frac{1}{\sqrt{n^3+1}} < \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$, so $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ converges by comparison with the convergent p – series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \left(p=\frac{3}{2}>1 \right)$.

17. Use the Limit Comparison Test with $a_n = \frac{1}{\sqrt{n^2+1}}$ and $b_n = \frac{1}{n}$:

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1>0$. Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$.

18. Use the Limit Comparison Test with $a_n = \frac{1}{2n+3}$ and $b_n = \frac{1}{n}$:

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{2n+3} = \lim_{n\to\infty} \frac{1}{2+(3/n)} = \frac{1}{2} >0$. Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty} \frac{1}{2n+3}$.

19. $\frac{2^n}{1+3^n} < \frac{2^n}{3^n} = \left(\frac{2}{3} \right)^n$. $\sum_{n=1}^{\infty} \left(\frac{2}{3} \right)^n$ is a convergent geometric series ($|r| = \frac{2}{3} <1$), so $\sum_{n=1}^{\infty} \frac{2^n}{1+3^n}$ converges by the Comparison Test.

20. Use the Limit Comparison Test with $a_n = \frac{1+2^n}{1+3^n}$ and $b_n = \frac{2^n}{3^n}$: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(1/2)^n+1}{(1/3)^n+1} = 1>0$.

Since $\sum_{n=1}^{\infty} b_n$ converges (geometric series with $|r| = \frac{2}{3} <1$), $\sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$ also converges.

21. Use the Limit Comparison Test with $a_n = \frac{1}{1+\sqrt{n}}$ and $b_n = \frac{1}{\sqrt{n}}$: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = 1>0$.

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p– series ($p=\frac{1}{2} \leq 1$), $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ also diverges.

22. Use the Limit Comparison Test with $a_n = \frac{n+2}{(n+1)^3}$ and $b_n = \frac{1}{n^2}$:

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2(n+2)}{(n+1)^3} = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{\left(1+\frac{1}{n}\right)^3} = 1>0$. Since $\sum_{n=3}^{\infty} \frac{1}{n^2}$ is a convergent (partial) p –series $(p=2>1)$, the series $\sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$ also converges.

23. Use the Limit Comparison Test with $a_n = \frac{5+2n}{(1+n^2)^2}$ and $b_n = \frac{1}{n^3}$:

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3(5+2n)}{(1+n^2)^2} = \lim_{n\to\infty} \frac{5n^3+2n^4}{(1+n^2)^2} \cdot \frac{1/n^4}{1/(n^2)^2} = \lim_{n\to\infty} \frac{\frac{5}{n}+2}{\left(\frac{1}{n^2}+1\right)^2} = 2>0$. Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p –series $(p=3>1)$, the series $\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$ also converges.

24. If $a_n = \frac{n^2-5n}{n^3+n+1}$ and $b_n = \frac{1}{n}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3-5n^2}{n^3+n+1} = \lim_{n\to\infty} \frac{1-5/n}{1+1/n^2+1/n^3} = 1>0$, so $\sum_{n=1}^{\infty} \frac{n^2-5n}{n^3+n+1}$ diverges by the Limit Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.
(Note that $a_n>0$ for $n \geq 6$.)

25. If $a_n = \frac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ and $b_n = \frac{1}{n}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+n^2+n^3}{\sqrt{1+n^2+n^6}} = \lim_{n\to\infty} \frac{1/n^2+1/n+1}{\sqrt{1/n^6+1/n^4+1}} = 1>0$, so $\sum_{n=1}^{\infty} \frac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ diverges by the Limit Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

26. If $a_n = \dfrac{n+5}{\sqrt[3]{n^7+n^2}}$ and $b_n = \dfrac{n}{\sqrt[3]{n^7}} = \dfrac{n}{n^{7/3}} = \dfrac{1}{n^{4/3}}$, then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^{7/3}+5n^{4/3}}{\left(n^7+n^2\right)^{1/3}}\cdot\frac{n^{-7/3}}{n^{-7/3}} = \lim_{n\to\infty}\frac{1+5/n}{\left[\left(n^7+n^2\right)/n^7\right]^{1/3}}$$

$$= \lim_{n\to\infty}\frac{1+5/n}{\left(1+1/n^5\right)^{1/3}} = \frac{1+0}{(1+0)^{1/3}} = 1>0,$$

so $\Sigma_{n=1}^{\infty}\dfrac{n+5}{\sqrt[3]{n^7+n^2}}$ converges by the Limit Comparison Test with the convergent p –series $\Sigma_{n=1}^{\infty}\dfrac{1}{n^{4/3}}$

.

27. Use the Limit Comparison Test with $a_n = \left(1+\dfrac{1}{n}\right)^2 e^{-n}$ and $b_n = e^{-n}$:

$\lim_{n\to\infty}\dfrac{a_n}{b_n} = \lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^2 = 1>0$. Since $\Sigma_{n=1}^{\infty} e^{-n} = \Sigma_{n=1}^{\infty}\dfrac{1}{e^n}$ is a convergent geometric series $\left(|r| = \dfrac{1}{e} < 1\right)$, the series $\Sigma_{n=1}^{\infty}\left(1+\dfrac{1}{n}\right)^2 e^{-n}$ also converges.

28. Use the Limit Comparison Test with $a_n = \dfrac{2n^2+7n}{3^n\left(n^2+5n-1\right)}$ and $b_n = \dfrac{1}{3^n}$.

$\lim_{n\to\infty}\dfrac{a_n}{b_n} = \lim_{n\to\infty}\dfrac{2n^2+7n}{n^2+5n-1} = 2>0$, and since $\Sigma_{n=1}^{\infty} b_n$ is a convergent geometric series ($|r| = \dfrac{1}{3} < 1$), $\Sigma_{n=1}^{\infty}\dfrac{2n^2+7n}{3^n(n^2+5n-1)}$ converges also.

29. Clearly $n! = n(n-1)(n-2)\cdot\cdot\cdot(3)(2) \geq 2\cdot 2\cdot 2\cdot\cdot\cdot\cdot\cdot 2\cdot 2 = 2^{n-1}$, so $\dfrac{1}{n!} \leq \dfrac{1}{2^{n-1}}$. $\Sigma_{n=1}^{\infty}\dfrac{1}{2^{n-1}}$ is a convergent geometric series ($|r| = \dfrac{1}{2} < 1$), so $\Sigma_{n=1}^{\infty}\dfrac{1}{n!}$ converges by the Comparison Test.

30. $\dfrac{n!}{n^n} = \dfrac{1\cdot 2\cdot 3\cdot\cdot\cdot\cdot\cdot(n-1)n}{n\cdot n\cdot n\cdot\cdot\cdot\cdot\cdot n\cdot n} \leq \dfrac{1}{n}\cdot\dfrac{2}{n}\cdot 1\cdot 1\cdot\cdot\cdot\cdot\cdot 1$ for $n\geq 2$, so since $\Sigma_{n=1}^{\infty}\dfrac{2}{n^2}$ converges ($p=2>1$),

$\sum_{n=1}^{\infty} \dfrac{n!}{n^n}$ converges also by the Comparison Test.

31. Use the Limit Comparison Test with $a_n = \sin\left(\dfrac{1}{n}\right)$ and $b_n = \dfrac{1}{n}$. Then $\sum a_n$ and $\sum b_n$ are series with positive terms and $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{\sin\,(1/n)}{1/n} = \lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} \sin\,(1/n)$ also diverges. (Note that we could also use l'Hospital's Rule to evaluate the limit: $\lim_{x\to\infty} \dfrac{\sin\,(1/x)}{1/x} = \lim_{x\to\infty} \dfrac{\cos\,(1/x)\cdot\left(-1/x^2\right)}{-1/x^2} = \lim_{x\to\infty} \cos\,\dfrac{1}{x} = \cos\,0 = 1$.)

32. Use the Limit Comparison Test with $a_n = \dfrac{1}{n^{1+1/n}}$ and $b_n = \dfrac{1}{n}$. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{n}{n^{1+1/n}} = \lim_{n\to\infty} \dfrac{1}{n^{1/n}} = 1$ (since $\lim_{x\to\infty} x^{1/x} = 1$ by l'Hospital's Rule), so $\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges (harmonic series) $\Rightarrow \sum_{n=1}^{\infty} \dfrac{1}{n^{1+1/n}}$ diverges.

33. $\sum_{n=1}^{10} \dfrac{1}{n^4+n^2} = \dfrac{1}{2} + \dfrac{1}{20} + \dfrac{1}{90} + \cdots + \dfrac{1}{10{,}100} \approx 0.567975$. Now $\dfrac{1}{n^4+n^2} < \dfrac{1}{n^4}$, so using the reasoning and notation of Example 5, the error is

$$R_{10} \le T_{10} = \sum_{n=11}^{\infty} \frac{1}{n^4} \le \int_{10}^{\infty} \frac{}{dx}\, x^4 = \lim_{t\to\infty} \left[-\frac{x^{-3}}{3} \right]_{10}^{t} = \frac{1}{3000} = 0.000\overline{3}\ .$$

34. $\sum_{n=1}^{10} \dfrac{1+\cos n}{n^5} = 1+\cos 1 + \dfrac{1+\cos 2}{32} + \dfrac{1+\cos 3}{243} + \cdots + \dfrac{1+\cos 10}{100{,}000} \approx 1.55972$. Now $\dfrac{1+\cos n}{n^5} \le \dfrac{2}{n^5}$,

so as in Example 5, $R_{10} \le T_{10} \le \displaystyle\int_{10}^{\infty} \frac{2}{x^5}\, dx = 2\lim_{t\to\infty} \left[-\frac{1}{4} x^{-4} \right]_{10}^{t} = 0.00005$.

35. $\sum_{n=1}^{10} \dfrac{1}{1+2^n} = \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{9} + \cdots + \dfrac{1}{1025} \approx 0.76352$. Now $\dfrac{1}{1+2^n} < \dfrac{1}{2^n}$, so the error is

$R_{10} \le T_{10} = \sum_{n=11}^{\infty} \frac{1}{2^n} = \frac{1/2^{11}}{1-1/2}$ (geometric series) ≈ 0.00098 .

36. $\sum_{n=1}^{10} \frac{n}{(n+1)3^n} = \frac{1}{6} + \frac{2}{27} + \frac{3}{108} + \cdots + \frac{10}{649{,}539} \approx 0.283597$. Now $\frac{n}{(n+1)3^n} < \frac{n}{n \cdot 3^n} = \frac{1}{3^n}$, so the error is $R_{10} \le T_{10} = \sum_{n=11}^{\infty} \frac{1}{3^n} = \frac{1/3^{11}}{1-1/3} \approx 0.0000085$.

37. Since $\frac{d_n}{10^n} \le \frac{9}{10^n}$ for each n , and since $\sum_{n=1}^{\infty} \frac{9}{10^n}$ is a convergent geometric series ($|r| = \frac{1}{10} < 1$), $0.d_1 d_2 d_3 \ldots = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$ will always converge by the Comparison Test.

38. Clearly, if $p<0$ then the series diverges, since $\lim_{n \to \infty} \frac{1}{n^p \ln n} = \infty$. If $0 \le p \le 1$, then $n^p \ln n \le n \ln n \Rightarrow \frac{1}{n^p \ln n} \ge \frac{1}{n \ln n}$ and $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges (Exercise .3.21), so $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ diverges. If $p>1$, use the Limit Comparison Test with $a_n = \frac{1}{n^p \ln n}$ and $b_n = \frac{1}{n^p}$. $\sum_{n=2}^{\infty} b_n$ converges, and $\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1}{\ln n} = 0$, so $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ also converges. (Or use the Comparison Test, since $n^p \ln n > n^p$ for $n > e$.) In summary, the series converges if and only if $p>1$.

39. Since $\sum a_n$ converges, $\lim_{n \to \infty} a_n = 0$, so there exists N such that $\left| a_n - 0 \right| < 1$ for all $n>N \Rightarrow 0 \le a_n < 1$ for all $n>N \Rightarrow 0 \le a_n^2 \le a_n$. Since $\sum a_n$ converges, so does $\sum a_n^2$ by the Comparison Test.

40. **(a)** Since $\lim_{n \to \infty} \left(a_n / b_n \right) = 0$, there is a number $N>0$ such that $\left| a_n / b_n - 0 \right| < 1$ for all $n>N$, and so $a_n < b_n$ since a_n and b_n are positive. Thus, since $\sum b_n$ converges, so does $\sum a_n$ by the Comparison Test.
(b)

(i) If $a_n = \frac{\ln n}{n^3}$ and $b_n = \frac{1}{n^2}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$, so $\Sigma_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges by part (a).

(ii) If $a_n = \frac{\ln n}{\sqrt{n}\, e^n}$ and $b_n = \frac{1}{e^n}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

Now $\Sigma\, b_n$ is a convergent geometric series with ratio $r=1/e\,(|r|<1)$, so $\Sigma\, a_n$ converges by part (a).

41. **(a)** Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, there is an integer N such that $\frac{a_n}{b_n} > 1$ whenever $n>N$. (Take $M=1$ in Definition .1.5.) Then $a_n > b_n$ whenever $n>N$ and since $\Sigma\, b_n$ is divergent, $\Sigma\, a_n$ is also divergent by the Comparison Test.

(b)

(c) If $a_n = \frac{1}{\ln n}$ and $b_n = \frac{1}{n}$ for $n \geq 2$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty$, so by part (a), $\Sigma_{n=2}^{\infty} \frac{1}{\ln n}$ is divergent.

(d) If $a_n = \frac{\ln n}{n}$ and $b_n = \frac{1}{n}$, then $\Sigma_{n=1}^{\infty} b_n$ is the divergent harmonic series and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \ln n = \lim_{x\to\infty} \ln x = \infty$, so $\Sigma_{n=1}^{\infty} a_n$ diverges by part (a).

42. Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n} = 0$, but $\Sigma\, b_n$ diverges while $\Sigma\, a_n$ converges.

43. $\lim_{n\to\infty} n a_n = \lim_{n\to\infty} \frac{a_n}{1/n}$, so we apply the Limit Comparison Test with $b_n = \frac{1}{n}$. Since $\lim_{n\to\infty} n a_n > 0$ we know that either both series converge or both series diverge, and we also know that

$\sum_{n=0}^{\infty} \frac{1}{n}$ diverges ($p-$ series with $p=1$). Therefore, $\sum a_n$ must be divergent.

44. First we observe that, by l'Hospital's Rule, $\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1}{1+x} = 1$. Also, if $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$ by Theorem .2.6. Therefore, $\lim_{n\to\infty} \frac{\ln\left(1+a_n\right)}{a_n} = \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1 > 0$. We are given that $\sum a_n$ is convergent and $a_n > 0$. Thus, $\sum \ln\left(1+a_n\right)$ is convergent by the Limit Comparison Test.

45. Yes. Since $\sum a_n$ is a convergent series with positive terms, $\lim_{n\to\infty} a_n = 0$ by Theorem .2.6, and $\sum b_n = \sum \sin\left(a_n\right)$ is a series with positive terms (for large enough n). We have $\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{\sin\left(a_n\right)}{a_n} = 1 > 0$ by Theorem 3. .2. Thus, $\sum b_n$ is also convergent by the Limit Comparison Test.

46. Yes. Since $\sum a_n$ converges, its terms approach 0 as $n\to\infty$, so for some integer N , $a_n \leq 1$ for all $n \geq N$. But then $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{N-1} a_n b_n + \sum_{n=N}^{\infty} a_n b_n \leq \sum_{n=1}^{N-1} a_n b_n + \sum_{n=N}^{\infty} b_n$. The first term is a finite sum, and the second term converges since $\sum_{n=1}^{\infty} b_n$ converges. So $\sum a_n b_n$ converges by the Comparison Test.

1. **(a)** An alternating series is a series whose terms are alternately positive and negative.
(b) An alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges if $0<b_{n+1}\leq b_n$ for all n and $\lim_{n\to\infty} b_n=0$. (This is the Alternating Series Test.)
(c) The error involved in using the partial sum s_n as an approximation to the total sum s is the remainder $R_n=s-s_n$ and the size of the error is smaller than b_{n+1} ; that is, $\left|R_n\right|\leq b_{n+1}$. (This is the Alternating Series Estimation Theorem.)

2. $-\frac{1}{3}+\frac{2}{4}-\frac{3}{5}+\frac{4}{6}-\frac{5}{7}+\cdots=\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+2}$. Here $a_n=(-1)^n\frac{n}{n+2}$. Since $\lim_{n\to\infty} a_n\neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

3. $\frac{4}{7}-\frac{4}{8}+\frac{4}{9}-\frac{4}{10}+\frac{4}{11}-\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{4}{n+6}$. Now $b_n=\frac{4}{n+6}>0$, $\left\{b_n\right\}$ is decreasing, and $\lim_{n\to\infty} b_n=0$, so the series converges by the Alternating Series Test.

4. $\sum_{n=2}^{\infty}(-1)^n\frac{1}{\ln n}$. $b_n=\frac{1}{\ln n}$ is positive and $\left\{b_n\right\}$ is decreasing; $\lim_{n\to\infty}\frac{1}{\ln n}=0$, so the series converges by the Alternating Series Test.

5. $b_n=\frac{1}{\sqrt{n}}>0$, $\left\{b_n\right\}$ is decreasing, and $\lim_{n\to\infty} b_n=0$, so the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the Alternating Series Test.

6. $b_n=\frac{1}{3n-1}>0$, $\left\{b_n\right\}$ is decreasing, and $\lim_{n\to\infty} b_n=0$, so the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3n-1}$ converges by the Alternating Series Test.

7. $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}(-1)^n\frac{3n-1}{2n+1}=\sum_{n=1}^{\infty}(-1)^n b_n$. Now $\lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{3-1/n}{2+1/n}=\frac{3}{2}\neq 0$. Since $\lim_{n\to\infty} a_n\neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

8. $b_n=\frac{2n}{4n^2+1}>0$, $\left\{b_n\right\}$ is decreasing , and $\lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{2/n}{4+1/n^2}=0$, so the series $\sum_{n=1}^{\infty}(-1)^n\frac{2n}{4n^2+1}$ converges by the Alternating Series Test. Alternatively, to show that $\left\{b_n\right\}$ is decreasing, we could verify that $\frac{d}{dx}\left(\frac{2x}{4x^2+1}\right)<0$ for $x\geq 1$.

9. $b_n = \dfrac{1}{4n^2+1} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the series $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{4n^2+1}$ converges by the Alternating Series Test.

10. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \dfrac{\sqrt{n}}{1+2\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^n b_n$. Now $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \dfrac{1}{2+1/\sqrt{n}} = \dfrac{1}{2} \neq 0$. Since $\lim_{n\to\infty} a_n \neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

11. $b_n = \dfrac{n^2}{n^3+4} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing for $n \geq 2$ since

$$\left(\frac{x^2}{x^3+4}\right)' = \frac{\left(x^3+4\right)(2x) - x^2\left(3x^2\right)}{\left(x^3+4\right)^2} = \frac{x\left(2x^3+8-3x^3\right)}{\left(x^3+4\right)^2} = \frac{x\left(8-x^3\right)}{\left(x^3+4\right)^2} < 0 \text{ for } x > 2 \text{ . Also,}$$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \dfrac{1/n}{1+4/n^3} = 0$. Thus, the series $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{n^2}{n^3+4}$ converges by the Alternating Series Test.

12. $b_n = \dfrac{e^{1/n}}{n} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing since

$\left(\dfrac{e^{1/x}}{x}\right)' = \dfrac{x \cdot e^{1/x}\left(-1/x^2\right) - e^{1/x} \cdot 1}{x^2} = \dfrac{-e^{1/x}(1+x)}{x^3} < 0$ for $x > 0$. Also, $\lim_{n\to\infty} b_n = 0$ since $\lim_{n\to\infty} e^{1/n} = 1$. Thus, the series $\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{e^{1/n}}{n}$ converges by the Alternating Series Test.

13. $\sum_{n=2}^{\infty} (-1)^n \dfrac{n}{\ln n}$. $\lim_{n\to\infty} \dfrac{n}{\ln n} = \lim_{x\to\infty} \dfrac{x}{\ln x} = \lim_{x\to\infty} \dfrac{1}{1/x} = \infty$, so the series diverges by the Test for Divergence.

14. $\sum_{n=1}^{\infty} (-1)^{n-1} \left(\dfrac{\ln n}{n}\right) = 0 + \sum_{n=2}^{\infty} (-1)^{n-1} \left(\dfrac{\ln n}{n}\right)$. $b_n = \dfrac{\ln n}{n} > 0$ for $n \geq 2$, and if $f(x) = \dfrac{\ln x}{x}$, then $f'(x) = \dfrac{1-\ln x}{x^2} < 0$ for $x > e$, so $\{b_n\}$ is eventually decreasing. Also,

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \dfrac{\ln n}{n} = \lim_{x\to\infty} \dfrac{\ln x}{x} = \lim_{x\to\infty} \dfrac{1/x}{1} = 0$, so the series converges by the Alternating Series Test.

15.

$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{3/4}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$. $b_n = \frac{1}{n^{3/4}}$ is decreasing and positive and $\lim_{n\to\infty} \frac{1}{n^{3/4}} = 0$, so the series converges by the Alternating Series Test.

16. $\sin\left(\frac{n\pi}{2}\right) = 0$ if n is even and $(-1)^k$ if $n=2k+1$, so the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$. $b_n = \frac{1}{(2n+1)!} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} \frac{1}{(2n+1)!} = 0$, so the series converges by the Alternating Series Test.

17. $\sum_{n=1}^{\infty} (-1)^n \sin \frac{\pi}{n}$. $b_n = \sin \frac{\pi}{n} > 0$ for $n \geq 2$ and $\sin \frac{\pi}{n} \geq \sin \frac{\pi}{n+1}$, and $\lim_{n\to\infty} \sin \frac{\pi}{n} = \sin 0 = 0$, so the series converges by the Alternating Series Test.

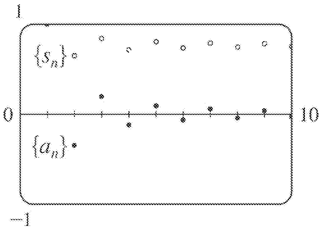
18. $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$. $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1$, so $\lim_{n\to\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$ does not exist and the series diverges by the Test for Divergence.

19. $\frac{n^n}{n!} = \frac{n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot \cdots \cdot n} \geq n \Rightarrow \lim_{n\to\infty} \frac{n^n}{n!} = \infty \Rightarrow \lim_{n\to\infty} \frac{(-1)^n n^n}{n!}$ does not exist. So the series diverges by the Test for Divergence.

20. $\sum_{n=1}^{\infty} \left(-\frac{n}{5}\right)^n$ diverges by the Test for Divergence since $\lim_{n\to\infty} \left(\frac{n}{5}\right)^n = \infty \Rightarrow \lim_{n\to\infty} \left(-\frac{n}{5}\right)^n$ does not exist.

21.

n	a_n	s_n
1	1	1
2	−0.35355	0.64645
3	0.19245	0.83890
4	−0.125	0.71390
5	0.08944	0.80334
6	−0.06804	0.73530
7	0.05399	0.78929
8	−0.04419	0.74510
9	0.03704	0.78214
10	−0.03162	0.75051

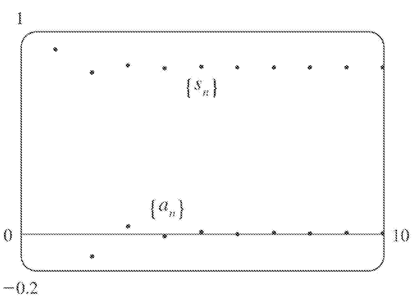


By the Alternating Series Estimation Theorem, the error in the approximation

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3/2}} \approx 0.75051$ is $\left|s-s_{10}\right| \leq b_{11}=1/(11)^{3/2} \approx 0.0275$ (to four decimal places, rounded up).

22.

n	a_n	s_n
1	1	1
2	−0.125	0.875
3	0.03704	0.91204
4	−0.01563	0.89641
5	0.008	0.90441
6	−0.00463	0.89978
7	0.00292	0.90270
8	−0.00195	0.90074
9	0.00137	0.90212
10	−0.001	0.90112



By the Alternating Series Estimation Theorem, the error in the approximation

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3}} \approx 0.90112$ is $\left|s-s_{10}\right| \leq b_{11}=1/11^{3} \approx 0.0007513$.

23. The series $\sum_{n=1}^{\infty}(-1)^{n-1} \frac{1}{n^2}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)^2} < \frac{1}{n^2}$ and (ii) $\lim_{n\to\infty} \frac{1}{n^2} = 0$, so the series is convergent. Now $b_{10} = \frac{1}{10^2} = 0.01$ and $b_{11} = \frac{1}{11^2} = \frac{1}{121} \approx 0.008 < 0.01$, so by the Alternating Series Estimation Theorem, $n=10$. (That is, since the 11 th term is less than the

desired error, we need to add the first 10 terms to get the sum to the desired accuracy.)

24. The series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^4}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)^4}<\frac{1}{n^4}$ and (ii) $\lim_{n\to\infty}\frac{1}{n^4}=0$, so the series is convergent. Now $b_5=1/5^4=0.0016>0.001$ and $b_6=1/6^4\approx 0.00077<0.001$, so by the Alternating Series Estimation Theorem, $n=5$.

25. The series $\sum_{n=1}^{\infty}\frac{(-2)^n}{n!}=\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n!}$ satisfies (i) of the Alternating Series Test because $b_{n+1}=\frac{2^{n+1}}{(n+1)!}=\frac{2\cdot 2^n}{(n+1)n!}=\frac{2}{n+1}\cdot\frac{2^n}{n!}=\frac{2}{n+1}\cdot b_n\leq b_n$ and (ii) $\lim_{n\to\infty}\frac{2^n}{n!}=\frac{2}{n}\cdot\frac{2}{n-1}\cdot\cdots\cdot\frac{2}{2}\cdot\frac{2}{1}=0$, so the series is convergent. Now $b_7=2^7/7!\approx 0.025>0.01$ and $b_8=2^8/8!\approx 0.006<0.01$, so by the Alternating Series Estimation Theorem, $n=7$. (That is, since the 8 th term is less than the desired error, we need to add the first 7 terms to get the sum to the desired accuracy.)

26. The series $\sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}=\sum_{n=1}^{\infty}(-1)^n\frac{n}{4^n}$ satisfies (i) of the Alternating Series Test because $b_{n+1}=\frac{n+1}{4^{n+1}}<\frac{n+3n}{4^n\cdot 4^1}=\frac{4n}{4\cdot 4^n}=\frac{n}{4^n}=b_n$ and (ii) $\lim_{n\to\infty}\frac{n}{4^n}=0$, so the series is convergent. Now $b_5=5/4^5\approx 0.0049>0.002$ and $b_6=6/4^6\approx 0.0015<0.002$, so by the Alternating Series Estimation Theorem, $n=5$.

27. $b_7=\frac{1}{7^5}=\frac{1}{16{,}807}\approx 0.0000595$, so $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}\approx s_6=\sum_{n=1}^{6}\frac{(-1)^{n+1}}{n^5}=1-\frac{1}{32}+\frac{1}{243}-\frac{1}{1024}+\frac{1}{3125}-\frac{1}{7776}\approx 0.972080$. Adding b_7 to s_6 does not change the fourth decimal place of s_6 , so the sum of the series, correct to four decimal places, is 0.9721 .

28. $b_6=\frac{6}{8^6}=\frac{6}{262{,}144}\approx 0.000023$, so

$\sum_{n=1}^{\infty} \frac{(-1)^n n}{8^n} \approx s_5 = \sum_{n=1}^{5} \frac{(-1)^n n}{8^n} = -\frac{1}{8} + \frac{2}{64} - \frac{3}{512} + \frac{4}{4096} - \frac{5}{32{,}768} \approx -0.098785$. Adding b_6 to s_5 does not change the fourth decimal place of s_5 , so the sum of the series, correct to four decimal places, is -0.0988 .

29. $b_7 = \frac{7^2}{10^7} = 0.0000049$, so

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{10^n} \approx s_6 = \sum_{n=1}^{6} \frac{(-1)^{n-1} n^2}{10^n} = \frac{1}{10} - \frac{4}{100} + \frac{9}{1000} - \frac{16}{10{,}000} + \frac{25}{100{,}000} - \frac{36}{1{,}000{,}000} = 0.067614$.
Adding b_7 to s_6 does not change the fourth decimal place of s_6 , so the sum of the series, correct to four decimal places, is 0.0676 .

30. $b_6 = \frac{1}{3^6 \cdot 6!} = \frac{1}{524{,}880} \approx 0.0000019$, so

$\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n!} \approx s_5 = \sum_{n=1}^{5} \frac{(-1)^n}{3^n n!} = -\frac{1}{3} + \frac{1}{18} - \frac{1}{162} + \frac{1}{1944} - \frac{1}{29{,}160} \approx -0.283471$. Adding b_6 to s_5 does not change the fourth decimal place of s_5 , so the sum of the series, correct to four decimal places, is -0.2835 .

31. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{49} - \frac{1}{50} + \frac{1}{51} - \frac{1}{52} + \cdots$. The 50 th partial sum of this series is an underestimate, since $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = s_{50} + \left(\frac{1}{51} - \frac{1}{52} \right) + \left(\frac{1}{53} - \frac{1}{54} \right) + \cdots$, and the terms in parentheses are all positive. The result can be seen geometrically in Figure 1.

32. If $p>0$, $\frac{1}{(n+1)^p} \le \frac{1}{n^p}$ $\left(\left\{ 1/n^p \right\} \right.$ is decreasing) and $\lim_{n\to\infty} \frac{1}{n^p} = 0$, so the series converges by the Alternating Series Test. If $p \le 0$, $\lim_{n\to\infty} \frac{(-1)^{n-1}}{n^p}$ does not exist, so the series diverges by the Test for Divergence. Thus, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ converges $\Leftrightarrow p>0$.

33. Clearly $b_n = \dfrac{1}{n+p}$ is decreasing and eventually positive and $\lim_{n\to\infty} b_n = 0$ for any p. So the series converges (by the Alternating Series Test) for any p for which every b_n is defined, that is, $n+p \neq 0$ for $n \geq 1$, or p is not a negative integer.

34. Let $f(x) = \dfrac{(\ln x)^p}{x}$. Then $f'(x) = \dfrac{(\ln x)^{p-1}(p - \ln x)}{x^2} < 0$ if $x > e^p$ so f is eventually decreasing for every p. Clearly $\lim_{n\to\infty} \dfrac{(\ln n)^p}{n} = 0$ if $p \leq 0$, and if $p > 0$ we can apply l'Hospital's Rule $[p+1]$ times to get a limit of 0 as well. So the series converges for all p (by the Alternating Series Test).

35. $\Sigma b_{2n} = \Sigma 1/(2n)^2$ clearly converges (by comparison with the p– series for $p = 2$). So suppose that $\Sigma (-1)^{n-1} b_n$ converges. Then by Theorem .2.8(ii), so does $\Sigma \left[(-1)^{n-1} b_n + b_n \right] = 2\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots \right) = 2\Sigma \dfrac{1}{2n-1}$. But this diverges by comparison with the harmonic series, a contradiction. Therefore, $\Sigma (-1)^{n-1} b_n$ must diverge. The Alternating Series Test does not apply since $\{ b_n \}$ is not decreasing.

36. **(a)** We will prove this by induction. Let $P(n)$ be the proposition that $s_{2n} = h_{2n} - h_n$. $P(1)$ is the statement $s_2 = h_2 - h_1$, which is true since $1 - \frac{1}{2} = \left(1 + \frac{1}{2}\right) - 1$. So suppose that $P(n)$ is true. We will show that $P(n+1)$ must be true as a consequence.

$$\begin{aligned} h_{2n+2} - h_{n+1} &= \left(h_{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}\right) - \left(h_n + \frac{1}{n+1}\right) = \left(h_{2n} - h_n\right) + \frac{1}{2n+1} - \frac{1}{2n+2} \\ &= s_{2n} + \frac{1}{2n+1} - \frac{1}{2n+2} = s_{2n+2} \end{aligned}$$

which is $P(n+1)$, and proves that $s_{2n} = h_{2n} - h_n$ for all n.

(b) We know that $h_{2n} - \ln(2n) \to \gamma$ and $h_n - \ln n \to \gamma$ as $n \to \infty$. So $s_{2n} = h_{2n} - h_n = [h_{2n} - \ln(2n)] - \left(h_n - \ln n\right) + [\ln(2n) - \ln n]$, and $\lim_{n\to\infty} s_{2n} = \gamma - \gamma + \lim_{n\to\infty} [\ln(2n) - \ln n] = \lim_{n\to\infty} (\ln 2 + \ln n - \ln n) = \ln 2$.

1. **(a)** Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=8>1$, part (b) of the Ratio Test tells us that the series Σa_n is divergent.

(b) Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=0.8<1$, part (a) of the Ratio Test tells us that the series Σa_n is absolutely convergent (and therefore convergent).

(c) Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$, the Ratio Test fails and the series Σa_n might converge or it might diverge.

2. The series $\Sigma_{n=1}^{\infty}\frac{n^2}{2^n}$ has positive terms and

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\left[\frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2}\right]=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2\cdot\frac{1}{2}=\frac{1}{2}<1$, so the series is absolutely convergent by the Ratio Test.

3. $\Sigma_{n=0}^{\infty}\frac{(-10)^n}{n!}$. Using the Ratio Test,

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-10)^{n+1}}{(n+1)!}\cdot\frac{n!}{(-10)^n}\right|=\lim_{n\to\infty}\left|\frac{-10}{n+1}\right|=0<1$, so the series is absolutely convergent.

4. $\Sigma_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n^4}$ diverges by the Test for Divergence. $\lim_{n\to\infty}\frac{2^n}{n^4}=\infty$, so $\lim_{n\to\infty}(-1)^{n-1}\frac{2^n}{n^4}$ does not exist.

5. $\Sigma_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt[4]{n}}$ converges by the Alternating Series Test, but $\Sigma_{n=1}^{\infty}\frac{1}{\sqrt{[4]n}}$ is a divergent p –series $\left(p=\frac{1}{4}\leq 1\right)$, so the given series is conditionally convergent.

6. $\Sigma_{n=1}^{\infty}\frac{1}{n^4}$ is a convergent p –series $(p=4>1)$, so $\Sigma_{n=1}^{\infty}\frac{(-1)^n}{n^4}$ is absolutely convergent.

7. $\lim_{n\to\infty} \left| a_n \right| = \lim_{n\to\infty} \frac{n}{5+n} = \lim_{n\to\infty} \frac{1}{5/n+1} = 1$, so $\lim_{n\to\infty} a_n \neq 0$. Thus, the given series is divergent by the Test for Divergence.

8. $\Sigma_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges by the Limit Comparison Test with the harmonic series:

$\lim_{n\to\infty} \frac{n/(n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$. But $\Sigma_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ converges by the Alternating Series Test:

$\left\{ \frac{n}{n^2+1} \right\}$ has positive terms, is decreasing since $\left(\frac{x}{x^2+1} \right)' = \frac{1-x^2}{\left(x^2+1\right)^2} \leq 0$ for $x \geq 1$, and

$\lim_{n\to\infty} \frac{n}{n^2+1} = 0$. Thus, $\Sigma_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ is conditionally convergent.

9.

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1/(2n+2)!}{1/(2n)!} = \lim_{n\to\infty} \frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty} \frac{(2n)!}{(2n+2)(2n+1)(2n)!} = \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0 < 1$

, so the series $\Sigma_{n=1}^{\infty} \frac{1}{(2n)!}$ is absolutely convergent by the Ratio Test. Of course, absolute convergence is the same as convergence for this series, since all of its terms are positive.

10. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!/e^{n+1}}{n!/e^n} \right| = \frac{1}{e} \lim_{n\to\infty} (n+1) = \infty$, so the series $\Sigma_{n=1}^{\infty} e^{-n} n!$ diverges by the Ratio Test.

11. Since $0 \leq \frac{e^{1/n}}{n^3} \leq \frac{e}{n^3} = e \left(\frac{1}{n^3} \right)$ and $\Sigma_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p –series $(p=3>1)$, $\Sigma_{n=1}^{\infty} \frac{e^{1/n}}{n^3}$ converges, and so $\Sigma_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$ is absolutely convergent.

12. $\left| \frac{\sin 4n}{4^n} \right| \leq \frac{1}{4^n}$, so $\Sigma_{n=1}^{\infty} \left| \frac{\sin 4n}{4^n} \right|$ converges by comparison with the convergent geometric series

$\sum_{n=1}^{\infty} \frac{1}{4^n} \left(|r| = \frac{1}{4} < 1 \right)$. Thus, $\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$ is absolutely convergent.

13. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[\frac{(n+1)3^{n+1}}{4^n} \cdot \frac{4^{n-1}}{n \cdot 3^n} \right] = \lim_{n\to\infty} \left(\frac{3}{4} \cdot \frac{n+1}{n} \right) = \frac{3}{4} < 1$, so the series $\sum_{n=1}^{\infty} \frac{n(-3)^n}{4^{n-1}}$ is absolutely convergent by the Ratio Test.

14. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[\frac{(n+1)^2 2^{n+1}}{(n+1)!} \cdot \frac{n!}{n^2 2^n} \right] = \lim_{n\to\infty} \left[\left(1 + \frac{1}{n} \right)^2 \cdot \frac{2}{n+1} \right] = 0$, so the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2 2^n}{n!}$ is absolutely convergent by the Ratio Test.

15. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[\frac{10^{n+1}}{(n+2)4^{2n+3}} \cdot \frac{(n+1)4^{2n+1}}{10^n} \right] = \lim_{n\to\infty} \left(\frac{10}{4^2} \cdot \frac{n+1}{n+2} \right) = \frac{5}{8} < 1$, so the series $\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$ is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

16. $n^{2/3} - 2 > 0$ for $n \geq 3$, so $\frac{3 - \cos n}{n^{2/3} - 2} > \frac{1}{n^{2/3} - 2} > \frac{1}{n^{2/3}}$ for $n \geq 3$. Since $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges $\left(p = \frac{2}{3} \leq 1 \right)$, so does $\sum_{n=1}^{\infty} \frac{3 - \cos n}{n^{2/3} - 2}$ by the Comparison Test.

17. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges by the Alternating Series Test since $\lim_{n\to\infty} \frac{1}{\ln n} = 0$ and $\left\{ \frac{1}{\ln n} \right\}$ is decreasing. Now $\ln n < n$, so $\frac{1}{\ln n} > \frac{1}{n}$, and since $\sum_{n=2}^{\infty} \frac{1}{n}$ is the divergent (partial) harmonic series, $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by the Comparison Test. Thus, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is conditionally convergent.

18.

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \lim_{n\to\infty}\frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\frac{1}{(1+1/n)^n} = \frac{1}{e} < 1$, so the series $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ converges absolutely by the Ratio Test.

19. $\frac{|\cos(n\pi/3)|}{n!} \le \frac{1}{n!}$ and $\sum_{n=1}^{\infty}\frac{1}{n!}$ converges (use the Ratio Test or the result of Exercise .4.29), so the series $\sum_{n=1}^{\infty}\frac{\cos(n\pi/3)}{n!}$ converges absolutely by the Comparison Test.

20. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{1}{\ln n} = 0 < 1$, so the series $\sum_{n=2}^{\infty}\frac{(-1)^n}{(\ln n)^n}$ converges absolutely by the Root Test.

21. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left(\frac{n^n}{3^{1+3n}}\right)^{1/n} = \lim_{n\to\infty}\frac{n}{\sqrt[n]{3}\cdot 3^3} = \infty$, so the series $\sum_{n=1}^{\infty}\frac{n^n}{3^{1+3n}}$ is divergent by the Root Test.

Or:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{(n+1)^{n+1}}{3^{4+3n}}\cdot\frac{3^{1+3n}}{n^n}\right] = \lim_{n\to\infty}\left[\frac{1}{3^3}\cdot\left(\frac{n+1}{n}\right)^n(n+1)\right]$$

$$= \frac{1}{27}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\lim_{n\to\infty}(n+1) = \frac{1}{27}e\lim_{n\to\infty}(n+1) = \infty,$$

so the series is divergent by the Ratio Test.

22. Since $\left\{\frac{1}{n\ln n}\right\}$ is decreasing and $\lim_{n\to\infty}\frac{1}{n\ln n} = 0$, the series $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$ converges by the Alternating Series Test. Since $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges by the Integral Test (Exercise .3.21), the series $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$ is conditionally convergent.

23. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{n^2+1}{2n^2+1} = \lim_{n\to\infty}\frac{1+1/n^2}{2+1/n^2} = \frac{1}{2} < 1$, so the series $\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$ is absolutely

convergent by the Root Test.

24. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \dfrac{1}{\arctan n} = \dfrac{1}{\pi/2} = \dfrac{2}{\pi} < 1$, so the series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{(\arctan n)^n}$ is absolutely convergent by the Root Test.

25. Use the Ratio Test with the series

$$1 - \frac{1\cdot 3}{3!} + \frac{1\cdot 3\cdot 5}{5!} - \frac{1\cdot 3\cdot 5\cdot 7}{7!} + \cdots + (-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{(2n-1)!} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{(2n-1)!}$$

.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)[2(n+1)-1]}{[2(n+1)-1]!}\cdot\frac{(2n-1)!}{(-1)^{n-1}\cdot 1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}\right| \\ &= \lim_{n\to\infty}\left|\frac{(-1)(2n+1)(2n-1)!}{(2n+1)(2n)(2n-1)!}\right| \\ &= \lim_{n\to\infty}\frac{1}{2n} = 0 < 1,\end{aligned}$$

so the given series is absolutely convergent and therefore convergent.

26. Use the Ratio Test with the series

$$\frac{2}{5} + \frac{2\cdot 6}{5\cdot 8} + \frac{2\cdot 6\cdot 10}{5\cdot 8\cdot 11} + \frac{2\cdot 6\cdot 10\cdot 14}{5\cdot 8\cdot 11\cdot 14} + \cdots = \sum_{n=1}^{\infty}\frac{2\cdot 6\cdot 10\cdot 14\cdot \cdots \cdot (4n-2)}{5\cdot 8\cdot 11\cdot 14\cdot \cdots \cdot (3n+2)} .$$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{2\cdot 6\cdot 10\cdot \cdots \cdot (4n-2)[4(n+1)-2]}{5\cdot 8\cdot 11\cdot \cdots \cdot (3n+2)[3(n+1)+2]}\cdot\frac{5\cdot 8\cdot 11\cdot \cdots \cdot (3n+2)}{2\cdot 6\cdot 10\cdot \cdots \cdot (4n-2)}\right| \\ &= \lim_{n\to\infty}\frac{4n+2}{3n+5} = \frac{4}{3} > 1,\end{aligned}$$

so the given series is divergent.

27. $\sum_{n=1}^{\infty}\dfrac{2\cdot 4\cdot 6\cdot \cdots \cdot (2n)}{n!} = \sum_{n=1}^{\infty}\dfrac{(2\cdot 1)\cdot(2\cdot 2)\cdot(2\cdot 3)\cdot \cdots \cdot (2\cdot n)}{n!} = \sum_{n=1}^{\infty}\dfrac{2^n n!}{n!} = \sum_{n=1}^{\infty} 2^n$, which diverges by the Test for Divergence since $\lim_{n\to\infty} 2^n = \infty$.

28.

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}(n+1)!}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+5)}}{\frac{2^n n!}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}}\right| = \lim_{n\to\infty}\frac{2(n+1)}{3n+5} = \frac{2}{3} < 1$, so the series converges absolutely by the Ratio Test.

29. By the recursive definition, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{5n+1}{4n+3}\right| = \frac{5}{4} > 1$, so the series diverges by the Ratio Test.

30. By the recursive definition, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2+\cos n}{\sqrt{n}}\right| = 0 < 1$, so the series converges absolutely by the Ratio Test.

31. **(a)** $\lim_{n\to\infty}\left|\frac{1/(n+1)^3}{1/n^3}\right| = \lim_{n\to\infty}\frac{n^3}{(n+1)^3} = \lim_{n\to\infty}\frac{1}{(1+1/n)^3} = 1$. Inconclusive.

(b) $\lim_{n\to\infty}\left|\frac{(n+1)}{2^{n+1}}\cdot\frac{2^n}{n}\right| = \lim_{n\to\infty}\frac{n+1}{2n} = \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{2n}\right) = \frac{1}{2}$. Conclusive (convergent).

(c) $\lim_{n\to\infty}\left|\frac{(-3)^n}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(-3)^{n-1}}\right| = 3\lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = 3\lim_{n\to\infty}\sqrt{\frac{1}{1+1/n}} = 3$. Conclusive (divergent).

(d) $\lim_{n\to\infty}\left|\frac{\sqrt{n+1}}{1+(n+1)^2}\cdot\frac{1+n^2}{\sqrt{n}}\right| = \lim_{n\to\infty}\left[\sqrt{1+\frac{1}{n}}\cdot\frac{1/n^2+1}{1/n^2+(1+1/n)^2}\right] = 1$. Inconclusive.

32. We use the Ratio Test:
$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[(n+1)!]^2}{[k(n+1)]!}\cdot\frac{(kn)!}{(n!)^2}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{[k(n+1)][k(n+1)-1]\cdots[kn+1]}\right|$ Now if $k=1$, then this is equal to $\lim_{n\to\infty}\left|\frac{(n+1)^2}{(n+1)}\right| = \infty$, so the series diverges; if $k=2$, the limit is $\lim_{n\to\infty}\left|\frac{(n+1)^2}{(2n+2)(2n+1)}\right| = \frac{1}{4} < 1$, so the series converges, and if $k>2$, then the highest power of n in the denominator is larger than 2 , and so the limit is 0 , indicating convergence. So the series converges for $k\geq 2$.

33. **(a)** $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|=\lim_{n\to\infty}\left|\frac{x}{n+1}\right|=|x|\lim_{n\to\infty}\frac{1}{n+1}=|x|\cdot 0=0<1$, so by the Ratio Test the series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges for all x .

(b) Since the series of part (a) always converges, we must have $\lim_{n\to\infty}\frac{x^n}{n!}=0$ by Theorem .2.6.

34. **(a)**

$$R_n=a_{n+1}+a_{n+2}+a_{n+3}+a_{n+4}+\cdots=a_{n+1}\left(1+\frac{a_{n+2}}{a_{n+1}}+\frac{a_{n+3}}{a_{n+1}}+\frac{a_{n+4}}{a_{n+1}}+\cdots\right)$$

$$=a_{n+1}\left(1+\frac{a_{n+2}}{a_{n+1}}+\frac{a_{n+3}}{a_{n+2}}\frac{a_{n+2}}{a_{n+1}}+\frac{a_{n+4}}{a_{n+3}}\frac{a_{n+3}}{a_{n+2}}\frac{a_{n+2}}{a_{n+1}}+\cdots\right)$$

$$=a_{n+1}\left(1+r_{n+1}+r_{n+2}r_{n+1}+r_{n+3}r_{n+2}r_{n+1}+\cdots\right)\ (*)$$

$$\le a_{n+1}\left(1+r_{n+1}+r_{n+1}^2+r_{n+1}^3+\cdots\right)=\frac{a_{n+1}}{1-r_{n+1}}$$

(b) Note that since $\{r_n\}$ is increasing and $r_n\to L$ as $n\to\infty$, we have $r_n<L$ for all n . So, starting with equation ($*$),

$$R_n=a_{n+1}\left(1+r_{n+1}+r_{n+2}r_{n+1}+r_{n+3}r_{n+2}r_{n+1}+\cdots\right)\le a_{n+1}\left(1+L+L^2+L^3+\cdots\right)=\frac{a_{n+1}}{1-L}$$

35. **(a)** $s_5=\sum_{n=1}^{5}\frac{1}{n2^n}=\frac{1}{2}+\frac{1}{8}+\frac{1}{24}+\frac{1}{64}+\frac{1}{160}=\frac{661}{960}\approx 0.68854$. Now the ratios $r_n=\frac{a_{n+1}}{a_n}=\frac{n2^n}{(n+1)2^{n+1}}=\frac{n}{2(n+1)}$ form an increasing sequence, since $r_{n+1}-r_n=\frac{n+1}{2(n+2)}-\frac{n}{2(n+1)}=\frac{(n+1)^2-n(n+2)}{2(n+1)(n+2)}=\frac{1}{2(n+1)(n+2)}>0$. So by Exercise 34(b), the error in

using s_5 is $R_5 \le \dfrac{a_6}{1-\lim_{n\to\infty} r_n} = \dfrac{1/\left(6\cdot 2^6\right)}{1-1/2} = \dfrac{1}{192} \approx 0.00521$.

(b) The error in using s_n as an approximation to the sum is $R_n = \dfrac{a_{n+1}}{1-\frac{1}{2}} = \dfrac{2}{(n+1)2^{n+1}}$. We want $R_n < 0.00005 \Leftrightarrow \dfrac{1}{(n+1)2^n} < 0.00005 \Leftrightarrow (n+1)2^n > 20{,}000$. To find such an n we can use trial and error or a graph. We calculate $(11+1)2^{11} = 24{,}576$, so $s_{11} = \sum_{n=1}^{11} \dfrac{1}{n2^n} \approx 0.693109$ is within 0.00005 of the actual sum.

36. $s_{10} = \sum_{n=1}^{10} \dfrac{n}{2^n} = \dfrac{1}{2} + \dfrac{2}{4} + \dfrac{3}{8} + \cdots + \dfrac{10}{1024} \approx 1.988$. The ratios $r_n = \dfrac{a_{n+1}}{a_n} = \dfrac{n+1}{2^{n+1}} \cdot \dfrac{2^n}{n} = \dfrac{n+1}{2n} = \dfrac{1}{2}\left(1+\dfrac{1}{n}\right)$ form a decreasing sequence, and $r_{11} = \dfrac{11+1}{2(11)} = \dfrac{12}{22} = \dfrac{6}{11} < 1$, so by Exercise 34(a), the error in using s_{10} to approximate the sum of the series $\sum_{n=1}^{\infty} \dfrac{n}{2^n}$ is

$$R_{10} \le \frac{a_{11}}{1-r_{11}} = \frac{\frac{11}{2048}}{1-\frac{6}{11}} = \frac{121}{10{,}240} \approx 0.0118 .$$

37. Summing the inequalities $-\left|a_i\right| \le a_i \le \left|a_i\right|$ for $i=1,2,\ldots,n$, we get $-\sum_{i=1}^{n}\left|a_i\right| \le \sum_{i=1}^{n} a_i \le \sum_{i=1}^{n}\left|a_i\right|$ $\Rightarrow -\lim_{n\to\infty}\sum_{i=1}^{n}\left|a_i\right| \le \lim_{n\to\infty}\sum_{i=1}^{n} a_i \le \lim_{n\to\infty}\sum_{i=1}^{n}\left|a_i\right| \Rightarrow -\sum_{n=1}^{\infty}\left|a_n\right| \le \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty}\left|a_n\right| \Rightarrow$ $\left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty}\left|a_n\right|$.

38. **(a)** Following the hint, we get that $\left|a_n\right| < r^n$ for $n \ge N$, and so since the geometric series $\sum_{n=1}^{\infty} r^n$ converges ($0<r<1$), the series $\sum_{n=N}^{\infty}\left|a_n\right|$ converges as well by the Comparison Test, and hence so does $\sum_{n=1}^{\infty}\left|a_n\right|$, so $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) If $\lim_{n\to\infty} \sqrt[n]{\left|a_n\right|} = L > 1$, then there is an integer N such that $\sqrt[n]{\left|a_n\right|} > 1$ for all $n \ge N$, so $\left|a_n\right| > 1$

for $n \geq N$. Thus, $\lim_{n\to\infty} a_n \neq 0$, so $\Sigma_{n=1}^{\infty} a_n$ diverges by the Test for Divergence.

39. **(a)** Since $\Sigma\, a_n$ is absolutely convergent, and since $\left|a_n^+\right| \leq \left|a_n\right|$ and $\left|a_n^-\right| \leq \left|a_n\right|$ (because a_n^+ and a_n^- each equal either a_n or 0), we conclude by the Comparison Test that both $\Sigma\, a_n^+$ and $\Sigma\, a_n^-$ must be absolutely convergent. (Or use Theorem .2.8.)

(b) We will show by contradiction that both $\Sigma\, a_n^+$ and $\Sigma\, a_n^-$ must diverge. For suppose that $\Sigma\, a_n^+$ converged. Then so would $\Sigma\left(a_n^+ - \frac{1}{2} a_n\right)$ by Theorem .2.8. But $\Sigma\left(a_n^+ - \frac{1}{2} a_n\right) = \Sigma\left[\frac{1}{2}\left(a_n + \left|a_n\right|\right) - \frac{1}{2} a_n\right] = \frac{1}{2}\Sigma\left|a_n\right|$, which diverges because $\Sigma\, a_n$ is only conditionally convergent. Hence, $\Sigma\, a_n^+$ can't converge. Similarly, neither can $\Sigma\, a_n^-$.

40. Let $\Sigma\, b_n$ be the rearranged series constructed in the hint. This series will have partial sums s_n that oscillate in value back and forth across r . Since $\lim_{n\to\infty} a_n = 0$ (by Theorem .2.6), and since the size of the oscillations $\left|s_n - r\right|$ is always less than $\left|a_n\right|$ because of the way $\Sigma\, b_n$ was constructed, we have that $\Sigma\, b_n = \lim_{n\to\infty} s_n = r$.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2-1}{n^2+1} = \lim_{n\to\infty} \frac{1-1/n^2}{1+1/n} = 1 \neq 0$, so the series $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$ diverges by the Test for Divergence.

2. If $a_n = \frac{n-1}{n^2+n}$ and $b_n = \frac{1}{n}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2-n}{n^2+n} = \lim_{n\to\infty} \frac{1-1/n}{1+1/n} = 1$, so the series $\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$ diverges by the Limit Comparison Test with the harmonic series.

3. $\frac{1}{n^2+n} < \frac{1}{n^2}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges by the Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a p – series that converges because $p=2>1$.

4. Let $b_n = \frac{n-1}{n^2+n}$. Then $b_1=0$, and $b_2=b_3=\frac{1}{6}$, but $b_n > b_{n+1}$ for $n \geq 3$ since

$$\left(\frac{x-1}{x^2+x} \right)' = \frac{\left(x^2+x\right)-(x-1)(2x+1)}{\left(x^2+x\right)^2} = \frac{-x^2+2x+1}{\left(x^2+x\right)^2} = \frac{2-(x-1)^2}{(x^2+x)^2} < 0 \text{ for } x \geq 3 \text{ . Thus,}$$

$\left\{ b_n \mid n \geq 3 \right\}$ is decreasing and $\lim_{n\to\infty} b_n = 0$, so $\sum_{n=3}^{\infty} (-1)^{n-1} \frac{n-1}{n^2+n}$ converges by the Alternating Series Test. Hence, the full series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n-1}{n^2+n}$ also converges.

5. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-3)^{n+2}}{2^{3(n+1)}} \cdot \frac{2^{3n}}{(-3)^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{-3 \cdot 2^{3n}}{2^{3n} \cdot 2^3} \right| = \lim_{n\to\infty} \frac{3}{2^3} = \frac{3}{8} < 1$, so the series $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$ is absolutely convergent by the Ratio Test.

6. $\lim_{n\to\infty} \sqrt[n]{\left| a_n \right|} = \lim_{n\to\infty} \frac{3n}{1+8n} = \lim_{n\to\infty} \frac{3}{1/n+8} = \frac{3}{8} < 1$, so $\sum_{n=1}^{\infty} \left(\frac{3n}{1+8n} \right)^n$ converges by the Root Test.

7. Let $f(x) = \frac{1}{x\sqrt{\ln x}}$. Then f is positive, continuous, and decreasing on $[2,\infty)]$, so we can apply the Integral Test. Since $\int \frac{1}{x\sqrt{\ln x}}\, dx \left[\begin{array}{c} \text{u=ln x,} \\ \text{du=dx/x} \end{array} \right] = \int u^{-1/2} du = 2u^{1/2} + C = 2\sqrt{\ln x} + C$, we find

$\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \int_2^{t} \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \left[2\sqrt{\ln x}\,\right]_2^t = \lim_{t\to\infty} \left(2\sqrt{\ln t} - 2\sqrt{\ln 2}\,\right) = \infty$. Since the integral diverges, the given series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges.

8. $\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!} = \sum_{k=1}^{\infty} \frac{2^k}{(k+1)(k+2)}$. Using the Ratio Test, we get

$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{2^{k+1}}{(k+2)(k+3)} \cdot \frac{(k+1)(k+2)}{2^k} \right| = \lim_{k\to\infty} \left(2\cdot \frac{k+1}{k+3} \right) = 2 > 1$, so the series diverges.

Or: Use the Test for Divergence.

9. $\sum_{k=1}^{\infty} k^2 e^{-k} = \sum_{k=1}^{\infty} \frac{k^2}{e^k}$. Using the Ratio Test, we get

$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2} \right| = \lim_{k\to\infty} \left[\left(\frac{k+1}{k} \right)^2 \cdot \frac{1}{e} \right] = 1^2 \cdot \frac{1}{e} = \frac{1}{e} < 1$, so the series converges.

10. Let $f(x) = x^2 e^{-x^3}$. Then f is continuous and positive on $[1,\infty)$, and $f'(x) = \frac{x(2-3x^3)}{e^{x^3}} < 0$ for $x \geq 1$, so f is decreasing on $[1,\infty)$ as well, and we can apply the Integral Test.

$\int_1^{\infty} x^2 e^{-x^3}\,dx = \lim_{t\to\infty} \left[-\frac{1}{3} e^{-x^3} \right]_1^t = \frac{1}{3e}$, so the integral converges, and hence, the series converges.

11. $b_n = \frac{1}{n\ln n} > 0$ for $n \geq 2$, $\{ b_n \}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the given series $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\ln n}$ converges by the Alternating Series Test.

12. Let $b_n = \frac{n}{n^2+25}$. Then $b_n > 0$, $\lim_{n\to\infty} b_n = 0$, and $b_n - b_{n+1} = \frac{n}{n^2+25} - \frac{n+1}{n^2+2n+26} = \frac{n^2+n-25}{(n^2+25)(n^2+2n+26)}$, which is positive for $n \geq 5$, so the sequence $\{ b_n \}$ decreases from $n=5$ on. Hence, the given series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+25}$ converges by the Alternating Series Test.

13. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(n+1)^2}{(n+1)!}\cdot\frac{n!}{3^n n^2}\right| = \lim_{n\to\infty}\left[\frac{3(n+1)^2}{(n+1)n^2}\right] = 3\lim_{n\to\infty}\frac{n+1}{n^2} = 0 < 1$, so the series $\sum_{n=1}^{\infty}\frac{3^n n^2}{n!}$ converges by the Ratio Test.

14. The series $\sum_{n=1}^{\infty}\sin n$ diverges by the Test for Divergence since $\lim_{n\to\infty}\sin n$ does not exist.

15.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n+2)[3(n+1)+2]}\cdot\frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n+2)}{n!}\right|$$

$$= \lim_{n\to\infty}\frac{n+1}{3n+5} = \frac{1}{3} < 1$$

so the series $\sum_{n=0}^{\infty}\frac{n!}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n+2)}$ converges by the Ratio Test.

16. Using the Limit Comparison Test with $a_n = \frac{n^2+1}{n^3+1}$ and $b_n = \frac{1}{n}$, we have $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{n^2+1}{n^3+1}\cdot\frac{n}{1}\right) = \lim_{n\to\infty}\frac{n^3+n}{n^3+1} = \lim_{n\to\infty}\frac{1+1/n^2}{1+1/n^3} = 1 > 0$. Since $\sum_{n=1}^{\infty}b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty}a_n$ is also divergent.

17. $\lim_{n\to\infty}2^{1/n} = 2^0 = 1$, so $\lim_{n\to\infty}(-1)^n 2^{1/n}$ does not exist and the series $\sum_{n=1}^{\infty}(-1)^n 2^{1/n}$ diverges by the Test for Divergence.

18. $b_n = \frac{1}{\sqrt{n}-1}$ for $n \geq 2$. $\{b_n\}$ is a decreasing sequence of positive numbers and $\lim_{n\to\infty}b_n = 0$, so $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}-1}$ converges by the Alternating Series Test.

19. Let $f(x) = \frac{\ln x}{\sqrt{x}}$. Then $f'(x) = \frac{2-\ln x}{2x^{3/2}} < 0$ when $\ln x > 2$ or $x > e^2$, so $\frac{\ln n}{\sqrt{n}}$ is decreasing for $n > e^2$.

By l'Hospital's Rule, $\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{n\to\infty} \frac{1/n}{1/(2\sqrt{n})} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$, so the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ converges by the Alternating Series Test.

20. $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{k+6}{5^{k+1}} \cdot \frac{5^k}{k+5} \right| = \frac{1}{5} \lim_{k\to\infty} \frac{k+6}{k+5} = \frac{1}{5} < 1$, so the series $\sum_{k=1}^{\infty} \frac{k+5}{5^k}$ converges by the Ratio Test.

21. $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n} = \sum_{n=1}^{\infty} \left(\frac{4}{n} \right)^n$. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{4}{n} = 0 < 1$, so the given series is absolutely convergent by the Root Test.

22. $\frac{\sqrt{n^2-1}}{n^3+2n^2+5} < \frac{n}{n^3+2n^2+5} < \frac{n}{n^3} = \frac{1}{n^2}$ for $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$ converges by the Comparison Test with the convergent p –series $\sum_{n=1}^{\infty} 1/n^2 \, (p=2>1)$.

23. Using the Limit Comparison Test with $a_n = \tan\left(\frac{1}{n} \right)$ and $b_n = \frac{1}{n}$, we have

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan(1/n)}{1/n} = \lim_{x\to\infty} \frac{\tan(1/x)}{1/x} = \lim_{x\to\infty} \frac{\sec^2(1/x)\cdot(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} \sec^2(1/x) = 1^2 = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} a_n$ is also divergent.

24. $\frac{|\cos(n/2)|}{n^2+4n} < \frac{1}{n^2+4n} < \frac{1}{n^2}$ and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p=2>1$), $\sum_{n=1}^{\infty} \frac{\cos(n/2)}{n^2+4n}$ converges absolutely by the Comparison Test.

25. Use the Ratio Test.

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!} \right| = \lim_{n\to\infty} \frac{(n+1)n! \cdot e^{n^2}}{e^{n^2+2n+1} n!} = \lim_{n\to\infty} \frac{n+1}{e^{2n+1}} = 0 < 1$, so $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$ converges.

26.

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{n^2+2n+2}{5^{n+1}}\cdot\frac{5^n}{n^2+1}\right) = \lim_{n\to\infty}\left(\frac{1+2/n+2/n^2}{1+1/n^2}\cdot\frac{1}{5}\right) = \frac{1}{5} < 1$, so $\sum_{n=1}^{\infty}\frac{n^2+1}{5^n}$ converges by the Ratio Test.

27. $\int_2^{\infty}\frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x}-\frac{1}{x}\right]_1^t$ (using integration by parts) $=1$. So $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ converges by the Integral Test, and since $\frac{k\ln k}{(k+1)^3} < \frac{k\ln k}{k^3} = \frac{\ln k}{k^2}$, the given series $\sum_{k=1}^{\infty}\frac{k\ln k}{(k+1)^3}$ converges by the Comparison Test.

28. Since $\left\{\frac{1}{n}\right\}$ is a decreasing sequence, $e^{1/n} \le e^{1/1} = e$ for all $n \ge 1$, and $\sum_{n=1}^{\infty}\frac{e}{n^2}$ converges ($p=2>1$), so $\sum_{n=1}^{\infty}\frac{e^{1/n}}{n^2}$ converges by the Comparison Test. (Or use the Integral Test.)

29. $0 < \frac{\tan^{-1}n}{n^{3/2}} < \frac{\pi/2}{n^{3/2}}$. $\sum_{n=1}^{\infty}\frac{\pi/2}{n^{3/2}} = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ which is a convergent $p-$ series ($p=\frac{3}{2}>1$), so $\sum_{n=1}^{\infty}\frac{\tan^{-1}n}{n^{3/2}}$ converges by the Comparison Test.

30. Let $f(x)=\frac{\sqrt{x}}{x+5}$. Then $f(x)$ is continuous and positive on $[1,\infty)$, and since $f'(x)=\frac{5-x}{2\sqrt{x}\,(x+5)^2} < 0$ for $x>5$, $f(x)$ is eventually decreasing, so we can use the Alternating Series Test.

$\lim_{n\to\infty}\frac{\sqrt{n}}{n+5} = \lim_{n\to\infty}\frac{1}{n^{1/2}+5n^{-1/2}} = 0$, so the series $\sum_{j=1}^{\infty}(-1)^j\frac{\sqrt{j}}{j+5}$ converges.

31. $\lim_{k\to\infty}a_k = \lim_{k\to\infty}\frac{5^k}{3^k+4^k} = \left[\text{ divide by } 4^k\right]\lim_{k\to\infty}\frac{(5/4)^k}{(3/4)^k+1} = \infty$ since $\lim_{k\to\infty}\left(\frac{3}{4}\right)^k = 0$ and $\lim_{k\to\infty}\left(\frac{5}{4}\right)^k = \infty$. Thus, $\sum_{k=1}^{\infty}\frac{5^k}{3^k+4^k}$ diverges by the Test for Divergence.

32. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2n}{n^2} = \lim_{n\to\infty} \frac{2}{n} = 0$, so the series $\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$ converges by the Root Test.

33. Let $a_n = \frac{\sin(1/n)}{\sqrt{n}}$ and $b_n = \frac{1}{n\sqrt{n}}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1 > 0$, so $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$ converges by limit comparison with the convergent p –series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ $(p=3/2>1)$.

34. $0 \le n\cos^2 n \le n$, so $\frac{1}{n+n\cos^2 n} \ge \frac{1}{n+n} = \frac{1}{2n}$. Thus, $\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2 n}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{2n}$, which is a constant multiple of the (divergent) harmonic series.

35. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^{n^2/n} = \lim_{n\to\infty} \frac{1}{[(n+1)/n]^n} = \frac{1}{\lim_{n\to\infty} (1+1/n)^n} = \frac{1}{e} < 1$, so the series $\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2}$ converges by the Root Test.

36. Note that $(\ln n)^{\ln n} = \left(e^{\ln \ln n}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln \ln n} = n^{\ln \ln n}$ and $\ln \ln n \to \infty$ as $n \to \infty$, so $\ln \ln n > 2$ for sufficiently large n . For these n we have $(\ln n)^{\ln n} > n^2$, so $\frac{1}{(\ln n)^{\ln n}} < \frac{1}{n^2}$. Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges ($p=2>1$), so does $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ by the Comparison Test.

37. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(2^{1/n} - 1\right) = 1 - 1 = 0 < 1$, so the series $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$ converges by the Root Test.

38. Use the Limit Comparison Test with $a_n = \sqrt[n]{2} - 1$ and $b_n = 1/n$. Then

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^{1/n} - 1}{1/n} = \lim_{x\to\infty} \frac{2^{1/x} - 1}{1/x} = \lim_{x\to\infty} \frac{2^{1/x} \cdot \ln 2 \cdot (-1/x^2)}{-1/x^2} = \lim_{x\to\infty} (2^{1/x} \cdot \ln 2) = 1 \cdot \ln 2 = \ln 2 > 0$. So since $\sum_{n=1}^{\infty} b_n$ diverges (harmonic series), so does $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$.

Alternate Solution:

$\sqrt[n]{2}-1=\dfrac{1}{2^{(n-1)/n}+2^{(n-2)/n}+2^{(n-3)/n}+\cdot\cdot\cdot+2^{1/n}+1}\geq\dfrac{1}{2n}$,

and since $\sum_{n=1}^{\infty}\dfrac{1}{2n}=\dfrac{1}{2}\sum_{n=1}^{\infty}\dfrac{1}{n}$ diverges (harmonic series), so does $\sum_{n=1}^{\infty}\left(\sqrt[n]{2}-1\right)$ by the Comparison Test.

1. A power series is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$, where x is a variable and the c_n 's are constants called the coefficients of the series.

More generally, a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$ is called a power series in $(x-a)$ or a power series centered at a or a power series about a , where a is a constant.

2. **(a)** Given the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, the radius of convergence is:

(a) 0 if the series converges only when $x=a$

(b) ∞ if the series converges for all x , or

(c) a positive number R such that the series converges if $|x-a|<R$ and diverges if $|x-a|>R$.

In most cases, R can be found by using the Ratio Test.

(b) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (a), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers; that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a-R$ and $a+R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

3. If $a_n = \frac{x^n}{\sqrt{n}}$, then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x} \right| = \lim_{n\to\infty} \left| \frac{x}{\sqrt{n+1}/\sqrt{n}} \right| = \lim_{n\to\infty} \frac{|x|}{\sqrt{1+1/n}} = |x| .$$

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$ converges when $|x|<1$, so the radius of convergence $R=1$. Now we'll check the endpoints, that is, $x=\pm 1$. When $x=1$, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges because it is a p –series with $p=\frac{1}{2} \leq 1$. When $x=-1$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. Thus, the interval of convergence is $I=[-1,1)$.

4. If $a_n = \frac{(-1)^n x^n}{n+1}$, then $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{n+2} \cdot \frac{n+1}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{1+1/(n+1)} = |x|$. By the Ratio Test, the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ converges when $|x|<1$, so $R=1$. When $x=-1$, the series diverges because it is the harmonic series; when $x=1$, it is the alternating harmonic series, which converges by

the Alternating Series Test. Thus, $I=(-1,1]$.

5. If $a_n=\frac{(-1)^{n-1}x^n}{n^3}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^n x^{n+1}}{(n+1)^3}\cdot\frac{n^3}{(-1)^{n-1}x^n}\right|=\lim_{n\to\infty}\left|\frac{(-1)xn^3}{(n+1)^3}\right|=\lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^3|x|\right]=1^3\cdot|x|=|x|$$

. By the Ratio Test, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n^3}$ converges when $|x|<1$, so the radius of convergence $R=1$. Now we'll check the endpoints, that is, $x=\pm 1$. When $x=1$, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}$ converges by the Alternating Series Test. When $x=-1$, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-1)^n}{n^3}=-\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges because it is a constant multiple of a convergent p –series $(p=3>1)$. Thus, the interval of convergence is $I=[-1,1]$.

6. $a_n=\sqrt{n}\,x^n$, so we need $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{\sqrt{n+1}\,|x|^{n+1}}{\sqrt{n}\,|x|^n}=\lim_{n\to\infty}\sqrt{1+\frac{1}{n}}\;|x|=|x|<1$ for convergence (by the Ratio Test), so $R=1$. When $x=\pm 1$, $\lim_{n\to\infty}|a_n|=\lim_{n\to\infty}\sqrt{n}=\infty$, so the series diverges by the Test for Divergence. Thus, $I=(-1,1)$.

7. If $a_n=\frac{x^n}{n!}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|=\lim_{n\to\infty}\left|\frac{x}{n+1}\right|=|x|\lim_{n\to\infty}\frac{1}{n+1}=|x|\cdot 0=0<1$$ for *all* real x .

So, by the Ratio Test, $R=\infty$, and $I=(-\infty,\infty)$.

8. Here the Root Test is easier. If $a_n=n^n x^n$ then $\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}n|x|=\infty$ if $x\neq 0$, so $R=0$ and $I=\{0\}$.

9. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{(n+1)4^{n+1}|x|^{n+1}}{n4^n|x|^n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)4|x|=4|x|$. Now $4|x|<1\Leftrightarrow|x|<\frac{1}{4}$, so

by the Ratio Test, $R=\frac{1}{4}$. When $x=\frac{1}{4}$, we get the divergent series $\sum_{n=1}^{\infty}(-1)^n n$, and when $x=-\frac{1}{4}$, we get the divergent series $\sum_{n=1}^{\infty}n$. Thus, $I=\left(-\frac{1}{4},\frac{1}{4}\right)$.

10. If $a_n=\frac{x^n}{n3^n}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)3^{n+1}}\cdot\frac{n3^n}{x^n}\right|=\lim_{n\to\infty}\left|\frac{xn}{(n+1)3}\right|=\frac{|x|}{3}\lim_{n\to\infty}\frac{n}{n+1}=\frac{|x|}{3}$$

. By the Ratio Test, the series converges when $\frac{|x|}{3}<1\Leftrightarrow|x|<3$, so $R=3$. When $x=-3$, the series is the alternating harmonic series, which converges by the Alternating Series Test. When $x=3$, it is the harmonic series, which diverges. Thus, $I=[-3,3)$.

11. $a_n=\frac{(-2)^n x^n}{\sqrt[4]{n}}$, so $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{2^{n+1}|x|^{n+1}}{\sqrt[4]{n+1}}\cdot\frac{\sqrt[4]{n}}{2^n|x|^n}=\lim_{n\to\infty}2|x|\sqrt[4]{\frac{n}{n+1}}=2|x|$, so by the Ratio Test, the series converges when $2|x|<1\Leftrightarrow|x|<\frac{1}{2}$, so $R=\frac{1}{2}$. When $x=-\frac{1}{2}$, we get the divergent

p –series $\sum_{n=1}^{\infty}\frac{1}{\sqrt[4]{n}}\left(p=\frac{1}{4}\le1\right)$. When $x=\frac{1}{2}$, we get the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[4]{n}}$, which converges by the Alternating Series Test. Thus, $I=\left(-\frac{1}{2},\frac{1}{2}\right]$.

12. $a_n=\frac{x^n}{5^n n^5}$, so $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{5^{n+1}(n+1)^5}\cdot\frac{5^n n^5}{x^n}\right|=\lim_{n\to\infty}\frac{|x|}{5}\left(\frac{n}{n+1}\right)^5=\frac{|x|}{5}$. By the Ratio Test, the series converges when $|x|/5<1\Leftrightarrow|x|<5$, so $R=5$. When $x=-5$, we get the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}$, which converges by the Alternating Series Test. When $x=5$, we get the convergent p –series $\sum_{n=1}^{\infty}\frac{1}{n^5}$ $(p=5>1)$. Thus, $I=[-5,5]$.

13. If $a_n=(-1)^n\frac{x^n}{4^n\ln n}$, then

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{4^{n+1}\ln(n+1)}\cdot\frac{4^n \ln n}{x^n}\right| = \frac{|x|}{4}\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} = \frac{|x|}{4}\cdot 1$ (by l'Hospital's Rule) $= \frac{|x|}{4}$. By the Ratio Test, the series converges when $\frac{|x|}{4} < 1 \Leftrightarrow |x| < 4$, so $R=4$. When $x=-4$, $\sum_{n=2}^{\infty}(-1)^n \frac{x^n}{4^n \ln n} = \sum_{n=2}^{\infty}\frac{[(-1)(-4)]^n}{4^n \ln n} = \sum_{n=2}^{\infty}\frac{1}{\ln n}$. Since $\ln n < n$ for $n \geq 2$, $\frac{1}{\ln n} > \frac{1}{n}$ and $\sum_{n=2}^{\infty}\frac{1}{n}$ is the divergent harmonic series (without the $n=1$ term), $\sum_{n=2}^{\infty}\frac{1}{\ln n}$ is divergent by the Comparison Test. When $x=4$, $\sum_{n=2}^{\infty}(-1)^n\frac{x^n}{4^n \ln n} = \sum_{n=2}^{\infty}(-1)^n\frac{1}{\ln n}$, which converges by the Alternating Series Test. Thus, $I=(-4,4]$.

14. $a_n = (-1)^n \frac{x^{2n}}{(2n)!}$, so $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{|x|^{2n}} = \lim_{n\to\infty}\frac{|x|^2}{(2n+1)(2n+2)} = 0$. Thus, by the Ratio Test, the series converges for *all* real x and we have $R=\infty$ and $I=(-\infty,\infty)$.

15. If $a_n = \sqrt{n}\,(x-1)^n$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+1}\,|x-1|^{n+1}}{\sqrt{n}\,|x-1|^n}\right| = \lim_{n\to\infty}\sqrt{1+\frac{1}{n}}\,|x-1| = |x-1|$. By the Ratio Test, the series converges when $|x-1| < 1 \Leftrightarrow -1 < x-1 < 1 \Leftrightarrow 0 < x < 2$. When $x=0$, the series becomes $\sum_{n=0}^{\infty}(-1)^n\sqrt{n}$, which diverges by the Test for Divergence. When $x=2$, the series becomes $\sum_{n=0}^{\infty}\sqrt{n}$, which also diverges by the Test for Divergence. Thus, $I=(0,2)$.

16. If $a_n = n^3(x-5)^n$, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3(x-5)^{n+1}}{n^3(x-5)^n}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^3|x-5| = |x-5|$. By the Ratio Test, the series converges when $|x-5| < 1 \Leftrightarrow -1 < x-5 < 1 \Leftrightarrow 4 < x < 6$. When $x=4$, the series becomes $\sum_{n=0}^{\infty}(-1)^n n^3$, which diverges by the Test for Divergence. When $x=6$, the series becomes $\sum_{n=0}^{\infty}n^3$, which also diverges by the Test for Divergence. Thus, $R=1$ and $I=(4,6)$.

17. If $a_n = (-1)^n\frac{(x+2)^n}{n2^n}$, then

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x+2|^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{|x+2|^n}\right] = \lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{|x+2|}{2} = \frac{|x+2|}{2}$. By the Ratio Test, the series converges when $\frac{|x+2|}{2}<1 \Leftrightarrow |x+2|<2 \Leftrightarrow -2<x+2<2 \Leftrightarrow -4<x<0$. When $x=-4$, the series becomes $\sum_{n=1}^{\infty}(-1)^n\frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{2^n}{n2^n} = \sum_{n=1}^{\infty}\frac{1}{n}$, which is the divergent harmonic series. When $x=0$, the series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, the alternating harmonic series, which converges by the Alternating Series Test. Thus, $I=(-4,0]$.

18. If $a_n = \frac{(-2)^n}{\sqrt{n}}(x+3)^n$, then

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(x+3)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(-2)^n(x+3)^n}\right| = \lim_{n\to\infty}\frac{2|x+3|}{\sqrt{1+1/n}} = 2|x+3|<1 \Leftrightarrow |x+3|<\frac{1}{2}$ $\left[\text{so} R=\frac{1}{2}\right] \Leftrightarrow -\frac{7}{2}<x<-\frac{5}{2}$. When $x=-\frac{7}{2}$, the series becomes $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, which diverges because it is a p –series with $p=\frac{1}{2}\le 1$. When $x=-\frac{5}{2}$, the series becomes $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating Series Test. Thus, $I=\left(-\frac{7}{2},-\frac{5}{2}\right]$.

19. If $a_n = \frac{(x-2)^n}{n^n}$, then $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{|x-2|}{n} = 0$, so the series converges for all x (by the Root Test). $R=\infty$ and $I=(-\infty,\infty)$.

20.

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3x-2)^{n+1}}{(n+1)3^{n+1}}\cdot\frac{n3^n}{(3x-2)^n}\right| = \lim_{n\to\infty}\left(\frac{|3x-2|}{3}\cdot\frac{1}{1+1/n}\right) = \frac{|3x-2|}{3} = \left|x-\frac{2}{3}\right|$, so by the Ratio Test, the series converges when $\left|x-\frac{2}{3}\right|<1 \Leftrightarrow -\frac{1}{3}<x<\frac{5}{3}$. $R=1$. When $x=-\frac{1}{3}$, the series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, the convergent alternating harmonic series. When $x=\frac{5}{3}$, the series becomes the divergent harmonic series. Thus, $I=\left[-\frac{1}{3},\frac{5}{3}\right)$.

21. $a_n = \dfrac{n}{b^n}(x-a)^n$, where $b>0$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)|x-a|^{n+1}}{b^{n+1}}\cdot\frac{b^n}{n|x-a|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{|x-a|}{b} = \frac{|x-a|}{b} .$$

By the Ratio Test, the series converges when $\dfrac{|x-a|}{b} < 1 \Leftrightarrow |x-a| < b$ [so $R=b$] $\Leftrightarrow$ $-b<x-a<b \Leftrightarrow a-b<x<a+b$. When $|x-a|=b$, $\lim_{n\to\infty}\left|a_n\right| = \lim_{n\to\infty} n = \infty$, so the series diverges. Thus, $I=(a-\text{b},\text{a}+b)$.

22. $a_n = \dfrac{n(x-4)^n}{n^3+1}$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)|x-4|^{n+1}}{(n+1)^3+1}\cdot\frac{n^3+1}{n|x-4|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{n^3+1}{n^3+3n^2+3n+2}|x-4| = |x-4| .$$

By the Ratio Test, the series converges when $|x-4|<1 \Leftrightarrow -1<x-4<1 \Leftrightarrow 3<x<5$.

When $|x-4|=1$, $\sum_{n=1}^{\infty}\left|a_n\right| = \sum_{n=1}^{\infty}\dfrac{n}{n^3+1}$, which converges by comparison with the convergent p – series $\sum_{n=1}^{\infty}\dfrac{1}{n^2}$ $(p=2>1)$. Thus, $I=[3,5]$.

23. If $a_n = \text{n}!(2x-1)^n$, then $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\dfrac{(n+1)!(2x-1)^{n+1}}{\text{n}!(2x-1)^n}\right| = \lim_{n\to\infty}(n+1)|2x-1| \to \infty$ as $n\to\infty$ for all $x\neq\dfrac{1}{2}$. Since the series diverges for all $x\neq\dfrac{1}{2}$, $R=0$ and $I=\left\{\dfrac{1}{2}\right\}$.

24. $a_n = \dfrac{n^2x^n}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)} = \dfrac{n^2x^n}{2^n\text{n}!} = \dfrac{nx^n}{2^n(n-1)!}$, so

$\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\dfrac{(n+1)|x|^{n+1}}{2^{n+1}\text{n}!}\cdot\dfrac{2^n(n-1)!}{n|x|^n} = \lim_{n\to\infty}\dfrac{n+1}{n^2}\dfrac{|x|}{2} = 0$. Thus, by the Ratio Test, the series converges for *all* real x and we have $R=\infty$ and $I=(-\infty,\infty)$.

25. $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\dfrac{|4x+1|^{n+1}}{(n+1)^2}\cdot\dfrac{n^2}{|4x+1|^n}\right] = \lim_{n\to\infty}\dfrac{|4x+1|}{(1+1/n)^2} = |4x+1|$, so by the Ratio

Test, the series converges when $|4x+1|<1 \Leftrightarrow -1<4x+1<1 \Leftrightarrow -2<4x<0 \Leftrightarrow -\frac{1}{2}<x<0$, so $R=\frac{1}{4}$. When $x=-\frac{1}{2}$, the series becomes $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$, which converges by the Alternating Series Test. When $x=0$, the series becomes $\sum_{n=1}^{\infty}\frac{1}{n^2}$, a convergent p –series ($p=2>1$). $I=\left[-\frac{1}{2},0\right]$.

26. If $a_n=\frac{(-1)^n(2x+3)^n}{n\ln n}$, then we need $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=|2x+3|\lim_{n\to\infty}\frac{n\ln n}{(n+1)\ln(n+1)}=|2x+3|<1$ for convergence, so $-2<x<-1$ and $R=\frac{1}{2}$. When $x=-2$, $\sum_{n=2}^{\infty}a_n=\sum_{n=2}^{\infty}\frac{1}{n\ln n}$, which diverges (Integral Test), and when $x=-1$, $\sum_{n=2}^{\infty}a_n=\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$, which converges (Alternating Series Test), so $I=(-2,-1]$.

27. If $a_n=\frac{x^n}{(\ln n)^n}$, then $\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\frac{|x|}{\ln n}=0<1$ for all x , so $R=\infty$ and $I=(-\infty,\infty)$ by the Root Test.

28. If $a_n=\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)x^n}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}$, then we need $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}|x|\left(\frac{2n+2}{2n+1}\right)=|x|<1$ for convergence, so $R=1$. If $x=\pm 1$, $\left|a_n\right|=\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}>1$ for all n since each integer in the numerator is larger than the corresponding one in the denominator, so $\sum a_n$ diverges in both cases by the Test for Divergence, and $I=(-1,1)$.

29. **(a)** We are given that the power series $\sum_{n=0}^{\infty}c_n x^n$ is convergent for $x=4$. So by Theorem 3, it must converge for at least $-4<x\leq 4$. In particular, it converges when $x=-2$; that is, $\sum_{n=0}^{\infty}c_n(-2)^n$ is convergent.

(b) It does not follow that $\sum_{n=0}^{\infty}c_n(-4)^n$ is necessarily convergent.

30. We are given that the power series $\sum_{n=0}^{\infty}c_n x^n$ is convergent for $x=-4$ and divergent when $x=6$. So

by Theorem 3 it converges for at least $-4\leq x<4$ and diverges for at least $x\geq 6$ and $x<-6$. Therefore:
(a) It converges when $x=1$; that is, Σc_n is convergent.

(b) It diverges when $x=8$; that is, $\Sigma c_n 8^n$ is divergent.

(c) It converges when $x=-3$; that is, $\Sigma c_n(-3)^n$ is convergent.

(d) It diverges when $x=-9$; that is, $\Sigma c_n(-9)^n=\Sigma(-1)^n c_n 9^n$ is divergent.

31. If $a_n=\dfrac{(\mathrm{n}!)^k}{(kn)!}x^n$, then

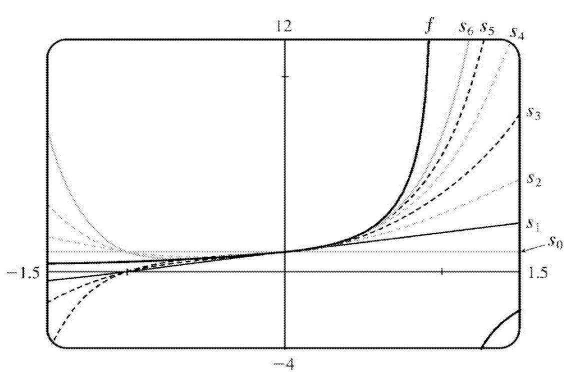
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{[(n+1)!]^k(kn)!}{(\mathrm{n}!)^k[k(n+1)]!}|x|=\lim_{n\to\infty}\frac{(n+1)^k}{(kn+k)(kn+k-1)\cdot\cdot\cdot(kn+2)(kn+1)}|x|$$

$$=\lim_{n\to\infty}\left[\frac{(n+1)}{(kn+1)}\frac{(n+1)}{(kn+2)}\cdot\cdot\cdot\frac{(n+1)}{(kn+k)}\right]|x|$$

$$=\lim_{n\to\infty}\left[\frac{n+1}{kn+1}\right]\lim_{n\to\infty}\left[\frac{n+1}{kn+2}\right]\cdot\cdot\cdot\lim_{n\to\infty}\left[\frac{n+1}{kn+k}\right]|x|=\left(\frac{1}{k}\right)^k|x|<1\Leftrightarrow$$

$|x|<k^k$ for convergence, and the radius of convergence is $R=k^k$.

32. The partial sums of the series $\Sigma_{n=0}^{\infty}x^n$ definitely do not converge to $f(x)=1/(1-x)$ for $x\geq 1$, since f is undefined at $x=1$ and negative on $(1,\infty)$, while all the partial sums are positive on this interval. The partial sums also fail to converge to f for $x\leq -1$, since $0<f(x)<1$ on this interval, while the partial sums are either larger than 1 or less than 0 . The partial sums seem to converge to f on $(-1,1)$. This graphical evidence is consistent with what we know about geometric series: convergence for $|x|<1$, divergence for $|x|\geq 1$ (see Example .2.5).

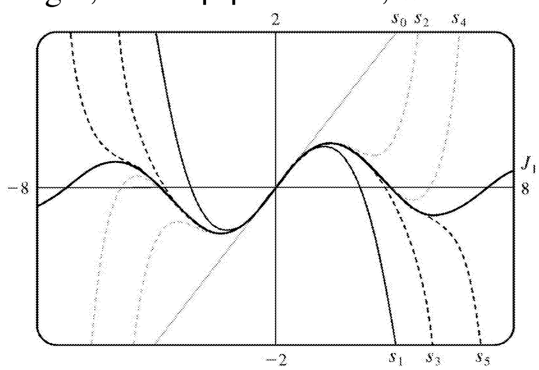


33. **(a)** If

$a_n = \dfrac{(-1)^n x^{2n+1}}{n!(n+1)!\,2^{2n+1}}$, then

$\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \dfrac{x^{2n+3}}{(n+1)!(n+2)!\,2^{2n+3}} \cdot \dfrac{n!(n+1)!\,2^{2n+1}}{x^{2n+1}} \right| = \left(\dfrac{x}{2} \right)^2 \lim_{n\to\infty} \dfrac{1}{(n+1)(n+2)} = 0$ for all x . So $J_1(x)$ converges for all x and its domain is $(-\infty, \infty)$.

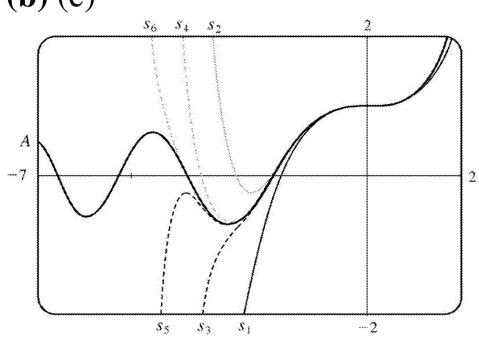
(b) p190pt (c) The initial terms of $J_1(x)$ up to n=5 are $a_0 = \dfrac{x}{2}$, $a_1 = -\dfrac{x^3}{16}$, $a_2 = \dfrac{x^5}{384}$, $a_3 = -\dfrac{x^7}{18{,}432}$, $a_4 = \dfrac{x^9}{1{,}474{,}560}$, and $a_5 = -\dfrac{x^{11}}{176{,}947{,}200}$. The partial sums seem to approximate $J_1(x)$ well near the origin, but as $|x|$ increases, we need to take a large number of terms to get a good approximation.



34. **(a)** $A(x) = 1 + \sum_{n=1}^{\infty} a_n$, where $a_n = \dfrac{x^{3n}}{2\cdot 3\cdot 5\cdot 6\cdot\cdots\cdot(3n-1)(3n)}$, so

$\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = |x|^3 \lim_{n\to\infty} \dfrac{1}{(3n+2)(3n+3)} = 0$ for all x , so the domain is R .

(b) (c)



s_0=1 has been omitted from the graph. The partial sums seem to approximate $A(x)$ well near the origin, but as $|x|$ increases, we need to take a large number of terms to get a good approximation.

To plot A , we must first define $A(x)$ for the CAS. Note that for $n\geq 1$, the denominator of a_n is

$2\cdot 3\cdot 5\cdot 6\cdot \cdots \cdot (3n-1)\cdot 3n=\dfrac{(3n)!}{1\cdot 4\cdot 7\cdot \cdots \cdot (3n-2)}=\dfrac{(3n)!}{\prod_{k=1}^{n}(3k-2)}$, so $a_n=\dfrac{\prod_{k=1}^{n}(3k-2)}{(3n)!}x^{3n}$ and thus

$A(x)=1+\sum_{n=1}^{\infty}\dfrac{\prod_{k=1}^{n}(3k-2)}{(3n)!}x^{3n}$. Both Maple and Mathematica are able to plot A if we define it this way, and Derive is able to produce a similar graph using a suitable partial sum of $A(x)$. Derive, Maple and Mathematica all have two initially known Airy functions, called AI_SERIES(z,m) and BI_SERIES(z,m) from BESSEL.MTH in Derive and AiryAi and AiryBi in Maple and Mathematica (just Ai and Bi in older versions of Maple). However, it is very difficult to solve for A in terms of the CAS's Airy functions, although in fact

$A(x)=\dfrac{\sqrt{3}\ \mathrm{AiryAi}(x)+\mathrm{AiryBi}(x)}{\sqrt{3}\ \mathrm{AiryAi}(0)+\mathrm{AiryBi}(0)}$.

35.

$$\begin{aligned} s_{2n-1} &=1+2x+x^2+2x^3+x^4+2x^5+\cdots+x^{2n-2}+2x^{2n-1} \\ &=1(1+2x)+x^2(1+2x)+x^4(1+2x)+\cdots+x^{2n-2}(1+2x) \\ &=(1+2x)\left(1+x^2+x^4+\cdots+x^{2n-2}\right) \\ &=(1+2x)\frac{1-x^{2n}}{1-x^2}\ \text{ with } r=x^2\ ] \to \frac{1+2x}{1-x^2}\ \text{ as } n\to\infty\ , \end{aligned}$$

when $|x|<1$. Also $s_{2n}=s_{2n-1}+x^{2n}\to\dfrac{1+2x}{1-x^2}$ since $x^{2n}\to 0$ for $|x|<1$. Therefore, $s_n\to\dfrac{1+2x}{1-x^2}$ since s_{2n} and s_{2n-1} both approach $\dfrac{1+2x}{1-x^2}$ as $n\to\infty$. Thus, the interval of convergence is $(-1,1)$ and $f(x)=\dfrac{1+2x}{1-x^2}$.

36.

$$s_{4n-1}=c_0+c_1x+c_2x^2+c_3x^3+c_0x^4+c_1x^5+c_2x^6+c_3x^7+\cdots+c_3x^{4n-1}$$

$$=\left(c_0+c_1x+c_2x^2+c_3x^3\right)\left(1+x^4+x^8+\cdot\cdot\cdot+x^{4n-4}\right)\to\frac{c_0+c_1x+c_2x^2+c_3x^3}{1-x^4}\text{ as }n\to\infty$$

for $\left|x^4\right|<1\Leftrightarrow|x|<1$. Also s_{4n} , s_{4n+1} , s_{4n+2} have the same limits (for example, $s_{4n}=s_{4n-1}+c_0x^{4n}$ and $x^{4n}\to0$ for $|x|<1$). So if at least one of c_0 , c_1 , c_2 , and c_3 is nonzero, then the interval of convergence is $(-1,1)$ and $f(x)=\dfrac{c_0+c_1x+c_2x^2+c_3x^3}{1-x^4}$.

37. We use the Root Test on the series Σc_nx^n . We need $\lim\limits_{n\to\infty}\sqrt[n]{\left|c_nx^n\right|}=|x|\lim\limits_{n\to\infty}\sqrt[n]{\left|c_n\right|}=c|x|<1$ for convergence, or $|x|<1/c$, so $R=1/c$.

38. Suppose $c_n\neq0$. Applying the Ratio Test to the series $\Sigma c_n(x-a)^n$, we find that

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right|=\lim_{n\to\infty}\frac{|x-a|}{\left|c_n/c_{n+1}\right|}\ (*)=\frac{|x-a|}{\lim\limits_{n\to\infty}\left|c_n/c_{n+1}\right|}$$

(if $\lim\limits_{n\to\infty}\left|c_n/c_{n+1}\right|\neq0$), so the series converges when $\dfrac{|x-a|}{\lim\limits_{n\to\infty}\left|c_n/c_{n+1}\right|}<1\Leftrightarrow|x-a|<\lim\limits_{n\to\infty}\left|\dfrac{c_n}{c_{n+1}}\right|$.

Thus, $R=\lim\limits_{n\to\infty}\left|\dfrac{c_n}{c_{n+1}}\right|$. If $\lim\limits_{n\to\infty}\left|\dfrac{c_n}{c_{n+1}}\right|=0$ and $|x-a|\neq0$, then $(*)$ shows that $L=\infty$ and so the series diverges, and hence, $R=0$. Thus, in all cases, $R=\lim\limits_{n\to\infty}\left|\dfrac{c_n}{c_{n+1}}\right|$.

39. For $2<x<3$, Σc_nx^n diverges and Σd_nx^n converges. By Exercise .2.61, $\Sigma\left(c_n+d_n\right)x^n$ diverges. Since both series converge for $|x|<2$, the radius of convergence of $\Sigma\left(c_n+d_n\right)x^n$ is 2 .

40. Since Σc_nx^n converges whenever $|x|<R$, $\Sigma c_nx^{2n}=\Sigma c_n\left(x^2\right)^n$ converges whenever $\left|x^2\right|<R\Leftrightarrow|x|<\sqrt{R}$, so the second series has radius of convergence $\sqrt{R}$.

1. If $f(x)=\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence 10 , then $f'(x)=\sum_{n=1}^{\infty} nc_n x^{n-1}$ also has radius of convergence 10 by Theorem 2.

2. If $f(x)=\sum_{n=0}^{\infty} b_n x^n$ converges on $(-2,2)$, then $\int f(x)dx=C+\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$ has the same radius of convergence (by Theorem 2), but may not have the same interval of convergence — it may happen that the integrated series converges at an endpoint (or both endpoints).

3. Our goal is to write the function in the form $\frac{1}{1-r}$, and then use Equation (1) to represent the function as a sum of a power series. $f(x)=\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}(-x)^n=\sum_{n=0}^{\infty}(-1)^n x^n$ with $|-x|<1 \Leftrightarrow |x|<1$, so $R=1$ and $I=(-1,1)$.

4. $f(x)=\frac{3}{1-x^4}=3\left(\frac{1}{1-x^4}\right)=3(1+x^4+x^8+x^{12}+\cdots)=3\sum_{n=0}^{\infty}(x^4)^n=\sum_{n=0}^{\infty}3x^{4n}$ with $\left|x^4\right|<1 \Leftrightarrow |x|<1$, so $R=1$ and $I=(-1,1)$.

5. Replacing x with x^3 in (1) gives $f(x)=\frac{1}{1-x^3}=\sum_{n=0}^{\infty}\left(x^3\right)^n=\sum_{n=0}^{\infty}x^{3n}$. The series converges when $\left|x^3\right|<1 \Leftrightarrow |x|^3<1 \Leftrightarrow |x|<\sqrt[3]{1} \Leftrightarrow |x|<1$. Thus, $R=1$ and $I=(-1,1)$.

6. $f(x)=\frac{1}{1+9x^2}=\frac{1}{1-\left(-9x^2\right)}=\sum_{n=0}^{\infty}\left(-9x^2\right)^n=\sum_{n=0}^{\infty}(-1)^n 3^{2n} x^{2n}$. The series converges when $\left|-9x^2\right|<1$; that is, when $|x|<\frac{1}{3}$, so $I=\left(-\frac{1}{3},\frac{1}{3}\right)$.

7. $f(x)=\frac{1}{x-5}=-\frac{1}{5}\left(\frac{1}{1-x/5}\right)=-\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n$ or equivalently, $-\sum_{n=0}^{\infty}\frac{1}{5^{n+1}}x^n$. The series converges when $\left|\frac{x}{5}\right|<1$; that is, when $|x|<5$, so $I=(-5,5)$.

8. $f(x)=\frac{x}{4x+1}=x\cdot\frac{1}{1-(-4x)}=x\sum_{n=0}^{\infty}(-4x)^n=\sum_{n=0}^{\infty}(-1)^n 2^{2n} x^{n+1}$. The series converges when $|-4x|<1$; that is, when $|x|<\frac{1}{4}$, so $I=\left(-\frac{1}{4},\frac{1}{4}\right)$.

9.

$$f(x)=\frac{x}{9+x^2}=\frac{x}{9}\left[\frac{1}{1+(x/3)^2}\right]=\frac{x}{9}\left[\frac{1}{1-\{-(x/3)^2\}}\right]=\frac{x}{9}\sum_{n=0}^{\infty}\left[-\left(\frac{x}{3}\right)^2\right]^n=\frac{x}{9}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{9^n}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{9^{n+1}}$$

. The geometric series $\sum_{n=0}^{\infty}\left[-\left(\frac{x}{3}\right)^2\right]^n$ converges when

$\left|-\left(\frac{x}{3}\right)^2\right|<1 \Leftrightarrow \frac{|x^2|}{9}<1 \Leftrightarrow |x|^2<9 \Leftrightarrow |x|<3$, so $R=3$ and $I=(-3,3)$.

10. $f(x)=\frac{x^2}{a^3-x^3}=\frac{x^2}{a^3}\cdot\frac{1}{1-x^3/a^3}=\frac{x^2}{a^3}\sum_{n=0}^{\infty}\left(\frac{x^3}{a^3}\right)^n=\sum_{n=0}^{\infty}\frac{x^{3n+2}}{a^{3n+3}}$. The series converges when $\left|x^3/a^3\right|<1 \Leftrightarrow \left|x^3\right|<\left|a^3\right| \Leftrightarrow |x|<|a|$, so $R=|a|$ and $I=(-|a|,|a|)$.

11. $f(x)=\frac{3}{x^2+x-2}=\frac{3}{(x+2)(x-1)}=\frac{A}{x+2}+\frac{B}{x-1} \Rightarrow 3=A(x-1)+B(x+2)$. Taking $x=-2$, we get $A=-1$. Taking $x=1$, we get $B=1$. Thus,

$$\frac{3}{x^2+x-2}=\frac{1}{x-1}-\frac{1}{x+2}=-\frac{1}{1-x}-\frac{1}{2}\,\frac{1}{1+x/2}=-\sum_{n=0}^{\infty}x^n-\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

$$=\sum_{n=0}^{\infty}\left[-1-\frac{1}{2}\left(-\frac{1}{2}\right)^n\right]x^n=\sum_{n=0}^{\infty}\left[-1+\left(-\frac{1}{2}\right)^{n+1}\right]x^n=\sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}}{2^{n+1}}-1\right]x^n$$

We represented the given function as the sum of two geometric series; the first converges for $x\in(-1,1)$ and the second converges for $x\in(-2,2)$. Thus, the sum converges for $x\in(-1,1)=I$.

12.

$$f(x)=\frac{7x-1}{3x^2+2x-1}=\frac{7x-1}{(3x-1)(x+1)}=\frac{A}{3x-1}+\frac{B}{x+1}=\frac{1}{3x-1}+\frac{2}{x+1}=2\cdot\frac{1}{1-(-x)}-\frac{1}{1-3x}$$

$$=2\sum_{n=0}^{\infty}(-x)^n-\sum_{n=0}^{\infty}(3x)^n=\sum_{n=0}^{\infty}\left[2(-1)^n-3^n\right]x^n$$

The series $\sum(-x)^n$ converges for $x\in(-1,1)$ and the series $\sum(3x)^n$ converges for $x\in\left(-\frac{1}{3},\frac{1}{3}\right)$, so their sum converges for $x\in\left(-\frac{1}{3},\frac{1}{3}\right)=I$.

13. **(a)**

$$f(x)=\frac{1}{(1+x)^2}=\frac{d}{dx}\left(\frac{-1}{1+x}\right)=-\frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]$$
$$=\sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1}=\sum_{n=0}^{\infty}(-1)^n(n+1)x^n \text{ with } R=1 .$$

In the last step, note that we *decreased* the initial value of the summation variable n by 1 , and then *increased* each occurrence of n in the term by 1 .

(b)

$$f(x)=\frac{1}{(1+x)^3}=-\frac{1}{2}\frac{d}{dx}\left[\frac{1}{(1+x)^2}\right]=-\frac{1}{2}\frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n(n+1)x^n\right]$$
$$=-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1}=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n \text{ with } R=1 .$$

(c)

$$f(x)=\frac{x^2}{(1+x)^3}=x^2\cdot\frac{1}{(1+x)^3}=x^2\cdot\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$$

$=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}$. To write the power series with x^n rather than x^{n+2} ,

we will *decrease* each occurrence of n in the term by 2 and *increase* the initial value of the summation variable by 2 . This gives us $\frac{1}{2}\sum_{n=2}^{\infty}(-1)^n(n)(n-1)x^n$.

14. **(a)** $\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}(-1)^n x^n$, so

$$f(x)=\ln(1+x)=\int\frac{dx}{1+x}=\int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx=C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$$
$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}\ C=0 \text{ since } f(0)=\ln 1=0],\text{ with } R=1$$

(b) $f(x)=x\ln(1+x)=x\left[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}\right]=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{n+1}}{n}=\sum_{n=2}^{\infty}\frac{(-1)^n x^n}{n-1}$ with $R=1$.

(c) $f(x)=\ln\left(x^2+1\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\left(x^2\right)^n}{n}$ [by part (a)] $=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n}$ with $R=1$.

15.

$$f(x)=\ln(5-x)=-\int\frac{dx}{5-x}=-\frac{1}{5}\int\frac{dx}{1-x/5}$$

$$=-\frac{1}{5}\int\left[\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n\right]dx=C-\frac{1}{5}\sum_{n=0}^{\infty}\frac{x^{n+1}}{5^n(n+1)}=C-\sum_{n=1}^{\infty}\frac{x^n}{n5^n}$$

Putting $x=0$, we get $C=\ln 5$. The series converges for $|x/5|<1\Leftrightarrow|x|<5$, so $R=5$.

16. We know that $\frac{1}{1-2x}=\sum_{n=0}^{\infty}(2x)^n$. Differentiating, we get

$\frac{2}{(1-2x)^2}=\sum_{n=1}^{\infty}2^n nx^{n-1}=\sum_{n=0}^{\infty}2^{n+1}(n+1)x^n$, so

$f(x)=\frac{x^2}{(1-2x)^2}=\frac{x^2}{2}\cdot\frac{2}{(1-2x)^2}=\frac{x^2}{2}\sum_{n=0}^{\infty}2^{n+1}(n+1)x^n=\sum_{n=0}^{\infty}2^n(n+1)x^{n+2}$ or $\sum_{n=2}^{\infty}2^{n-2}(n-1)x^n$, with $R=\frac{1}{2}$.

17. $\frac{1}{2-x}=\frac{1}{2(1-x/2)}=\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}x^n$ for $\left|\frac{x}{2}\right|<1\Leftrightarrow|x|<2$. Now

$\frac{1}{(x-2)^2}=\frac{d}{dx}\left(\frac{1}{2-x}\right)=\frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}x^n\right)=\sum_{n=1}^{\infty}\frac{n}{2^{n+1}}x^{n-1}=\sum_{n=0}^{\infty}\frac{n+1}{2^{n+2}}x^n$. So

$f(x)=\frac{x^3}{(x-2)^2}=x^3\sum_{n=0}^{\infty}\frac{n+1}{2^{n+2}}x^n=\sum_{n=0}^{\infty}\frac{n+1}{2^{n+2}}x^{n+3}$ or $\sum_{n=3}^{\infty}\frac{n-2}{2^{n-1}}x^n$ for $|x|<2$. Thus, $R=2$ and $I=(-2,2)$.

18. From Example 7, $g(x)=\arctan x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$. Thus,

$f(x)=\arctan(x/3)=\sum_{n=0}^{\infty}(-1)^n\frac{(x/3)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{2n+1}(2n+1)}x^{2n+1}$ for $\left|\frac{x}{3}\right|<1\Leftrightarrow|x|<3$, so $R=3$.

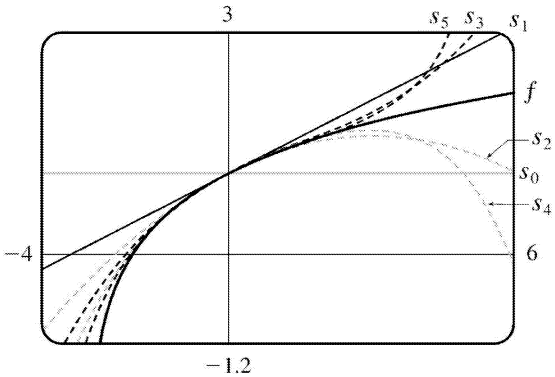
19.

$$f(x)=\ln(3+x)=\int\frac{dx}{3+x}=\frac{1}{3}\int\frac{dx}{1+x/3}=\frac{1}{3}\int\frac{dx}{1-(-x/3)}=\frac{1}{3}\int\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n dx$$

$=C+\frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)3^n}x^{n+1}=\ln 3+\frac{1}{3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n3^{n-1}}x^n$

$=\ln 3+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n3^n}x^n$. The series converges when $|-x/3|<1 \Leftrightarrow |x|<3$, so $R=3$.

The terms of the series are $a_0=\ln 3, a_1=\frac{x}{3}, a_2=-\frac{x^2}{18}, a_3=\frac{x^3}{81}, a_4=-\frac{x^4}{324}, a_5=\frac{x^5}{1215}, \ldots$.



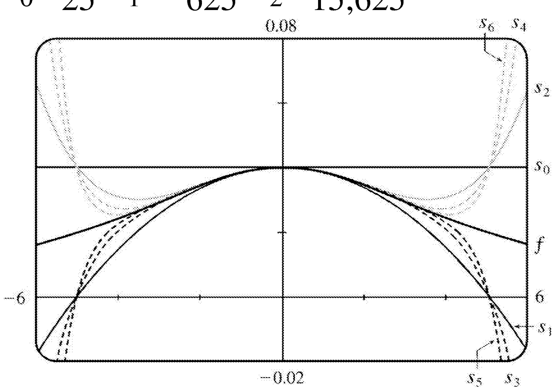
As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-3,3)$.

20.

$f(x)=\frac{1}{x^2+25}=\frac{1}{25}\left(\frac{1}{1+x^2/25}\right)=\frac{1}{25}\left(\frac{1}{1-\left(-x^2/25\right)}\right)=\frac{1}{25}\sum_{n=0}^{\infty}\left(-\frac{x^2}{25}\right)^n=\frac{1}{25}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{5}\right)^{2n}$

. The series converges when $\left|-x^2/25\right|<1 \Leftrightarrow x^2<25 \Leftrightarrow |x|<5$, so $R=5$. The terms of the series are

$a_0=\frac{1}{25}, a_1=-\frac{x^2}{625}, a_2=\frac{x^4}{15{,}625}, \ldots$.

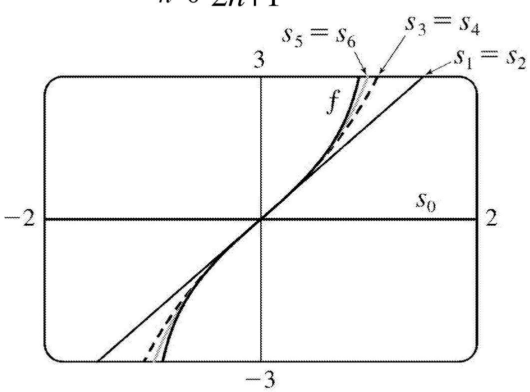


As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-5,5)$.

21.

$$f(x)=\ln\left(\frac{1+x}{1-x}\right)=\ln(1+x)-\ln(1-x)=\int\frac{dx}{1+x}+\int\frac{dx}{1-x}$$
$$=\int\frac{dx}{1-(-x)}+\int\frac{dx}{1-x}=\int\left[\sum_{n=0}^{\infty}(-1)^n x^n+\sum_{n=0}^{\infty}x^n\right]dx$$
$$=\int\left[\left(1-x+x^2-x^3+x^4-\cdots\right)+\left(1+x+x^2+x^3+x^4+\cdots\right)\right]dx$$
$$=\int\left(2+2x^2+2x^4+\cdots\right)dx=\int\sum_{n=0}^{\infty}2x^{2n}\,dx=C+\sum_{n=0}^{\infty}\frac{2x^{2n+1}}{2n+1}$$

But $f(0)=\ln\frac{1}{1}=0$, so $C=0$ and we have $f(x)=\sum_{n=0}^{\infty}\frac{2x^{2n+1}}{2n+1}$ with $R=1$. If $x=\pm 1$, then $f(x)=\pm 2\sum_{n=0}^{\infty}\frac{1}{2n+1}$, which both diverge by the Limit Comparison Test with $b_n=\frac{1}{n}$.

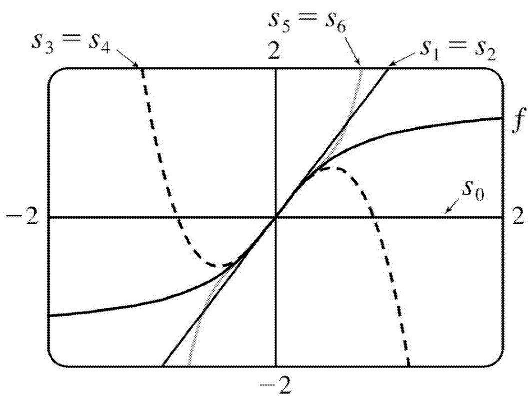


As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-1,1)$.

22.

$$f(x)=\tan^{-1}(2x)=2\int\frac{dx}{1+4x^2}=2\int\sum_{n=0}^{\infty}(-1)^n\left(4x^2\right)^n dx=2\int\sum_{n=0}^{\infty}(-1)^n 4^n x^{2n}\,dx$$
$$=C+2\sum_{n=0}^{\infty}\frac{(-1)^n 4^n x^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}x^{2n+1}}{2n+1}\quad[\,f(0)=\tan^{-1}0=0\text{, so }C=0\,].$$

The series converges when $\left|4x^2\right|<1\Leftrightarrow|x|<\frac{1}{2}$, so $R=\frac{1}{2}$. If $x=\pm\frac{1}{2}$, then $f(x)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}$ and $f(x)=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{2n+1}$, respectively. Both series converge by the Alternating Series Test.



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

23. $\frac{t}{1-t^8} = t\cdot\frac{1}{1-t^8} = t\sum_{n=0}^{\infty}(t^8)^n = \sum_{n=0}^{\infty} t^{8n+1} \Rightarrow \int \frac{t}{1-t^8}\,dt = C + \sum_{n=0}^{\infty}\frac{t^{8n+2}}{8n+2}$. The series for $\frac{1}{1-t^8}$ converges when $\left|t^8\right| < 1 \Leftrightarrow |t| < 1$, so $R=1$ for that series and also the series for $t/(1-t^8)$. By Theorem 2, the series for $\int \frac{t}{1-t^8}\,dt$ also has $R=1$.

24. By Example 6, $\ln(1-t) = -\sum_{n=1}^{\infty}\frac{t^n}{n}$ for $|t|<1$, so $\frac{\ln(1-t)}{t} = -\sum_{n=1}^{\infty}\frac{t^{n-1}}{n}$ and $\int \frac{\ln(1-t)}{t}\,dt = C - \sum_{n=1}^{\infty}\frac{t^n}{n^2}$. By Theorem 2, $R=1$.

25. By Example 7, $\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ with $R=1$, so

$$x - \tan^{-1}x = x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) = \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{2n+1}$$ and

$$\frac{x - \tan^{-1}x}{x^3} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-2}}{2n+1},$$ so

$$\int \frac{x-\tan^{-1}x}{x^3}\,dx = C + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{(2n+1)(2n-1)} = C + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{4n^2-1}.$$ By Theorem 2, $R=1$.

26. By Example 7, $\int \tan^{-1}\left(x^2\right)dx = \int \sum_{n=0}^{\infty}(-1)^n\frac{\left(x^2\right)^{2n+1}}{2n+1}\,dx = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+3}}{(2n+1)(4n+3)}$ with $R=1$.

27. $\frac{1}{1+x^5}=\frac{1}{1-\left(-x^5\right)}=\sum_{n=0}^{\infty}\left(-x^5\right)^n=\sum_{n=0}^{\infty}(-1)^n x^{5n}\Rightarrow$

$\int \frac{1}{1+x^5}\,dx=\int\sum_{n=0}^{\infty}(-1)^n x^{5n}\,dx=C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{5n+1}}{5n+1}$. Thus,

$I=\int_0^{0.2}\frac{1}{1+x^5}\,dx=\left[x-\frac{x^6}{6}+\frac{x^{11}}{11}-\cdots\right]_0^{0.2}=0.2-\frac{(0.2)^6}{6}+\frac{(0.2)^{11}}{11}-\cdots$. The series is alternating, so if we use the first two terms, the error is at most $(0.2)^{11}/11\approx 1.9\times 10^{-9}$. So $I\approx 0.2-(0.2)^6/6\approx 0.199989$ to six decimal places.

28. From Example 6 we know $\ln(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n}$, so

$\ln\left(1+x^4\right)=\ln\left[1-\left(-x^4\right)\right]=-\sum_{n=1}^{\infty}\frac{\left(-x^4\right)^n}{n}=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{4n}}{n}\Rightarrow$

$\int\ln\left(1+x^4\right)dx=\int\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{4n}}{n}\,dx=C+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{4n+1}}{n(4n+1)}$. Thus,

$I=\int_0^{0.4}\ln\left(1+x^4\right)dx=\left[\frac{x^5}{5}-\frac{x^9}{18}+\frac{x^{13}}{39}-\frac{x^{17}}{68}+\cdots\right]_0^{0.4}=\frac{(0.4)^5}{5}-\frac{(0.4)^9}{18}+\frac{(0.4)^{13}}{39}-\frac{(0.4)^{17}}{68}+\cdots$. The series is alternating, so if we use the first three terms, the error is at most $(0.4)^{17}/68\approx 2.5\times 10^{-9}$. So $I\approx(0.4)^5/5-(0.4)^9/18+(0.9)^{13}/39\approx 0.002034$ to six decimal places.

29. We substitute x^4 for x in Example 7, and find that

$$\int x^2\tan^{-1}\left(x^4\right)dx=\int x^2\sum_{n=0}^{\infty}(-1)^n\frac{\left(x^4\right)^{2n+1}}{2n+1}\,dx$$

$$=\int\sum_{n=0}^{\infty}(-1)^n\frac{x^{8n+6}}{2n+1}\,dx=C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{8n+7}}{(2n+1)(8n+7)}$$

So $\int_0^{1/3}x^2\tan^{-1}\left(x^4\right)dx=\left[\frac{x^7}{7}-\frac{x^{15}}{45}+\cdots\right]_0^{1/3}=\frac{1}{7\cdot 3^7}-\frac{1}{45\cdot 3^{15}}+\cdots$. The series is alternating, so if we use only one term, the error is at most $1/\left(45\cdot 3^{15}\right)\approx 1.5\times 10^{-9}$. So $\int_0^{1/3}x^2\tan^{-1}\left(x^4\right)dx\approx 1/\left(7\cdot 3^7\right)\approx 0.000065$ to six decimal places.

30. We substitute x^4 for x in Example 7, and find that

$$\int x^2 \tan^{-1}\left(x^4\right) dx = \int x^2 \Sigma_{n=0}^{\infty} (-1)^n \frac{\left(x^4\right)^{2n+1}}{2n+1}\, dx$$

$$= \int \Sigma_{n=0}^{\infty} (-1)^n \frac{x^{8n+6}}{2n+1}\, dx = C + \Sigma_{n=0}^{\infty} (-1)^n \frac{x^{8n+7}}{(2n+1)(8n+7)}$$

So $\int_0^{1/3} x^2 \tan^{-1}\left(x^4\right) dx = \left[\frac{x^7}{7} - \frac{x^{15}}{45} + \cdots\right]_0^{1/3} = \frac{1}{7\cdot 3^7} - \frac{1}{45\cdot 3^{15}} + \cdots$. The series is alternating, so if we use only one term, the error is at most $1/\left(45\cdot 3^{15}\right) \approx 1.5\times 10^{-9}$. So $\int_0^{1/3} x^2 \tan^{-1}\left(x^4\right) dx \approx 1/\left(7\cdot 3^7\right) \approx 0.000065$ to six decimal places.

31. Using the result of Example 6, $\ln(1-x) = -\Sigma_{n=1}^{\infty} \frac{x^n}{n}$, with $x=-0.1$, we have $\ln 1.1 = \ln\left[1-(-0.1)\right] = 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} + \frac{0.00001}{5} - \cdots$. The series is alternating, so if we use only the first four terms, the error is at most $\frac{0.00001}{5} = 0.000002$. So $\ln 1.1 \approx 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} \approx 0.09531$.

32. $f(x) = \Sigma_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow f'(x) = \Sigma_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{(2n)!}$, so

$$f''(x) = \Sigma_{n=1}^{\infty} \frac{(-1)^n (2n)(2n-1) x^{2n-2}}{(2n)!} = \Sigma_{n=1}^{\infty} \frac{(-1)^n x^{2(n-1)}}{[2(n-1)]!}$$

$$= \Sigma_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!} \quad [\text{ subsistuting } n+1 \text{ for } n\ ]$$

$$= -\Sigma_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = -f(x) \Rightarrow f''(x) + f(x) = 0.$$

33. **(a)** $J_0(x) = \Sigma_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$, $J_0'(x) = \Sigma_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2}$, and $J_0''(x) = \Sigma_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1) x^{2n-2}}{2^{2n}(n!)^2}$, so

$$x^2 J_0''(x)+xJ_0'(x)+x^2 J_0(x)=\sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2}+\sum_{n=1}^{\infty} \frac{(-1)^n 2nx^{2n}}{2^{2n}(n!)^2}+\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^{2n}(n!)^2}$$

$$=\sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2}+\sum_{n=1}^{\infty} \frac{(-1)^n 2nx^{2n}}{2^{2n}(n!)^2}+\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{2^{2n-2}[(n-1)!]^2}$$

$$=\sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2}+\sum_{n=1}^{\infty} \frac{(-1)^n 2nx^{2n}}{2^{2n}(n!)^2}+\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^{-1}2^2 n^2 x^{2n}}{2^{2n}(n!)^2}$$

$$=\sum_{n=1}^{\infty}(-1)^n \left[\frac{2n(2n-1)+2n-2^2 n^2}{2^{2n}(n!)^2} \right] x^{2n}=\sum_{n=1}^{\infty}(-1)^n \left[\frac{4n^2-2n+2n-4n^2}{2^{2n}(n!)^2} \right] x^{2n}=0$$

(b)

$$\int_0^1 J_0(x)\,dx=\int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \right] dx=\int_0^1 \left(1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\cdots \right) dx$$

$$=\left[x-\frac{x^3}{3\cdot 4}+\frac{x^5}{5\cdot 64}-\frac{x^7}{7\cdot 2304}+\cdots \right]_0^1=1-\frac{1}{12}+\frac{1}{320}-\frac{1}{16{,}128}+\cdots$$

Since $\frac{1}{16{,}128}\approx 0.000062$, it follows from The Alternating Series Estimation Theorem that, correct to three decimal places, $\int_0^1 J_0(x)\,dx\approx 1-\frac{1}{12}+\frac{1}{320}\approx 0.920$.

34. **(a)** $J_1(x)=\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$, $J_1'(x)=\sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)x^{2n}}{n!(n+1)!2^{2n+1}}$, and

$$J_1''(x)=\sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n)x^{2n-1}}{n!(n+1)!2^{2n+1}} .$$

$$x^2 J_1''(x)+xJ_1'(x)+\left(x^2-1\right) J_1(x)$$

$$=\sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n)x^{2n+1}}{n!(n+1)!2^{2n+1}}+\sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

$$+\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{n!(n+1)!2^{2n+1}}-\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)(2n)x^{2n+1}}{n!(n+1)!2^{2n+1}}+\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

$$-\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+1}}{(n-1)!n!2^{2n-1}}-\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}\left[\begin{array}{c}\text{Replace n with n}-1\\ \text{in the third term}\end{array}\right]$$

$$=\frac{x}{2}-\frac{x}{2}+\sum_{n=1}^{\infty}(-1)^n\left[\frac{(2n+1)(2n)+(2n+1)-(n)(n+1)2^2-1}{n!(n+1)!2^{2n+1}}\right]x^{2n+1}=0$$

(b) $J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\Rightarrow$

$$J_0'(x)=\sum_{n=1}^{\infty}\frac{(-1)^n(2n)x^{2n-1}}{2^{2n}(n!)^2}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}2(n+1)x^{2n+1}}{2^{2n+2}[(n+1)!]^2}\quad[\text{ Replace } n \text{ with } n+1\ ]$$

$$=-\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^{2n+1}(n+1)!n!}\quad[\text{ cancel 2 and } n+1\text{ ; take }-1\text{ outside sum}]=-J_1(x)$$

35. **(a)** $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}\Rightarrow f'(x)=\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n!}=\sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}=\sum_{n=0}^{\infty}\frac{x^n}{n!}=f(x)$

(b) By Theorem .4.2, the only solution to the differential equation $df(x)/dx=f(x)$ is $f(x)=Ke^x$, but $f(0)=1$, so $K=1$ and $f(x)=e^x$.
Or: We could solve the equation $df(x)/dx=f(x)$ as a separable differential equation.

36. $\frac{|\sin nx|}{n^2}\le\frac{1}{n^2}$, so $\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}$ converges by the Comparison Test. $\frac{d}{dx}\left(\frac{\sin nx}{n^2}\right)=\frac{\cos nx}{n}$, so when $x=2k\pi$ (k an integer), $\sum_{n=1}^{\infty}f_n'(x)=\sum_{n=1}^{\infty}\frac{\cos(2kn\pi)}{n}=\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges (harmonic series). $f_n''(x)=-\sin nx$, so $\sum_{n=1}^{\infty}f_n''(x)=-\sum_{n=1}^{\infty}\sin nx$, which converges only if $\sin nx=0$, or $x=k\pi$ (k an integer).

37. If $a_n=\frac{x^n}{n^2}$, then by the Ratio Test,

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2}\cdot\frac{n^2}{x^n}\right| = |x|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = |x| < 1$ for convergence, so $R=1$.

When $x=\pm 1$, $\sum_{n=1}^{\infty}\left|\frac{x^n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$ which is a convergent $p-$ series ($p=2>1$), so the interval of convergence for f is $[-1,1]$. By Theorem 2, the radii of convergence of f' and f'' are both 1 , so we need

only check the endpoints. $f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2} \Rightarrow f'(x)=\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n^2}=\sum_{n=0}^{\infty}\frac{x^n}{n+1}$, and this series diverges for $x=1$ (harmonic series) and converges for $x=-1$ (Alternating Series Test), so the interval of convergence is $[-1,1)$. $f''(x)=\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n+1}$ diverges at both 1 and -1 (Test for Divergence) since $\lim_{n\to\infty}\frac{n}{n+1}=1\neq 0$, so its interval of convergence is $(-1,1)$.

38. **(a)** $\sum_{n=1}^{\infty}nx^{n-1}=\sum_{n=0}^{\infty}\frac{d}{dx}x^n=\frac{d}{dx}\left[\sum_{n=0}^{\infty}x_n\right]=\frac{d}{dx}\left[\frac{1}{1-x}\right]=-\frac{1}{(1-x)^2}(-1)=\frac{1}{(1-x)^2}$, $|x|<1$.

(b)

(i) $\sum_{n=1}^{\infty}nx^n=x\sum_{n=1}^{\infty}nx^{n-1}=x\left[\frac{1}{(1-x)^2}\right]=\frac{x}{(1-x)^2}$ for $|x|<1$.

(ii) Put $x=\frac{1}{2}$ in (i): $\sum_{n=1}^{\infty}\frac{n}{2^n}=\sum_{n=1}^{\infty}n\left(\frac{1}{2}\right)^n=\frac{1/2}{(1-1/2)^2}=2$.

(c)

(i)

$$\sum_{n=2}^{\infty}n(n-1)x^n=x^2\sum_{n=2}^{\infty}n(n-1)x^{n-2}=x^2\frac{d}{dx}\left[\sum_{n=1}^{\infty}nx^{n-1}\right]=x^2\frac{d}{dx}\frac{1}{(1-x)^2}$$

$$=x^2\frac{2}{(1-x)^3}=\frac{2x^2}{(1-x)^3}\text{ for }|x|<1 .$$

(ii) Put $x=\frac{1}{2}$ in (i): $\sum_{n=2}^{\infty}\frac{n^2-n}{2^n}=\sum_{n=2}^{\infty}n(n-1)\left(\frac{1}{2}\right)^n=\frac{2(1/2)^2}{(1-1/2)^3}=4$.

(iii)

From (b)(ii) and (c)(ii), we have $\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \sum_{n=1}^{\infty} \frac{n^2-n}{2^n} + \sum_{n=1}^{\infty} \frac{n}{2^n} = 4+2=6$.

39. By Example 7, $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $|x|<1$. In particular, for $x = \frac{1}{\sqrt{3}}$, we have

$\frac{\pi}{6} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(1/\sqrt{3}\right)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{3}\right)^n \frac{1}{\sqrt{3}} \frac{1}{2n+1}$, so

$\pi = \frac{6}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$.

40. **(a)** $\int_0^{1/2} \frac{dx}{x^2-x+1} = \int_0^{1/2} \frac{dx}{(x-1/2)^2+3/4} \left[\begin{array}{l} x-1/2=\left(\sqrt{3}/2\right) u,\ u=\left(2/\sqrt{3}\right)(x-1/2) \\ dx=\left(\sqrt{3}/2\right) du \end{array}\right]$

$= \int_{-1/\sqrt{3}}^{0} \frac{\left(\sqrt{3}/2\right) du}{(3/4)(u^2+1)} = \frac{2\sqrt{3}}{3} \left[\tan^{-1} u\right]_{-1/\sqrt{3}}^{0} = \frac{2}{\sqrt{3}} \left[0 - \left(-\frac{\pi}{6}\right)\right] = \frac{\pi}{3\sqrt{3}}$

(b) $\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} \Rightarrow \frac{1}{x^2-x+1} = (x+1)\left(\frac{1}{1+x^3}\right) = (x+1)\frac{1}{1-(-x^3)}$

$= (x+1)\sum_{n=0}^{\infty} (-1)^n x^{3n} = \sum_{n=0}^{\infty} (-1)^n x^{3n+1} + \sum_{n=0}^{\infty} (-1)^n x^{3n}$ for $|x|<1 \Rightarrow \int \frac{dx}{x^2-x+1}$

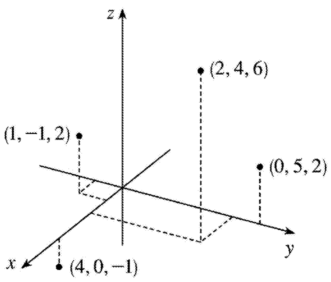
$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{3n+2} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1}$ for $|x|<1 \Rightarrow \int_0^{1/2} \frac{dx}{x^2-x+1}$

$= \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{4\cdot 8^n(3n+2)} + \frac{1}{2\cdot 8^n(3n+1)}\right] = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} \left(\frac{2}{3n+1} + \frac{1}{3n+2}\right)$

By part (a), this equals $\frac{\pi}{3\sqrt{3}}$, so $\pi = \frac{3\sqrt{3}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} \left(\frac{2}{3n+1} + \frac{1}{3n+2}\right)$.

1. We start at the origin, which has coordinates $(0,0,0)$. First we move 4 units along the positive x –axis, affecting only the x –coordinate, bringing us to the point $(4,0,0)$. We then move 3 units straight downward, in the negative z –direction. Thus only the z –coordinate is affected, and we arrive at $(4,0,-3)$.

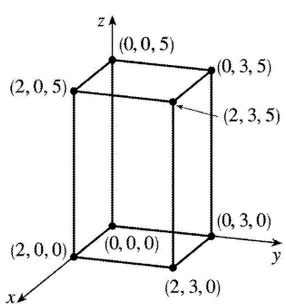
2.



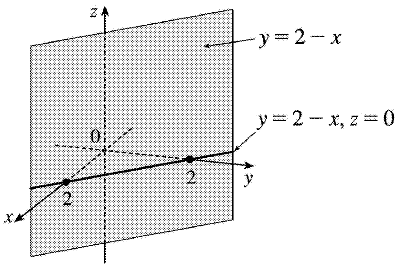
3. The distance from a point to the xz –plane is the absolute value of the y –coordinate of the point. $Q(-5,-1,4)$ has the y –coordinate with the smallest absolute value, so Q is the point closest to the xz –plane. $R(0,3,8)$ must lie in the yz –plane since the distance from R to the yz –plane, given by the x –coordinate of R, is 0.

4. The projection of $(2,3,5)$ on the xy –plane is $(2,3,0)$;
on the yz –plane, $(0,3,5)$; on the xz –plane, $(2,0,5)$.
The length of the diagonal of the box is the distance between
the origin and $(2,3,5)$ given by

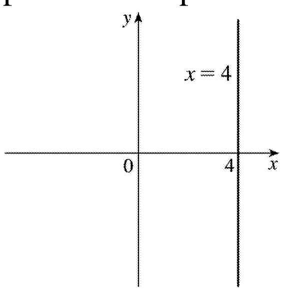
$$\sqrt{(2-0)^2+(3-0)^2+(5-0)^2}=\sqrt{38}\approx 6.16$$

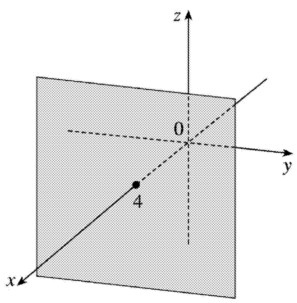


5. The equation $x+y=2$ represents the set of all points in R^3 whose x – and y –coordinates have a sum of 2 , or equivalently where $y=2-x$. This is the set $\{(x,2-x,z) \mid x\in R, z\in R\}$ which is a vertical plane that intersects the xy –plane in the line $y=2-x$, $z=0$.

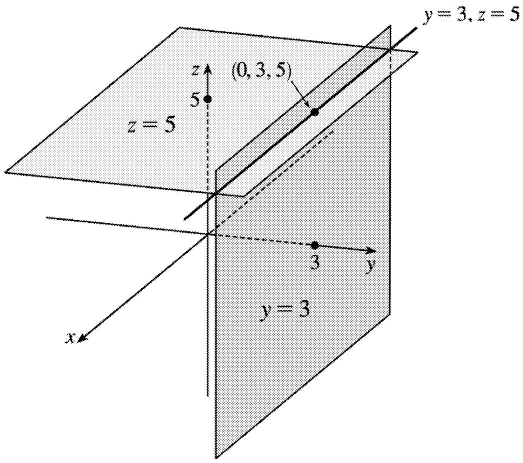


6. **(a)** In R^2 , the equation $x=4$ represents a line parallel to the y –axis. In R^3 , the equation $x=4$ represents the set $\{ (x,y,z) \mid x=4\}$, the set of all points whose x –coordinate is 4 . This is the vertical plane that is parallel to the yz –plane and 4 units in front of it.





(b) In R^3 , the equation $y=3$ represents a vertical plane that is parallel to the xz –plane and 3 units to the right of it. The equation $z=5$ represents a horizontal plane parallel to the xy –plane and 5 units above it. The pair of equations $y=3$, $z=5$ represents the set of points that are simultaneously on both planes, or in other words, the line of intersection of the planes $y=3$, $z=5$. This line can also be described as the set $\{ (x,3,5) \mid x\in R\}$, which is the set of all points in R^3 whose x –coordinate may vary but whose y – and z –coordinates are fixed at 3 and 5 , respectively. Thus the line is parallel to the x –axis and intersects the yz –plane in the point $(0,3,5)$.



7. We first find the lengths of the sides of the triangle by using the distance formula between pairs of vertices:

$$|PQ| = \sqrt{[1-(-2)]^2+(2-4)^2+(-1-0)^2} = \sqrt{9+4+1} = \sqrt{14}$$
$$|QR| = \sqrt{(-1-1)^2+(1-2)^2+[2-(-1)]^2} = \sqrt{4+1+9} = \sqrt{14}$$
$$|PR| = \sqrt{[-1-(-2)]^2+(1-4)^2+(2-0)^2} = \sqrt{1+9+4} = \sqrt{14}$$

Since all three sides have the same length, *PQR* is an equilateral triangle.

8. We can find the lengths of the sides of the triangle by using the distance formula between pairs of vertices:

$$|AB| = \sqrt{(3-1)^2+(4-2)^2+[-2-(-3)]^2} = \sqrt{4+4+1} = 3$$
$$|BC| = \sqrt{(3-3)^2+(-2-4)^2+[1-(-2)]^2} = \sqrt{0+36+9} = \sqrt{45} = 3\sqrt{5}$$
$$|AC| = \sqrt{(3-1)^2+(-2-2)^2+[1-(-3)]^2} = \sqrt{4+16+16} = 6$$

Since the Pythagorean Theorem is satisfied by $|AB|^2+|AC|^2=|BC|^2$, *ABC* is a right triangle. *ABC* is not ososceles, as no two sides have the same length.

9. **(a)** First we find the distances between points:

$$|AB| = \sqrt{(7-5)^2+(9-1)^2+(-1-3)^2} = \sqrt{84} = 2\sqrt{21}$$
$$|BC| = \sqrt{(1-7)^2+(-15-9)^2+[11-(-1)]^2} = \sqrt{756} = 6\sqrt{21}$$
$$|AC| = \sqrt{(1-5)^2+(-15-1)^2+(11-3)^2} = \sqrt{336} = 4\sqrt{21}$$

In order for the points to lie on a straight line, the sum of the two shortest distances must be equal to the longest distance. Since $|AB|+|AC|=|BC|$, the three points lie on a straight line.

(b) The distances between points are

$$|KL| = \sqrt{(1-0)^2+(2-3)^2+[-2-(-4)]^2}=\sqrt{6}$$
$$|LM| = \sqrt{(3-1)^2+(0-2)^2+[1-(-2)]^2}=\sqrt{17}$$
$$|KM| = \sqrt{(3-0)^2+(0-3)^2+[1-(-4)]^2}=\sqrt{43}$$

Since $\sqrt{6}+\sqrt{17}\neq\sqrt{43}$, the three points do not lie on a straight line.

10. **(a)** The distance from a point to the xy –plane is the absolute value of the z –coordinate of the point. Thus, the distance is $|-5|=5$.
(b) Similarly, the distance is the absolute value of the x –coordinate of the point: $|3|=3$.
(c) The distance is the absolute value of the y –coordinate of the point: $|7|=7$.
(d) The point on the x –axis closest to $(3,7,-5)$ is the point $(3,0,0)$. (Approach the x –axis perpendicularly.) The distance from $(3,7,-5)$ to the x –axis is the distance between these two points: $\sqrt{(3-3)^2+(7-0)^2+(-5-0)^2}=\sqrt{74}\approx 8.60$.
(e) The point on the y –axis closest to $(3,7,-5)$ is $(0,7,0)$. The distance between these points is $\sqrt{(3-0)^2+(7-7)^2+(-5-0)^2}=\sqrt{34}\approx 5.83$.
(f) The point on the z –axis closest to $(3,7,-5)$ is $(0,0,-5)$. The distance between these points is $\sqrt{(3-0)^2+(7-0)^2+[-5-(-5)]^2}=\sqrt{58}\approx 7.62$.

11. An equation of the sphere with center $(1,-4,3)$ and radius 5 is $(x-1)^2+[y-(-4)]^2+(z-3)^2=5^2$ or $(x-1)^2+(y+4)^2+(z-3)^2=25$. The intersection of this sphere with the xz –plane is the set of points on the sphere whose y –coordinate is 0 . Putting $y=0$ into the equation, we have $(x-1)^2+4^2+(z-3)^2=25, y=0$ or $(x-1)^2+(z-3)^2=9, y=0$, which represents a circle in the xz –plane with center $(1,0,3)$ and radius 3 .

12. An equation of the sphere with center $(6,5,-2)$ and radius $\sqrt{7}$ is $(x-6)^2+(y-5)^2+[z-(-2)]^2=(\sqrt{7})^2$ or $(x-6)^2+(y-5)+(z+2)^2=7$. The intersection of this sphere with the xy –plane is the set of points on the sphere whose z –coordinate is 0 . Putting $z=0$ into the equation, we have $(x-6)^2+(y-5)^2=3, z=0$ which represents a circle in the xy –plane with center $(6,5,0)$ and radius $\sqrt{3}$. To find the intersection with the xz –plane, we set $y=0$: $(x-6)^2+(z+2)^2=-18$. Since no points satisfy this equation, the sphere does not intersect the xz –plane. (Also note that the distance from the center of the sphere to the xz –plane is greater than the radius of the sphere.) Similarly, the sphere does not intersect the yz –plane since substituting $x=0$ into the equation gives $(y-5)^2+(z+2)^2=-29$.

13. The radius of the sphere is the distance between $(4,3,-1)$ and $(3,8,1)$: $r=\sqrt{(3-4)^2+(8-3)^2+[1-(-1)]^2}=\sqrt{30}$. Thus, an equation of the sphere is $(x-3)^2+(y-8)^2+(z-1)^2=30$.

14. If the sphere passes through the origin, the radius of the sphere must be the distance from the origin to the point $(1,2,3)$: $r=\sqrt{(1-0)^2+(2-0)^2+(3-0)^2}=\sqrt{14}$. Then an equation of the sphere is $(x-1)^2+(y-2)^2+(z-3)^2=14$.

15. Completing squares in the equation $x^2+y^2+z^2-6x+4y-2z=11$ gives
$(x^2-6x+9)+(y^2+4y+4)+(z^2-2z+1)=11+9+4+1 \Rightarrow$
$(x-3)^2+(y+2)^2+(z-1)^2=25$ which we recognize as an equation of a sphere with center $(3,-2,1)$ and radius 5 .

16. Completing squares in the equation gives $(x^2-4x+4)+(y^2+2y+1)+z^2=0+4+1 \Rightarrow (x-2)^2+(y+1)^2+z^2=5$ which we recognize as an equation of a sphere with center $(2,-1,0)$ and radius $\sqrt{5}$.

17. Completing squares in the equation gives
$\left(x^2-x+\frac{1}{4}\right)+\left(y^2-y+\frac{1}{4}\right)+\left(z^2-z+\frac{1}{4}\right)=\frac{1}{4}+\frac{1}{4}+\frac{1}{4} \Rightarrow$
$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2+\left(z-\frac{1}{2}\right)^2=\frac{3}{4}$ which we recognize as an equation of a sphere with center $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ and radius $\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$.

18. Completing squares in the equation gives $4\left(x^2-2x+1\right)+4\left(y^2+4y+4\right)+4z^2=1+4+16 \Rightarrow$
$4(x-1)^2+4(y+2)^2+4z^2=21 \Rightarrow (x-1)^2+(y+2)^2+z^2=\frac{21}{4}$, which we recognize as an equation of a sphere with center $(1,-2,0)$ and radius $\sqrt{\frac{21}{4}}=\frac{\sqrt{21}}{2}$.

19. **(a)** If the midpoint of the line segment from $P_1\left(x_1,y_1,z_1\right)$ to $P_2\left(x_2,y_2,z_2\right)$ is $Q=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$, then the distances $\left|P_1Q\right|$ and $\left|QP_2\right|$ are equal, and each is half of $\left|P_1P_2\right|$. We verify that this is the case:

$$\left|P_1P_2\right| = \sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2}$$

$$\begin{aligned}\left|P_1Q\right| &= \sqrt{\left[\frac{1}{2}\left(x_1+x_2\right)-x_1\right]^2+\left[\frac{1}{2}\left(y_1+y_2\right)-y_1\right]^2+\left[\frac{1}{2}\left(z_1+z_2\right)-z_1\right]^2} \\ &= \sqrt{\left(\frac{1}{2}x_2-\frac{1}{2}x_1\right)^2+\left(\frac{1}{2}y_2-\frac{1}{2}y_1\right)^2+\left(\frac{1}{2}z_2-\frac{1}{2}z_1\right)^2} \\ &= \sqrt{\left(\frac{1}{2}\right)^2\left[\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2\right]} \\ &= \frac{1}{2}\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2} \\ &= \frac{1}{2}\left|P_1P_2\right|\end{aligned}$$

$$\begin{aligned}\left|QP_2\right| &= \sqrt{\left[x_2-\frac{1}{2}\left(x_1+x_2\right)\right]^2+\left[y_2-\frac{1}{2}\left(y_1+y_2\right)\right]^2+\left[z_2-\frac{1}{2}\left(z_1+z_2\right)\right]^2} \\ &= \sqrt{\left(\frac{1}{2}x_2-\frac{1}{2}x_1\right)^2+\left(\frac{1}{2}y_2-\frac{1}{2}y_1\right)^2+\left(\frac{1}{2}z_2-\frac{1}{2}z_1\right)^2} \\ &= \sqrt{\left(\frac{1}{2}\right)^2\left[\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2\right]} \\ &= \frac{1}{2}\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2} \\ &= \frac{1}{2}\left|P_1P_2\right|\end{aligned}$$

So Q is indeed the midpoint of P_1P_2.

(b) By part (a), the midpoints of sides AB, BC and CA are $P_1\left(-\frac{1}{2},1,4\right)$, $P_2\left(1,\frac{1}{2},5\right)$ and $P_3\left(\frac{5}{2},\frac{3}{2},4\right)$. (Recall that a median of a triangle is a line segment from a vertex to the midpoint of the opposite side.) Then the lengths of the medians are:

$$\left|AP_2\right| = \sqrt{0^2+\left(\frac{1}{2}-2\right)^2+(5-3)^2}=\sqrt{\frac{9}{4}+4}=\sqrt{\frac{25}{4}}=\frac{5}{2}$$

$$\left|BP_3\right| = \sqrt{\left(\frac{5}{2}+2\right)^2+\left(\frac{3}{2}\right)^2+(4-5)^2}=\sqrt{\frac{81}{4}+\frac{9}{4}+1}=\sqrt{\frac{94}{4}}=\frac{1}{2}\sqrt{94}$$

$$\left|CP_1\right| = \sqrt{\left(-\frac{1}{2}-4\right)^2+(1-1)^2+(4-5)^2} = \sqrt{\frac{81}{4}+1} = \frac{1}{2}\sqrt{85}$$

20. By Exercise 19(a), the midpoint of the diameter (and thus the center of the sphere) is $C(3,2,7)$. The radius is half the diameter, so $r=\frac{1}{2}\sqrt{(4-2)^2+(3-1)^2+(10-4)^2}=\frac{1}{2}\sqrt{44}=\sqrt{11}$. Therefore an equation of the sphere is $(x-3)^2+(y-2)^2+(z-7)^2=11$.

21. **(a)** Since the sphere touches the xy –plane, its radius is the distance from its center, $(2,-3,6)$, to the xy –plane, namely 6 . Therefore $r=6$ and an equation of the sphere is $(x-2)^2+(y+3)^2+(z-6)^2=6^2=36$.

(b) The radius of this sphere is the distance from its center $(2,-3,6)$ to the yz –plane, which is 2 . Therefore, an equation is $(x-2)^2+(y+3)^2+(z-6)^2=4$.

(c) Here the radius is the distance from the center $(2,-3,6)$ to the xz –plane, which is 3 . Therefore, an equation is $(x-2)^2+(y+3)^2+(z-6)^2=9$.

22. The largest sphere contained in the first octant must have a radius equal to the minimum distance from the center $(5,4,9)$ to any of the three coordinate planes. The shortest such distance is to the xz –plane, a distance of 4 . Thus an equation of the sphere is $(x-5)^2+(y-4)^2+(z-9)^2=16$.

23. The equation $y=-4$ represents a plane parallel to the xz –plane and 4 units to the left of it.

24. The equation $x=10$ represents a plane parallel to the yz –plane and 10 units in front of it.

25. The inequality $x>3$ represents a half–space consisting of all points in front of the plane $x=3$.

26. The inequality $y\geq 0$ represents a half–space consisting of all points on or to the right of the xz –plane.

27. The inequality $0\leq z\leq 6$ represents all points on or between the horizontal planes $z=0$ (the xy –plane) and $z=6$.

28. The equation $y=z$ represents a plane perpendicular to the yz – plane and intersecting the yz –plane in the line $y=z$, $x=0$.

29. The inequality $x^2+y^2+z^2>1$ is equivalent to $\sqrt{x^2+y^2+z^2}>1$, so the region consists of those points whose distance from the origin is greater than 1 . This is the set of all points outside the sphere with radius 1 and center $(0,0,0)$.

30. The inequality $1 \leq x^2+y^2+z^2 \leq 25$ is equivalent to $1 \leq \sqrt{x^2+y^2+z^2} \leq 5$, so the region consists of those points whose distance from the origin is at least 1 and at most 5 . This is the set of all points on or between the concentric spheres with radii 1 and 5 and center $(0,0,0)$.

31. Completing the square in z gives $x^2+y^2+(z^2-2z+1)<3+1$ or $x^2+y^2+(z-1)^2<4$, which is equivalent to $\sqrt{x^2+y^2+(z-1)^2}<2$. Thus the region consists of those points whose distance from the point $(0,0,1)$ is less than 2 . This is the set of all points inside the sphere with radius 2 and center $(0,0,1)$.

32. The equation $x^2+y^2=1$ represents the set of all points in R^3 where $x^2+y^2=1$, a surface that intersects the xy –plane in the circle $x^2+y^2=1, z=0$. Since z can vary, the surface is a circular cylinder of radius 1 . Thus, the equation represents the region consisting of all points on a circular cylinder of radius 1 with axis the z –axis.

33. Here $x^2+z^2 \leq 9$ or equivalently $\sqrt{x^2+z^2} \leq 3$ which describes the set of all points in R^3 whose distance from the y –axis is at most 3 . Thus, the inequality represents the region consisting of all points on or inside a circular cylinder of radius 3 with axis the y –axis.

34. The equation $xyz=0$ is satisfied when any of x , y , or z is 0 . Thus, the equation represents the region consisting of all points on the three coordinate planes $x=0$, $y=0$, and $z=0$.

35. This describes all points with negative y –coordinates, that is, $y<0$.

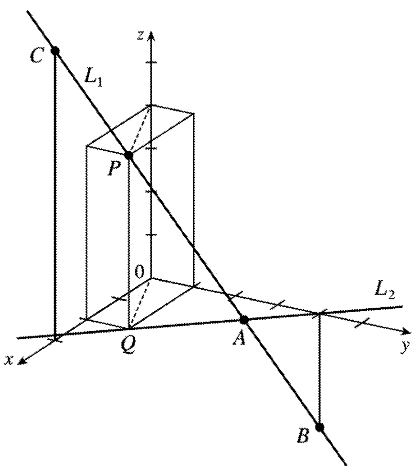
36. Because the box lies in the first quadrant, each point must comprise only nonnegative coordinates. So inequalities describing the region are $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq 3$.

37. This describes a region all of whose points have a distance to the origin which is greater than r , but smaller than R . So inequalities describing the region are $r<\sqrt{x^2+y^2+z^2}<R$, or $r^2<x^2+y^2+z^2<R^2$.

38. The solid sphere itself is represented by $\sqrt{x^2+y^2+z^2} \leq 2$. Since we want only the upper hemisphere, we restrict the z –coordinate to nonnegative values. Then inequalities describing the region are $\sqrt{x^2+y^2+z^2} \leq 2$, $z \geq 0$, or $x^2+y^2+z^2 \leq 4$, $z \geq 0$.

39. **(a)** To find the x – and y –coordinates of the point P , we project it onto L_2 and project the resulting point Q onto the x – and y –axes. To find the z –coordinate, we project P onto either the xz –plane or the yz –plane (using our knowledge of its x – or
y –coordinate) and then project the resulting point onto the z –axis. (Or, we could draw a line parallel to QO from P to the z –axis.) The coordinates of P are $(2,1,4)$.

(b) A is the intersection of L_1 and L_2 , B is directly below the y –intercept of L_2 , and C is directly above the x –intercept of L_2 .

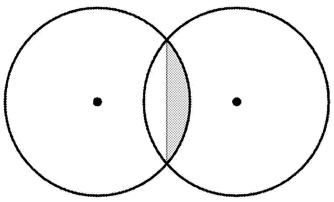


40. Let $P=(x,y,z)$. Then $2|PB|=|PA| \Leftrightarrow 4|PB|^2=|PA|^2 \Leftrightarrow$
$4\left((x-6)^2+(y-2)^2+(z+2)^2\right)=(x+1)^2+(y-5)^2+(z-3)^2 \Leftrightarrow$
$4\left(x^2-12x+36\right)-x^2-2x+4\left(y^2-4y+4\right)-y^2+10y+4\left(z^2+4z+4\right)-z^2+6z=35 \Leftrightarrow$
$3x^2-50x+3y^2-6y+3z^2+22z=35-144-16-16 \Leftrightarrow x^2-\frac{50}{3}x+y^2-2y+z^2+\frac{22}{3}z=-\frac{141}{3}$. By completing the square three times we get $\left(x-\frac{25}{3}\right)^2+(y-1)^2+\left(z+\frac{11}{3}\right)^2=\frac{332}{9}$, which is an equation of a sphere with center $\left(\frac{25}{3},1,-\frac{11}{3}\right)$ and radius $\frac{\sqrt{332}}{3}$.

41. We need to find a set of points $\{P(x,y,z)\,|\,|AP|=|BP|\ \}$.
$\sqrt{(x+1)^2+(y-5)^2+(z-3)^2}=\sqrt{(x-6)^2+(y-2)^2+(z+2)^2} \Rightarrow (x+1)^2+(y-5)+(z-3)^2=(x-6)^2+(y-2)^2+(z+2)^2 \Rightarrow$
$x^2+2x+1+y^2-10y+25+z^2-6z+9=x^2-12x+36+y^2-4y+4+z^2+4z+4 \Rightarrow 14x-6y-10z=9$. Thus the set of points is a plane perpendicular to the line segment joining A and B (since this plane must contain the perpendicular bisector of the line segment AB).

42. Completing the square three times in the first equation gives $(x+2)^2+(y-1)^2+(z+2)^2=2^2$, a sphere with center $(-2,1,2)$ and radius 2 . The second equation is that of a sphere with center $(0,0,0)$ and radius 2 . The distance between the centers of the spheres is $\sqrt{(-2-0)^2+(1-0)^2+(-2-0)^2}=\sqrt{4+1+4}=3$.

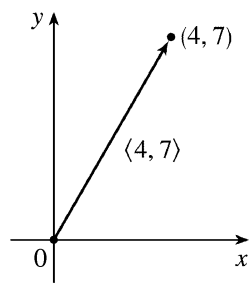
Since the spheres have the same radius, the volume inside both spheres is symmetrical about the plane containing the circle of intersection of the spheres. The distance from this plane to the center of the circles is $\frac{3}{2}$. So the region inside both spheres consists of two caps of spheres of height $h=2-\frac{3}{2}=\frac{1}{2}$. From Exercise 6.2.49 [ET 6.2.49], the volume of a cap of a sphere is $V=\frac{1}{3}\pi h^2(3r-h)=\frac{1}{3}\pi\left(\frac{1}{2}\right)^2\left(3\cdot 2-\frac{1}{2}\right)=\frac{11\pi}{24}$. So the total volume is $2\cdot\frac{11\pi}{24}=\frac{11\pi}{12}$.



1. **(a)** The cost of a theater ticket is a scalar, because it has only magnitude.
(b) The current in a river is a vector, because it has both magnitude (the speed of the current) and direction at any given location.
(c) If we assume that the initial path is linear, the initial flight path from Houston to Dallas is a vector, because it has both magnitude (distance) and direction.
(d) The population of the world is a scalar, because it has only magnitude.

2.

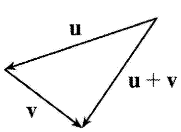
If the initial point of the vector $\langle 4,7\rangle$ is placed at the origin, then $\langle 4,7\rangle$ is the position vector of the point $(4,7)$.



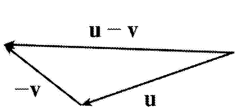
3. Vectors are equal when they share the same length and direction (but not necessarily location). Using the symmetry of the parallelogram as a guide, we see that $\overrightarrow{AB}=\overrightarrow{DC}$, $\overrightarrow{DA}=\overrightarrow{CB}$, $\overrightarrow{DE}=\overrightarrow{EB}$, and $\overrightarrow{EA}=\overrightarrow{CE}$.

4. **(a)** The initial point of $\overrightarrow{QR}$ is positioned at the terminal point of $\overrightarrow{PQ}$, so by the Triangle Law the sum $\overrightarrow{PQ}+\overrightarrow{QR}$ is the vector with initial point P and terminal point R , namely $\overrightarrow{PR}$.
(b) By the Triangle Law, $\overrightarrow{RP}+\overrightarrow{PS}$ is the vector with initial point R and terminal point S , namely $\overrightarrow{RS}$.
(c) First we consider $\overrightarrow{QS}-\overrightarrow{PS}$ as $\overrightarrow{QS}+\left(-\overrightarrow{PS}\right)$. Then since $-\overrightarrow{PS}$ has the same length as $\overrightarrow{PS}$ but points in the opposite direction, we have $-\overrightarrow{PS}=\overrightarrow{SP}$ and so $\overrightarrow{QS}-\overrightarrow{PS}=\overrightarrow{QS}+\overrightarrow{SP}=\overrightarrow{QP}$.
(d) We use the Triangle Law twice: $\overrightarrow{RS}+\overrightarrow{SP}+\overrightarrow{PQ}=\left(\overrightarrow{RS}+\overrightarrow{SP}\right)+\overrightarrow{PQ}=\overrightarrow{RP}+\overrightarrow{PQ}=\overrightarrow{RQ}$

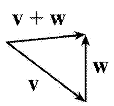
5. **(a)**



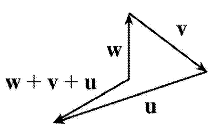
(b)



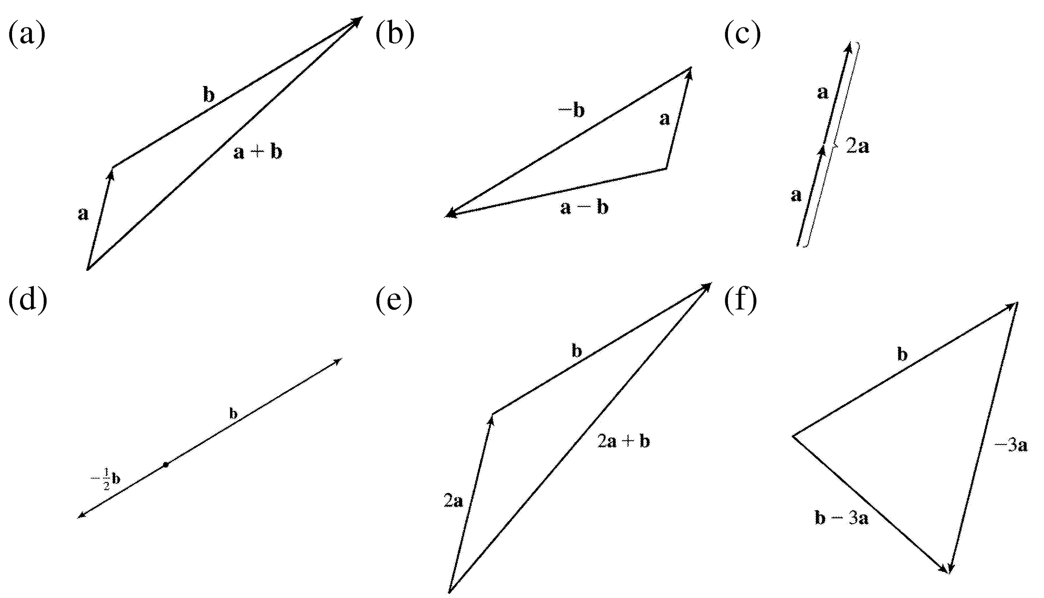
(c)



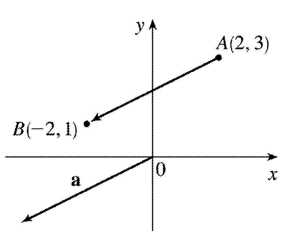
(d)



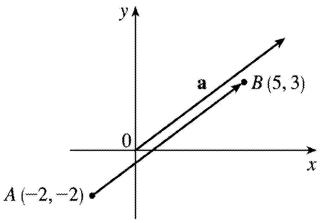
6.



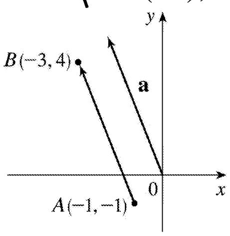
7. $\boldsymbol{a}=\langle -2-2, 1-3\rangle=\langle -4,-2\rangle$



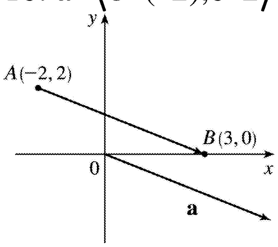
8. $\boldsymbol{a}=\langle 5-(-2), 3-(-2)\rangle=\langle 7,5\rangle$



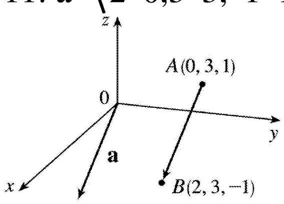
9. $\boldsymbol{a}=\left\langle -3-(-1),4-(-1)\right\rangle=\left\langle -2,5\right\rangle$



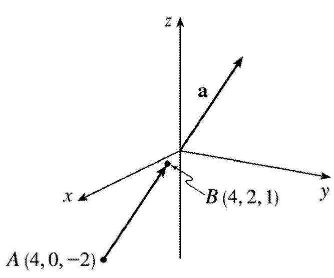
10. $\boldsymbol{a}=\left\langle 3-(-2),0-2\right\rangle=\left\langle 5,-2\right\rangle$



11. $\boldsymbol{a}=\left\langle 2-0,3-3,-1-1\right\rangle=\left\langle 2,0,-2\right\rangle$

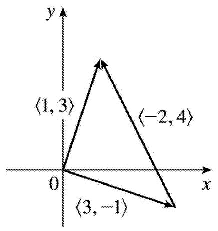


12. $\boldsymbol{a}=\left\langle 4-4,2-0,1-(-2)\right\rangle=\left\langle 0,2,3\right\rangle$



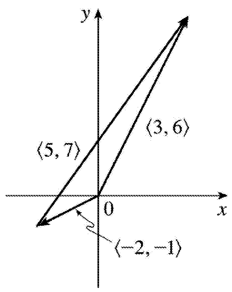
13.

$$\begin{aligned}\left\langle 3,-1\right\rangle+\left\langle -2,4\right\rangle &=\left\langle 3+(-2),-1+4\right\rangle\\ &=\left\langle 1,3\right\rangle\end{aligned}$$



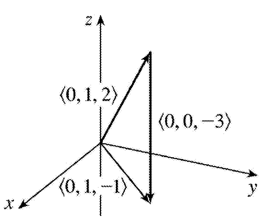
14.

$$\langle -2,-1\rangle + \langle 5,7\rangle = \langle -2+5,-1+7\rangle$$
$$= \langle 3,6\rangle$$



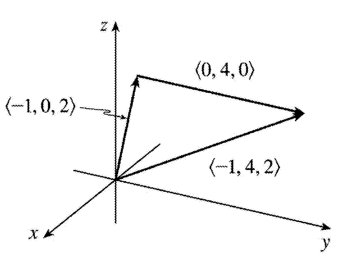
15.

$$\langle 0,1,2\rangle + \langle 0,0,-3\rangle = \langle 0+0,1+0,2+(-3)\rangle$$
$$= \langle 0,1,-1\rangle$$



16.

$$\langle -1,0,2\rangle + \langle 0,4,0\rangle = \langle -1+0,0+4,2+0\rangle$$
$$= \langle -1,4,2\rangle$$



17. $|\boldsymbol{a}|=\sqrt{(-4)^2+3^2}=\sqrt{25}=5$
$\boldsymbol{a}+\boldsymbol{b}=\langle -4+6,3+2\rangle=\langle 2,5\rangle$
$\boldsymbol{a}-\boldsymbol{b}=\langle -4-6,3-2\rangle=\langle -10,1\rangle$
$2\boldsymbol{a}=\langle 2(-4),2(3)\rangle=\langle -8,6\rangle$
$3\boldsymbol{a}+4\boldsymbol{b}=\langle -12,9\rangle+\langle 24,8\rangle=\langle 12,17\rangle$

18. $|\boldsymbol{a}|=\sqrt{2^2+(-3)^2}=\sqrt{13}$
$\boldsymbol{a}+\boldsymbol{b}=(2\boldsymbol{i}-3\boldsymbol{j})+(\boldsymbol{i}+5\boldsymbol{j})=3\boldsymbol{i}+2\boldsymbol{j}$
$\boldsymbol{a}-\boldsymbol{b}=(2\boldsymbol{i}-3\boldsymbol{j})-(\boldsymbol{i}+5\boldsymbol{j})=\boldsymbol{i}-8\boldsymbol{j}$
$2\boldsymbol{a}=2\,(2\boldsymbol{i}-3\boldsymbol{j})=4\boldsymbol{i}-6\boldsymbol{j}$
$$\begin{aligned}3\boldsymbol{a}+4\boldsymbol{b}&=3\,(2\boldsymbol{i}-3\boldsymbol{j})+4\,(\boldsymbol{i}+5\boldsymbol{j})\\&=6\boldsymbol{i}-9\boldsymbol{j}+4\boldsymbol{i}+20\boldsymbol{j}=10\boldsymbol{i}+11\boldsymbol{j}\end{aligned}$$

19. $|\boldsymbol{a}|=\sqrt{6^2+2^2+3^2}=\sqrt{49}=7$
$\boldsymbol{a}+\boldsymbol{b}=\langle 6+(-1),2+5,3+(-2)\rangle=\langle 5,7,1\rangle$
$$\begin{aligned}\boldsymbol{a}-\boldsymbol{b}&=\langle 6-(-1),2-5,3-(-2)\rangle\\&=\langle 7,-3,5\rangle\end{aligned}$$
$2\boldsymbol{a}=\langle 2(6),2(2),2(3)\rangle=\langle 12,4,6\rangle$
$$\begin{aligned}3\boldsymbol{a}+4\boldsymbol{b}&=\langle 18,6,9\rangle+\langle -4,20,-8\rangle\\&=\langle 14,26,1\rangle\end{aligned}$$

20. $|\boldsymbol{a}|=\sqrt{(-3)^2+(-4)^2+(-1)^2}=\sqrt{26}$
$$\begin{aligned}\boldsymbol{a}+\boldsymbol{b}&=\langle -3+6,-4+2,-1+(-3)\rangle\\&=\langle 3,-2,-4\rangle\end{aligned}$$

$$\begin{aligned}\boldsymbol{a}-\boldsymbol{b}&=\langle -3-6,-4-2,-1-(-3)\rangle\\&=\langle -9,-6,2\rangle\end{aligned}$$

$2\boldsymbol{a}=\langle 2(-3),2(-4),2(-1)\rangle=\langle -6,-8,-2\rangle$
$$\begin{aligned}3\boldsymbol{a}+4\boldsymbol{b}&=\langle -9,-12,-3\rangle+\langle 24,8,-12\rangle\\&=\langle 15,-4,-15\rangle\end{aligned}$$

21. $|\boldsymbol{a}|=\sqrt{1^2+(-2)^2+1^2}=\sqrt{6}$

$\boldsymbol{a}+\boldsymbol{b}=(\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k})+(\boldsymbol{j}+2\boldsymbol{k})=\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$
$\boldsymbol{a}-\boldsymbol{b}=(\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k})-(\boldsymbol{j}+2\boldsymbol{k})=\boldsymbol{i}-3\boldsymbol{j}-\boldsymbol{k}$
$2\boldsymbol{a}=2(\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k})=2\boldsymbol{i}-4\boldsymbol{j}+2\boldsymbol{k}$

$$\begin{aligned} 3\boldsymbol{a}+4\boldsymbol{b} &= 3(\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k})+4(\boldsymbol{j}+2\boldsymbol{k}) \\ &= 3\boldsymbol{i}-6\boldsymbol{j}+3\boldsymbol{k}+4\boldsymbol{j}+8\boldsymbol{k} \\ &= 3\boldsymbol{i}-2\boldsymbol{j}+11\boldsymbol{k} \end{aligned}$$

22. $|\boldsymbol{a}|=\sqrt{3^2+0^2+(-2)^2}=\sqrt{13}$
$\boldsymbol{a}+\boldsymbol{b}=(3\boldsymbol{i}-2\boldsymbol{k})+(\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k})=4\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}$
$\boldsymbol{a}-\boldsymbol{b}=(3\boldsymbol{i}-2\boldsymbol{k})-(\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k})=2\boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k}$
$2\boldsymbol{a}=2(3\boldsymbol{i}-2\boldsymbol{k})=6\boldsymbol{i}-4\boldsymbol{k}$

$$\begin{aligned} 3\boldsymbol{a}+4\boldsymbol{b} &= 3(3\boldsymbol{i}-2\boldsymbol{k})+4(\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}) \\ &= 9\boldsymbol{i}-6\boldsymbol{k}+4\boldsymbol{i}-4\boldsymbol{j}+4\boldsymbol{k} \\ &= 13\boldsymbol{i}-4\boldsymbol{j}-2\boldsymbol{k} \end{aligned}$$

23. $|\langle 9,-5\rangle|=\sqrt{9^2+(-5)^2}=\sqrt{106}$, so $\boldsymbol{u}=\frac{1}{\sqrt{106}}\langle 9,-5\rangle=\left\langle \frac{9}{\sqrt{106}},\frac{-5}{\sqrt{106}}\right\rangle$.

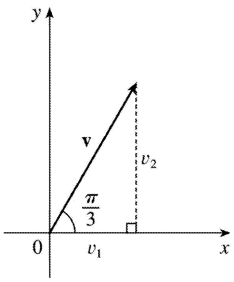
24. $|12\boldsymbol{i}-5\boldsymbol{j}|=\sqrt{12^2+(-5)^2}=\sqrt{169}=13$, so $\boldsymbol{u}=\frac{1}{13}(12\boldsymbol{i}-5\boldsymbol{j})=\frac{12}{13}\boldsymbol{i}-\frac{5}{13}\boldsymbol{j}$.

25. The vector $8\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k}$ has length $|8\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k}|=\sqrt{8^2+(-1)^2+4^2}=\sqrt{81}=9$, so by Equation 4 the unit vector with the same direction is $\frac{1}{9}(8\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k})=\frac{8}{9}\boldsymbol{i}-\frac{1}{9}\boldsymbol{j}+\frac{4}{9}\boldsymbol{k}$.

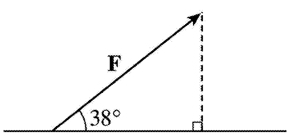
26. $|\langle -2,4,2\rangle|=\sqrt{(-2)^2+4^2+2^2}=\sqrt{24}=2\sqrt{6}$, so a unit vector in the direction of $\langle -2,4,2\rangle$ is $\boldsymbol{u}=\frac{1}{2\sqrt{6}}\langle -2,4,2\rangle$. A vector in the same direction but with length 6 is

$6\boldsymbol{u}=6\cdot\frac{1}{2\sqrt{6}}\langle -2,4,2\rangle=\left\langle -\frac{6}{\sqrt{6}},\frac{12}{\sqrt{6}},\frac{6}{\sqrt{6}}\right\rangle$ or $\langle -\sqrt{6},2\sqrt{6},\sqrt{6}\rangle$.

27. From the figure, we see that the x –component of $\boldsymbol{v}$ is $v_1=|\boldsymbol{v}|\cos(\pi/3)=4\cdot\frac{1}{2}=2$ and the y –component is $v_2=|\boldsymbol{v}|\sin(\pi/3)=4\cdot\frac{\sqrt{3}}{2}=2\sqrt{3}$. Thus $\boldsymbol{v}=\langle v_1,v_2\rangle=\langle 2,2\sqrt{3}\rangle$.



28. From the figure, we see that the horizontal component of the force $\boldsymbol{F}$ is $|\boldsymbol{F}|\cos 38°=50\cos 38°\approx 39.4$ N, and the vertical component is $|\boldsymbol{F}|\sin 38°=50\sin 38°\approx 30.8$ N.



29. $\left|\boldsymbol{F}_1\right|=10$ lb and $\left|\boldsymbol{F}_2\right|=12$ lb.

$$\begin{aligned}\boldsymbol{F}_1 &=-\left|\boldsymbol{F}_1\right|\cos 45°\boldsymbol{i}+\left|\boldsymbol{F}_1\right|\sin 45°\boldsymbol{j}=-10\cos 45°\boldsymbol{i}+10\sin 45°\boldsymbol{j}\\ &=-5\sqrt{2}\,\boldsymbol{i}+5\sqrt{2}\,\boldsymbol{j}\end{aligned}$$

$$\boldsymbol{F}_2=\left|\boldsymbol{F}_2\right|\cos 30°\boldsymbol{i}+\left|\boldsymbol{F}_2\right|\sin 30°\boldsymbol{j}=12\cos 30°\boldsymbol{i}+12\sin 30°\boldsymbol{j}=6\sqrt{3}\,\boldsymbol{i}+6\boldsymbol{j}$$

$$\boldsymbol{F}=\boldsymbol{F}_1+\boldsymbol{F}_2=\left(6\sqrt{3}-5\sqrt{2}\right)\boldsymbol{i}+\left(6+5\sqrt{2}\right)\boldsymbol{j}\approx 3.32\boldsymbol{i}+13.07\boldsymbol{j}$$

$$|\boldsymbol{F}|\approx\sqrt{(3.32)^2+(13.07)^2}\approx 13.5 \text{ lb. } \tan\theta=\frac{6+5\sqrt{2}}{6\sqrt{3}-5\sqrt{2}}\Rightarrow\theta=\tan^{-1}\frac{6+5\sqrt{2}}{6\sqrt{3}-5\sqrt{2}}\approx 76° .$$

30. Set up the coordinate axes so that north is the positive y –direction, and east is the positive x –direction. The wind is blowing at 50 km / h from the direction N45°W , so that its velocity vector is 50km/hS45°E , which can be written as $\boldsymbol{v}_{\text{wind}}=50(\cos 45°\boldsymbol{i}-\sin 45°\boldsymbol{j})$. With respect to the still air, the velocity vector of the plane is 250km/hN60°E , or equivalently $\boldsymbol{v}_{\text{plane}}=250(\cos 30°\boldsymbol{i}+\sin 30°\boldsymbol{j})$. The velocity of the plane relative to the ground is

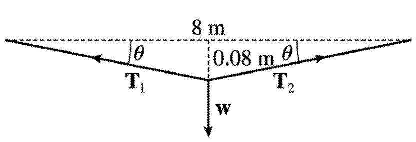
$$\begin{aligned}\boldsymbol{v} &= \boldsymbol{v}_{\text{wind}}+\boldsymbol{v}_{\text{plane}}=(50\cos 45°+250\cos 30°)\boldsymbol{i}+(-50\sin 45°+250\sin 30°)\boldsymbol{j}\\ &= \left(25\sqrt{2}+125\sqrt{3}\right)\boldsymbol{i}+\left(125-25\sqrt{2}\right)\boldsymbol{j}\approx 251.9\boldsymbol{i}+89.6\boldsymbol{j}\end{aligned}$$

The ground speed is $|\boldsymbol{v}|\approx\sqrt{(251.9)^2+(89.6)^2}\approx 267\text{km/h}$. The angle the velocity vector makes with the x –axis is $\theta\approx\tan^{-1}\left(\frac{89.6}{251.9}\right)\approx 20°$. Therefore, the true course of the plane is about N(90–20)°E=N70°E .

31. With respect to the water's surface, the woman's velocity is the vector sum of the velocity of the ship with respect to the water, and the woman's velocity with respect to the ship. If we let north be the positive y –direction, then $\boldsymbol{v}=\langle 0,22\rangle+\langle -3,0\rangle=\langle -3,22\rangle$. The woman's speed is $|\boldsymbol{v}|=\sqrt{9+484}\approx 22.2$ mi/h . The vector $\boldsymbol{v}$ makes an angle θ with the east, where $\theta=\tan^{-1}\left(\frac{22}{-3}\right)\approx 98°$. Therefore, the woman's direction is about $\mathrm{N}(98-90)°\mathrm{W}=\mathrm{N}8°\mathrm{W}$.

32. Call the two tensile forces $\boldsymbol{T}_3$ and $\boldsymbol{T}_5$, corresponding to the ropes of length 3 m and 5 m. In terms of vertical and horizontal components,
$\boldsymbol{T}_3=-\left|\boldsymbol{T}_3\right|\cos 52°\boldsymbol{i}+\left|\boldsymbol{T}_3\right|\sin 52°\boldsymbol{j}$ (1) and $\boldsymbol{T}_5=\left|\boldsymbol{T}_5\right|\cos 40°\boldsymbol{i}+\left|\boldsymbol{T}_5\right|\sin 40°\boldsymbol{j}$ (2)
The resultant of these forces, $\boldsymbol{T}_3+\boldsymbol{T}_5$, counterbalances the force of gravity acting on the decoration .
So $\boldsymbol{T}_3+\boldsymbol{T}_5=49\boldsymbol{j}$. Hence
$\boldsymbol{T}_3+\boldsymbol{T}_5=\left(-\left|\boldsymbol{T}_3\right|\cos 52°+\left|\boldsymbol{T}_5\right|\cos 40°\right)\boldsymbol{i}+\left(\left|\boldsymbol{T}_3\right|\sin 52°+\left|\boldsymbol{T}_5\right|\sin 40°\right)\boldsymbol{j}=49\boldsymbol{j}$. Thus
$-\left|\boldsymbol{T}_3\right|\cos 52°+\left|\boldsymbol{T}_5\right|\cos 40°=0$ and $\left|\boldsymbol{T}_3\right|\sin 52°+\left|\boldsymbol{T}_5\right|\sin 40°=49$.
From the first of these two equations $\left|\boldsymbol{T}_3\right|=\left|\boldsymbol{T}_5\right|\dfrac{\cos 40°}{\cos 52°}$. Substituting this into the second equation gives $\left|\boldsymbol{T}_5\right|=\dfrac{49}{\cos 40°\tan 52°+\sin 40°}\approx 30$ N. Therefore, $\left|\boldsymbol{T}_3\right|=\left|\boldsymbol{T}_5\right|\dfrac{\cos 40°}{\cos 52°}\approx 38\mathrm{N}$. Finally, from (1) and (2), $\boldsymbol{T}_3\approx -23\boldsymbol{i}+30\boldsymbol{j}$, and $\boldsymbol{T}_5\approx 23\boldsymbol{i}+19\boldsymbol{j}$.

33. Let $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ represent the tension vectors in each side of the clothesline as shown in the figure. $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ have equal vertical components and opposite horizontal components, so we can write

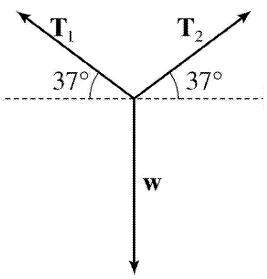


$\boldsymbol{T}_1=-a\boldsymbol{i}+b\boldsymbol{j}$ and $\boldsymbol{T}_2=a\boldsymbol{i}+b\boldsymbol{j}$ ($a,b>0$). By similar triangles, $\dfrac{b}{a}=\dfrac{0.08}{4}\Rightarrow a=50b$. The force due to gravity acting on the shirt has magnitude $0.8g\approx(0.8)(9.8)=7.84$ N, hence we have $\boldsymbol{w}=-7.84\boldsymbol{j}$. The resultant $\boldsymbol{T}_1+\boldsymbol{T}_2$ of the tensile forces counterbalances $\boldsymbol{w}$, so $\boldsymbol{T}_1+\boldsymbol{T}_2=-\boldsymbol{w}\Rightarrow(-a\boldsymbol{i}+b\boldsymbol{j})+(a\boldsymbol{i}+b\boldsymbol{j})=7.84\boldsymbol{j}$
$\Rightarrow(-50b\boldsymbol{i}+b\boldsymbol{j})+(50b\boldsymbol{i}+b\boldsymbol{j})=2b\boldsymbol{j}=7.84\boldsymbol{j}\Rightarrow b=\dfrac{7.84}{2}=3.92$ and $a=50b=196$. Thus the tensions are
$\boldsymbol{T}_1=-a\boldsymbol{i}+b\boldsymbol{j}=-196\boldsymbol{i}+3.92\boldsymbol{j}$ and $\boldsymbol{T}_2=a\boldsymbol{i}+b\boldsymbol{j}=196\boldsymbol{i}+3.92\boldsymbol{j}$.
Alternatively, we can find the value of θ and proceed as in Example 7.

34. We can consider the weight of the chain to be concentrated at its midpoint. The forces acting on the chain then are the tension vectors $\boldsymbol{T}_1$, $\boldsymbol{T}_2$ in each end of the chain and the weight $\boldsymbol{w}$, as shown in the figure. We know $\left|\boldsymbol{T}_1\right|=\left|\boldsymbol{T}_2\right|=25$ N so, in terms of vertical and horizontal components, we have

$\boldsymbol{T}_1 = -25\cos 37°\boldsymbol{i}+25\sin 37°\boldsymbol{j}$

$\boldsymbol{T}_2 = 25\cos 37°\boldsymbol{i}+25\sin 37°\boldsymbol{j}$



The resultant vector $\boldsymbol{T}_1+\boldsymbol{T}_2$ of the tensions counterbalances the weight $\boldsymbol{w}$, giving $\boldsymbol{T}_1+\boldsymbol{T}_2=-\boldsymbol{w}$. Since $\boldsymbol{w}=-|\boldsymbol{w}|\,\boldsymbol{j}$, we have $(-25\cos37°\boldsymbol{i}+25\sin37°\boldsymbol{j})+(25\cos37°\boldsymbol{i}+25\sin37°\boldsymbol{j})=|\boldsymbol{w}|\,\boldsymbol{j} \Rightarrow 50\sin37°\boldsymbol{j}=|\boldsymbol{w}|\,\boldsymbol{j} \Rightarrow$ $|\boldsymbol{w}|=50\sin37°\approx30.1$ So the weight is 30.1 N , and since $w=mg$, the mass is $\dfrac{30.1}{9.8}\approx3.07\text{kg}$.

35. By the Triangle Law, $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$. Then $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\overrightarrow{AC}+\overrightarrow{CA}$, but $\overrightarrow{AC}+\overrightarrow{CA}=\overrightarrow{AC}+\left(-\overrightarrow{AC}\right)=\mathbf{0}$. So $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\mathbf{0}$.

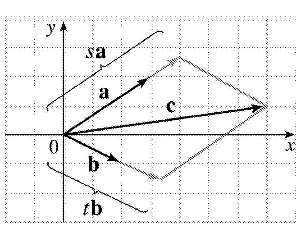
36. $\overrightarrow{AC}=\dfrac{1}{3}\overrightarrow{AB}$ and $\overrightarrow{BC}=\dfrac{2}{3}\overrightarrow{BA}$. $\boldsymbol{c}=\overrightarrow{OA}+\overrightarrow{AC}=\boldsymbol{a}+\dfrac{1}{3}\overrightarrow{AB} \Rightarrow \overrightarrow{AB}=3\boldsymbol{c}-3\boldsymbol{a}$.

$\boldsymbol{c}=\overrightarrow{OB}+\overrightarrow{BC}=\overrightarrow{OA}+\dfrac{2}{3}\overrightarrow{BA} \Rightarrow \overrightarrow{BA}=\dfrac{3}{2}\boldsymbol{c}-\dfrac{3}{2}\boldsymbol{b}$. $\overrightarrow{BA}=-\overrightarrow{AB}$, so $\dfrac{3}{2}\boldsymbol{c}-\dfrac{3}{2}\boldsymbol{b}=3\boldsymbol{a}-3\boldsymbol{c} \Leftrightarrow$

$\boldsymbol{c}+2\boldsymbol{c}=2\boldsymbol{a}+\boldsymbol{b} \Leftrightarrow \boldsymbol{c}=\dfrac{2}{3}\boldsymbol{a}+\dfrac{1}{3}\boldsymbol{b}$.

37.

(a), (b)

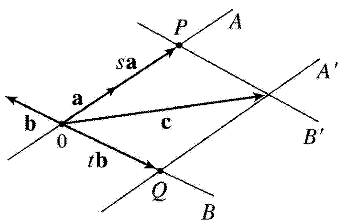


(c) From the sketch, we estimate that $s\approx1.3$ and $t\approx1.6$.

(d) $\boldsymbol{c}=s\,\boldsymbol{a}+t\,\boldsymbol{b} \Leftrightarrow 7=3s+2t$ and $1=2s-t$.

Solving these equations gives $s=\frac{9}{7}$ and $t=\frac{11}{7}$.

38. Draw $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ emanating from the origin. Extend $\boldsymbol{a}$ and $\boldsymbol{b}$ to form lines A and B , and draw lines $A^{/}$ and $B^{/}$ parallel to these two lines through the terminal point of $\boldsymbol{c}$.



Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel, A and $B^{/}$ must meet (at P), and $A^{/}$ and B must also meet (at Q). Now we see that $\overrightarrow{OP}+\overrightarrow{OQ}=\boldsymbol{c}$, so if $s=\frac{\left|\overrightarrow{OP}\right|}{|\boldsymbol{a}|}$ (or its negative, if $\boldsymbol{a}$ points in the direction opposite $\overrightarrow{OP}$) and $t=\frac{\left|\overrightarrow{OQ}\right|}{|\boldsymbol{b}|}$ (or its negative, as in the diagram), then $c=s\boldsymbol{a}+t\boldsymbol{b}$, as required.

Argument using components: Since $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ all lie in the same plane, we can consider them to be vectors in two dimensions. Let $\boldsymbol{a}=\left\langle a_1,a_2\right\rangle$, $\boldsymbol{b}=\left\langle b_1,b_2\right\rangle$, and $\boldsymbol{c}=\left\langle c_1,c_2\right\rangle$. We need $sa_1+tb_1=c_1$ and $sa_2+tb_2=c_2$. Multiplying the first equation by a_2 and the second by a_1 and subtracting, we get $t=\frac{c_2a_1-c_1a_2}{b_2a_1-b_1a_2}$. Similarly $s=\frac{b_2c_1-b_1c_2}{b_2a_1-b_1a_2}$. Since $\boldsymbol{a}\neq\boldsymbol{0}$ and $\boldsymbol{b}\neq\boldsymbol{0}$ and $\boldsymbol{a}$ is not a scalar multiple of $\boldsymbol{b}$, the denominator is not zero.

39. $\left|\boldsymbol{r}-\boldsymbol{r}_0\right|$ is the distance between the points (x,y,z) and $\left(x_0,y_0,z_0\right)$, so the set of points is a sphere with radius 1 and center $\left(x_0,y_0,z_0\right)$.

Alternate method: $\left|\boldsymbol{r}-\boldsymbol{r}_0\right|=1\Leftrightarrow\sqrt{\left(x-x_0\right)^2+\left(y-y_0\right)^2+\left(z-z_0\right)^2}=1\Leftrightarrow\left(x-x_0\right)^2+\left(y-y_0\right)^2+\left(z-z_0\right)^2=1$, which is the equation of a sphere with radius 1 and center $\left(x_0,y_0,z_0\right)$.

40. Let P_1 and P_2 be the points with position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ respectively. Then $\left|\boldsymbol{r}-\boldsymbol{r}_1\right|+\left|\boldsymbol{r}-\boldsymbol{r}_2\right|$ is

the sum of the distances from (x,y) to P_1 and P_2 . Since this sum is constant, the set of points (x,y) represents an ellipse with foci P_1 and P_2 . The condition $k>\left|\boldsymbol{r}_1-\boldsymbol{r}_2\right|$ assures us that the ellipse is not degenerate.

41.

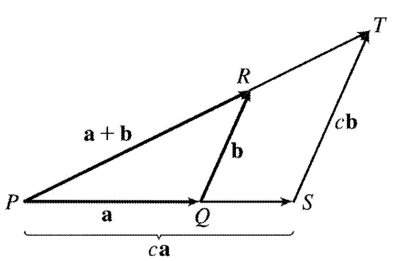
$$\begin{aligned}\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c}) &=\left\langle a_1,a_2\right\rangle+\left(\left\langle b_1,b_2\right\rangle+\left\langle c_1,c_2\right\rangle\right)=\left\langle a_1,a_2\right\rangle+\left\langle b_1+c_1,b_2+c_2\right\rangle\\ &=\left\langle a_1+b_1+c_1,a_2+b_2+c_2\right\rangle=\left\langle\left(a_1+b_1\right)+c_1,\left(a_2+b_2\right)+c_2\right\rangle\\ &=\left\langle a_1+b_1,a_2+b_2\right\rangle+\left\langle c_1,c_2\right\rangle=\left(\left\langle a_1,a_2\right\rangle+\left\langle b_1,b_2\right\rangle\right)+\left\langle c_1,c_2\right\rangle\\ &=(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}\end{aligned}$$

42. *Algebraically:*

$$\begin{aligned}c(\boldsymbol{a}+\boldsymbol{b}) &=c\left(\left\langle a_1,a_2,a_3\right\rangle+\left\langle b_1,b_2,b_3\right\rangle\right)=c\left\langle a_1+b_1,a_2+b_2,a_3+b_3\right\rangle\\ &=\left\langle c\left(a_1+b_1\right),c\left(a_2+b_2\right),c\left(a_3+b_3\right)\right\rangle=\left\langle ca_1+cb_1,ca_2+cb_2,ca_3+cb_3\right\rangle\\ &=\left\langle ca_1,ca_2,ca_3\right\rangle+\left\langle cb_1,cb_2,cb_3\right\rangle=c\,\boldsymbol{a}+c\,\boldsymbol{b}\end{aligned}$$

Geometrically:

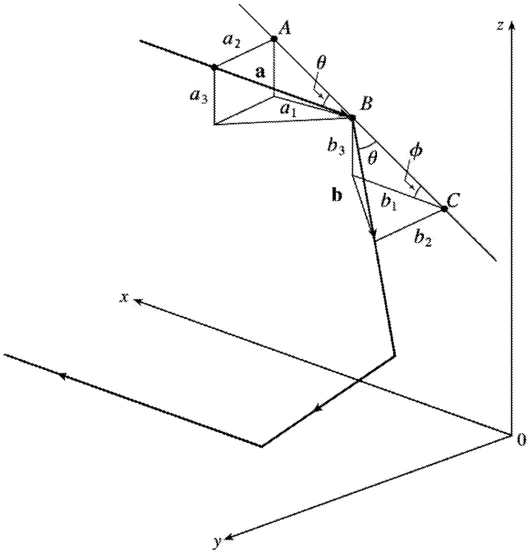
According to the Triangle Law, if $\boldsymbol{a}=\overrightarrow{PQ}$ and $\boldsymbol{b}=\overrightarrow{QR}$, then $\boldsymbol{a}+\boldsymbol{b}=\overrightarrow{PR}$. Construct triangle PST as shown so that $\overrightarrow{PS}=c\,\boldsymbol{a}$ and $\overrightarrow{ST}=c\,\boldsymbol{b}$. (We have drawn the case where $c>1$.) By the Triangle Law, $\overrightarrow{PT}=c\,\boldsymbol{a}+c\,\boldsymbol{b}$. But triangle PQR and triangle PST are similar triangles because $c\,\boldsymbol{b}$ is parallel to $\boldsymbol{b}$. Therefore, $\overrightarrow{PR}$ and $\overrightarrow{PT}$ are parallel and, in fact, $\overrightarrow{PT}=c\overrightarrow{PR}$. Thus, $c\,\boldsymbol{a}+c\,\boldsymbol{b}=c(\boldsymbol{a}+\boldsymbol{b})$.



43. Consider triangle ABC , where D and E are the midpoints of AB and BC . We know that $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ (1) and $\overrightarrow{DB}+\overrightarrow{BE}=\overrightarrow{DE}$ (2). However, $\overrightarrow{DB}=\frac{1}{2}\overrightarrow{AB}$, and $\overrightarrow{BE}=\frac{1}{2}\overrightarrow{BC}$. Substituting these expressions for $\overrightarrow{DB}$ and $\overrightarrow{BE}$ into (2) gives

$\frac{1}{2}\overrightarrow{AB}+\frac{1}{2}\overrightarrow{BC}=\overrightarrow{DE}$. Comparing this with (1) gives $\overrightarrow{DE}=\frac{1}{2}\overrightarrow{AC}$. Therefore $\overrightarrow{AC}$ and $\overrightarrow{DE}$ are parallel and $\left|\overrightarrow{DE}\right|=\frac{1}{2}\left|\overrightarrow{AC}\right|$.

44.



The question states that the light ray strikes all three mirrors, so it is not parallel to any of them and $a_1\neq 0$, $a_2\neq 0$ and $a_3\neq 0$. Let $\boldsymbol{b}=\left\langle b_1,b_2,b_3\right\rangle$, as in the diagram. We can let $|\boldsymbol{b}|=|\boldsymbol{a}|$, since only its direction is important. Then $\frac{\left|b_2\right|}{|\boldsymbol{b}|}=\sin\theta=\frac{\left|a_2\right|}{|\boldsymbol{a}|}\Rightarrow\left|b_2\right|=\left|a_2\right|$. From the diagram $b_2\boldsymbol{j}$ and $a_2\boldsymbol{j}$ point in opposite directions, so $b_2=-a_2$. $|AB|=|BC|$, so $\left|b_3\right|=\sin\phi\ |BC|=\sin\phi\ |AB|=\left|a_3\right|$, and $\left|b_1\right|=\cos\phi\ |BC|=\cos\phi\ |AB|=\left|a_1\right|$.

$b_3\boldsymbol{k}$ and $a_3\boldsymbol{k}$ have the same direction, as do $b_1\boldsymbol{i}$ and $a_1\boldsymbol{i}$, so $\boldsymbol{b}=\left\langle a_1,-a_2,a_3\right\rangle$. When the ray hits the other mirrors, similar arguments show that these reflections will reverse the signs of the other two coordinates, so the final reflected ray will be $\left\langle -a_1,-a_2,-a_3\right\rangle=-\boldsymbol{a}$, which is parallel to $\boldsymbol{a}$.

1. **(a)** $\boldsymbol{a}\cdot\boldsymbol{b}$ is a scalar, and the dot product is defined only for vectors, so $(\boldsymbol{a}\cdot\boldsymbol{b})\cdot\boldsymbol{c}$ has no meaning.
(b) $(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$ is a scalar multiple of a vector, so it does have meaning.
(c) Both $|\boldsymbol{a}|$ and $\boldsymbol{b}\cdot\boldsymbol{c}$ are scalars, so $|\boldsymbol{a}|\,(\boldsymbol{b}\cdot\boldsymbol{c})$ is an ordinary product of real numbers, and has meaning.
(d) Both $\boldsymbol{a}$ and $\boldsymbol{b}+\boldsymbol{c}$ are vectors, so the dot product $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})$ has meaning.
(e) $\boldsymbol{a}\cdot\boldsymbol{b}$ is a scalar, but $\boldsymbol{c}$ is a vector, and so the two quantities cannot be added and this expression has no meaning.
(f) $|\boldsymbol{a}|$ is a scalar, and the dot product is defined only for vectors, so $|\boldsymbol{a}|\cdot(\boldsymbol{b}+\boldsymbol{c})$ has no meaning.

2. Let the vectors be $\boldsymbol{a}$ and $\boldsymbol{b}$. Then by Theorem 3, $\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta=(6)\left(\frac{1}{3}\right)\cos\frac{\pi}{4}=\frac{6}{3\sqrt{2}}=\sqrt{2}$.

3. $\boldsymbol{a}\cdot\boldsymbol{b}=\langle 4,-1\rangle\cdot\langle 3,6\rangle=(4)\,(3)+(-1)\,(6)=6$

4. $\boldsymbol{a}\cdot\boldsymbol{b}=\left\langle \frac{1}{2},4\right\rangle\cdot\langle -8,-3\rangle=\left(\frac{1}{2}\right)(-8)+(4)(-3)=-16$

5. $\boldsymbol{a}\cdot\boldsymbol{b}=\langle 5,0,-2\rangle\cdot\langle 3,-1,10\rangle=(5)(3)+(0)(-1)+(-2)(10)=-5$

6. $\boldsymbol{a}\cdot\boldsymbol{b}=\langle s,2s,3s\rangle\cdot\langle t,-t,5t\rangle=(s)(t)+(2s)(-t)+(3s)(5t)=st-2st+15st=14st$

7. $\boldsymbol{a}\cdot\boldsymbol{b}=(\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k})\cdot(5\boldsymbol{i}+9\boldsymbol{k})=(1)(5)+(-2)(0)+(3)(9)=32$

8. $\boldsymbol{a}\cdot\boldsymbol{b}=(4\boldsymbol{j}-3\boldsymbol{k})\cdot(2\boldsymbol{i}+4\boldsymbol{j}+6\boldsymbol{k})=(0)(2)+(4)(4)+(-3)(6)=-2$

9. Use Theorem 3: $\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta=(12)(15)\cos\frac{\pi}{6}=180\cdot\frac{\sqrt{3}}{2}=90\sqrt{3}\approx 155.9$

10. Use Theorem 3: $\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta=(4)(10)\cos 120^\circ=40\left(-\frac{1}{2}\right)=-20$

11. $\boldsymbol{u}$,$\boldsymbol{v}$, and $\boldsymbol{w}$ are all unit vectors, so the triangle is an equilateral triangle. Thus the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is 60° and $\boldsymbol{u}\cdot\boldsymbol{v}=|\boldsymbol{u}|\,|\boldsymbol{v}|\cos 60^\circ=(1)(1)\left(\frac{1}{2}\right)=\frac{1}{2}$. If $\boldsymbol{w}$ is moved so it has the same initial point as $\boldsymbol{u}$, we can see that the angle between them is 120° and we have
$\boldsymbol{u}\cdot\boldsymbol{w}=|\boldsymbol{u}|\,|\boldsymbol{w}|\cos 120^\circ=(1)(1)\left(-\frac{1}{2}\right)=-\frac{1}{2}$.

12. $\boldsymbol{u}$ is a unit vector, so $\boldsymbol{w}$ is also a unit vector, and $|\boldsymbol{v}|$ can be determined by examining the right triangle formed by $\boldsymbol{u}$ and $\boldsymbol{v}$. Since the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is 45° , we have

$|\boldsymbol{v}|=|\boldsymbol{u}|\cos 45°=\frac{\sqrt{2}}{2}$. Then $\boldsymbol{u}\cdot\boldsymbol{v}=|\boldsymbol{u}|\,|\boldsymbol{v}|\cos 45°=(1)\left(\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2}=\frac{1}{2}$. Since $\boldsymbol{u}$ and $\boldsymbol{w}$ are orthogonal, $\boldsymbol{u}\cdot\boldsymbol{w}=0$.

13. **(a)** $\boldsymbol{i}\cdot\boldsymbol{j}=\langle 1,0,0\rangle\cdot\langle 0,1,0\rangle=(1)(0)+(0)(1)+(0)(0)=0$. Similarly $\boldsymbol{j}\cdot\boldsymbol{k}=(0)(0)+(1)(0)+(0)(1)=0$ and $\boldsymbol{k}\cdot\boldsymbol{i}=(0)(1)+(0)(0)+(1)(0)=0$.
Another method: Because $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are mutually perpendicular, the cosine factor in each dot product (see Theorem 3) is $\cos\frac{\pi}{2}=0$.
(b) By Property 1 of the dot product, $\boldsymbol{i}\cdot\boldsymbol{i}=|\boldsymbol{i}|^2=1^2=1$ since $\boldsymbol{i}$ is a unit vector. Similarly, $\boldsymbol{j}\cdot\boldsymbol{j}=|\boldsymbol{j}|^2=1$ and $\boldsymbol{k}\cdot\boldsymbol{k}=|\boldsymbol{k}|^2=1$.

14. The dot product $\boldsymbol{A}\cdot\boldsymbol{P}$ is
$\langle a,b,c\rangle\cdot\langle 2,1.5,1\rangle=a(2)+b(1.5)+c(1)$
= (number of hamburgers sold) (price per hamburger)
+ (number of hot dogs sold) (price per hot dog)
+ (number of soft drinks sold) (price per soft drink)

so it is equal to the vendor's total revenue for that day.

15. $|\boldsymbol{a}|=\sqrt{3^2+4^2}=5$, $|\boldsymbol{b}|=\sqrt{5^2+12^2}=13$, and $\boldsymbol{a}\cdot\boldsymbol{b}=(3)(5)+(4)(12)=63$. Using Corollary 6, we have $\cos\theta=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}=\frac{63}{5\cdot 13}=\frac{63}{65}$. So the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\theta=\cos^{-1}\left(\frac{63}{65}\right)\approx 14^\circ$.

16. $|\boldsymbol{a}|=\sqrt{(\sqrt{3})^2+1^2}=2$, $|\boldsymbol{b}|=\sqrt{0+25}=5$, and $\boldsymbol{a}\cdot\boldsymbol{b}=(\sqrt{3})(0)+(1)(5)=5$. Using Corollary 6, we have $\cos\theta=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}=\frac{5}{2\cdot 5}=\frac{1}{2}$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\cos^{-1}\left(\frac{1}{2}\right)=60°$.

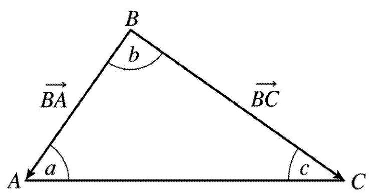
17. $|\boldsymbol{a}|=\sqrt{1^2+2^2+3^2}=\sqrt{14}$, $|\boldsymbol{b}|=\sqrt{4^2+0^2+(-1)^2}=\sqrt{17}$, and $\boldsymbol{a}\cdot\boldsymbol{b}=(1)(4)+(2)(0)+(3)(-1)=1$. Then $\cos\theta=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}=\frac{1}{\sqrt{14}\cdot\sqrt{17}}=\frac{1}{\sqrt{238}}$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\theta=\cos^{-1}\left(\frac{1}{\sqrt{238}}\right)\approx 86^\circ$.

18. $|\boldsymbol{a}|=\sqrt{6^2+(-3)^2+2^2}=7$, $|\boldsymbol{b}|=\sqrt{2^2+1^2+(-2)^2}=3$, and $\boldsymbol{a}\cdot\boldsymbol{b}=(6)(2)+(-3)(1)+(2)(-2)=5$. Then $\cos\theta=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}=\frac{5}{7\cdot 3}=\frac{5}{21}$ and $\theta=\cos^{-1}\left(\frac{5}{21}\right)\approx 76^\circ$.

19. $|\boldsymbol{a}|=\sqrt{0^2+1^2+1^2}=\sqrt{2}$, $|\boldsymbol{b}|=\sqrt{1^2+2^2+(-3)^2}=\sqrt{14}$, and $\boldsymbol{a}\cdot\boldsymbol{b}=(0)(1)+(1)(2)+(1)(-3)=-1$. Then $\cos\theta=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}=\frac{-1}{\sqrt{2}\cdot\sqrt{14}}=\frac{-1}{2\sqrt{7}}$ and $\theta=\cos^{-1}\left(-\frac{1}{2\sqrt{7}}\right)\approx 101^\circ$.

20. $|\mathbf{a}|=\sqrt{2^2+(-1)^2+1^2}=\sqrt{6}$, $|\mathbf{b}|=\sqrt{3^2+2^2+(-1)^2}=\sqrt{14}$, and $\mathbf{a}\cdot\mathbf{b}=(2)(3)+(-1)(2)+(1)(-1)=3$. Then $\cos\theta=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}=\dfrac{3}{\sqrt{6}\cdot\sqrt{14}}=\dfrac{3}{2\sqrt{21}}$ and $\theta=\cos^{-1}\left(\dfrac{3}{2\sqrt{21}}\right)\approx 71^\circ$.

21. Let a , b , and c be the angles at vertices A , B , and C respectively. Then a is the angle between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, b is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and c is the angle between vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$.



Thus $\cos a=\dfrac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}|\,|\overrightarrow{AC}|}=\dfrac{\langle 2,6\rangle\cdot\langle -2,4\rangle}{\sqrt{2^2+6^2}\sqrt{(-2)^2+4^2}}=\dfrac{1}{\sqrt{40}\sqrt{20}}(-4+24)=\dfrac{20}{\sqrt{800}}=\dfrac{\sqrt{2}}{2}$ and

$a=\cos^{-1}\left(\dfrac{\sqrt{2}}{2}\right)=45^\circ$. Similarly,

$\cos b=\dfrac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}|\,|\overrightarrow{BC}|}=\dfrac{\langle -2,-6\rangle\cdot\langle -4,-2\rangle}{\sqrt{4+36}\sqrt{16+4}}=\dfrac{1}{\sqrt{40}\sqrt{20}}(8+12)=\dfrac{20}{\sqrt{800}}=\dfrac{\sqrt{2}}{2}$ so $b=\cos^{-1}\left(\dfrac{\sqrt{2}}{2}\right)=45^\circ$ and $c=180^\circ-(45^\circ+45^\circ)=90^\circ$.

Alternate solution: Apply the Law of Cosines three times as follows: $\cos a=\dfrac{|\overrightarrow{BC}|^2-|\overrightarrow{AB}|^2-|\overrightarrow{AC}|^2}{2|\overrightarrow{AB}|\,|\overrightarrow{AC}|}$,

$\cos b=\dfrac{|\overrightarrow{AC}|^2-|\overrightarrow{AB}|^2-|\overrightarrow{BC}|^2}{2|\overrightarrow{AB}|\,|\overrightarrow{BC}|}$, and $\cos c=\dfrac{|\overrightarrow{AB}|^2-|\overrightarrow{AC}|^2-|\overrightarrow{BC}|^2}{2|\overrightarrow{AC}|\,|\overrightarrow{BC}|}$.

22. As in Exercise 21, let d , e , and f be the angles at vertices D , E , and F . Then d is the angle between vectors $\overrightarrow{DE}$ and $\overrightarrow{DF}$, e is the angle between vectors $\overrightarrow{ED}$ and $\overrightarrow{EF}$, and f is the angle between vectors $\overrightarrow{FD}$ and $\overrightarrow{FE}$. Thus

$\cos d=\dfrac{\overrightarrow{DE}\cdot\overrightarrow{DF}}{|\overrightarrow{DE}|\,|\overrightarrow{DF}|}=\dfrac{\langle -2,3,2\rangle\cdot\langle 1,1,-2\rangle}{\sqrt{(-2)^2+3^2+2^2}\sqrt{1^2+1^2+(-2)^2}}=\dfrac{1}{\sqrt{17}\sqrt{6}}(-2+3-4)=-\dfrac{3}{\sqrt{102}}$

and

$d=\cos^{-1}\left(-\frac{3}{\sqrt{102}}\right)\approx 107°$. Similarly,

$\cos e=\frac{\overrightarrow{ED}\cdot\overrightarrow{EF}}{|\overrightarrow{ED}|\,|\overrightarrow{EF}|}=\frac{\langle 2,-3,-2\rangle\cdot\langle 3,-2,-4\rangle}{\sqrt{4+9+4}\sqrt{9+4+16}}=\frac{1}{\sqrt{17}\sqrt{29}}(6+6+8)=\frac{20}{\sqrt{493}}$ so

$e=\cos^{-1}\left(\frac{20}{\sqrt{493}}\right)\approx 26°$ and $f\approx 180°-(107°+26°)=47°$.

Alternate solution: Apply the Law of Cosines three times as follows: $\cos d=\frac{|\overrightarrow{EF}|^2-|\overrightarrow{DE}|^2-|\overrightarrow{DF}|^2}{2|\overrightarrow{DE}|\,|\overrightarrow{DF}|}$,

$\cos e=\frac{|\overrightarrow{DF}|^2-|\overrightarrow{DE}|^2-|\overrightarrow{EF}|^2}{2|\overrightarrow{DE}|\,|\overrightarrow{EF}|}$, and $\cos f=\frac{|\overrightarrow{DE}|^2-|\overrightarrow{DF}|^2-|\overrightarrow{EF}|^2}{2|\overrightarrow{DF}|\,|\overrightarrow{EF}|}$.

23. **(a)** $\boldsymbol{a}\cdot\boldsymbol{b}=(-5)(6)+(3)(-8)+(7)(2)=-40\neq 0$, so $\boldsymbol{a}$ and $\boldsymbol{b}$ are not orthogonal. Also, since $\boldsymbol{a}$ is not a scalar multiple of $\boldsymbol{b}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel.
(b) $\boldsymbol{a}\cdot\boldsymbol{b}=(4)(-3)+(6)(2)=0$, so $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal (and not parallel).
(c) $\boldsymbol{a}\cdot\boldsymbol{b}=(-1)(3)+(2)(4)+(5)(-1)=0$, so $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal (and not parallel).
(d) Because $\boldsymbol{a}=-\frac{2}{3}\boldsymbol{b}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

24. **(a)** Because $\boldsymbol{u}=-\frac{3}{4}\boldsymbol{v}$, $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel vectors (and thus not orthogonal).
(b) $\boldsymbol{u}\cdot\boldsymbol{v}=(1)(2)+(-1)(-1)+(2)(1)=5\neq 0$, so $\boldsymbol{u}$ and $\boldsymbol{v}$ are not orthogonal. Also, $\boldsymbol{u}$ is not a scalar multiple of $\boldsymbol{v}$, so $\boldsymbol{u}$ and $\boldsymbol{v}$ are not parallel.
(c) $\boldsymbol{u}\cdot\boldsymbol{v}=(a)(-b)+(b)(a)+(c)(0)=-ab+ab+0=0$, so $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal (and not parallel).

25. $\overrightarrow{QP}=\langle -1,-3,2\rangle$, $\overrightarrow{QR}=\langle 4,-2,-1\rangle$, and $\overrightarrow{QP}\cdot\overrightarrow{QR}=-4+6-2=0$. Thus $\overrightarrow{QP}$ and $\overrightarrow{QR}$ are orthogonal, so the angle of the triangle at vertex Q is a right angle.

26. $\langle -6,b,2\rangle$ and $\langle b,b^2,b\rangle$ are orthogonal when $\langle -6,b,2\rangle\cdot\langle b,b^2,b\rangle=0\Leftrightarrow(-6)(b)+(b)(b^2)+(2)(b)=0$ $\Leftrightarrow b^3-4b=0\Leftrightarrow b(b+2)(b-2)=0\Leftrightarrow b=0$ or $b=\pm 2$.

27. Let $\boldsymbol{a}=a_1\boldsymbol{i}+a_2\boldsymbol{j}+a_3\boldsymbol{k}$ be a vector orthogonal to both $\boldsymbol{i}+\boldsymbol{j}$ and $\boldsymbol{i}+\boldsymbol{k}$. Then $\boldsymbol{a}\cdot(\boldsymbol{i}+\boldsymbol{j})=0\Leftrightarrow a_1+a_2=0$ and $\boldsymbol{a}\cdot(\boldsymbol{i}+\boldsymbol{k})=0\Leftrightarrow a_1+a_3=0$, so $a_1=-a_2=-a_3$. Furthermore $\boldsymbol{a}$ is to be a unit vector, so $1=a_1^2+a_2^2+a_3^2=3a_1^2$ implies $a_1=\pm\frac{1}{\sqrt{3}}$. Thus $\boldsymbol{a}=\frac{1}{\sqrt{3}}\boldsymbol{i}-\frac{1}{\sqrt{3}}\boldsymbol{j}-\frac{1}{\sqrt{3}}\boldsymbol{k}$ and $\boldsymbol{a}=-\frac{1}{\sqrt{3}}\boldsymbol{i}+\frac{1}{\sqrt{3}}\boldsymbol{j}+\frac{1}{\sqrt{3}}\boldsymbol{k}$ are two such unit vectors.

28. Let

$\boldsymbol{u}=\langle a,b\rangle$ be a unit vector. By Theorem 3 we need $\boldsymbol{u}\cdot\boldsymbol{v}=|\boldsymbol{u}|\,|\boldsymbol{v}|\cos 60°\Leftrightarrow 3a+4b=(1)(5)\frac{1}{2}\Leftrightarrow b=\frac{5}{8}-\frac{3}{4}a$. Since $\boldsymbol{u}$ is a unit vector, $|\boldsymbol{u}|=\sqrt{a^2+b^2}=1\Leftrightarrow a^2+b^2=1\Leftrightarrow a^2+\left(\frac{5}{8}-\frac{3}{4}a\right)^2=1\Leftrightarrow \frac{25}{16}a^2-\frac{15}{16}a+\frac{25}{64}=1\Leftrightarrow$ $100a^2-60a-39=0$. By the quadratic formula,

$a=\frac{-(-60)\pm\sqrt{(-60)^2-4(100)(-39)}}{2(100)}=\frac{60\pm\sqrt{19{,}200}}{200}=\frac{3\pm 4\sqrt{3}}{10}$. If $a=\frac{3+4\sqrt{3}}{10}$ then $b=\frac{5}{8}-\frac{3}{4}\left(\frac{3+4\sqrt{3}}{10}\right)=\frac{4-3\sqrt{3}}{10}$, and if $a=\frac{3-4\sqrt{3}}{10}$ then $b=\frac{5}{8}-\frac{3}{4}\left(\frac{3-4\sqrt{3}}{10}\right)=\frac{4+3\sqrt{3}}{10}$. Thus the two unit vectors are $\left\langle\frac{3+4\sqrt{3}}{10},\frac{4-3\sqrt{3}}{10}\right\rangle\approx\langle 0.9928,-0.1196\rangle$ and $\left\langle\frac{3-4\sqrt{3}}{10},\frac{4+3\sqrt{3}}{10}\right\rangle\approx\langle -0.3928,0.9196\rangle$.

29. Since $|\langle 3,4,5\rangle|=\sqrt{9+16+25}=\sqrt{50}=5\sqrt{2}$, using Equations 8 and 9 we have $\cos\alpha=\frac{3}{5\sqrt{2}}$, $\cos\beta=\frac{4}{5\sqrt{2}}$, and $\cos\gamma=\frac{5}{5\sqrt{2}}=\frac{1}{\sqrt{2}}$. The direction angles are given by $\alpha=\cos^{-1}\left(\frac{3}{5\sqrt{2}}\right)\approx 65°$, $\beta=\cos^{-1}\left(\frac{4}{5\sqrt{2}}\right)\approx 56°$, and $\gamma=\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=45°$.

30. Since $|\langle 1,-2,-1\rangle|=\sqrt{1+4+1}=\sqrt{6}$, using Equations 8 and 9 we have $\cos\alpha=\frac{1}{\sqrt{6}}$, $\cos\beta=\frac{-2}{\sqrt{6}}$, and $\cos\gamma=\frac{-1}{\sqrt{6}}$. The direction angles are given by $\alpha=\cos^{-1}\left(\frac{1}{\sqrt{6}}\right)\approx 66°$, $\beta=\cos^{-1}\left(-\frac{2}{\sqrt{6}}\right)\approx 145°$, and $\gamma=\cos^{-1}\left(-\frac{1}{\sqrt{6}}\right)\approx 114°$.

31. Since $|2\boldsymbol{i}+3\boldsymbol{j}-6\boldsymbol{k}|=\sqrt{4+9+36}=\sqrt{49}=7$, Equations 8 and 9 give $\cos\alpha=\frac{2}{7}$, $\cos\beta=\frac{3}{7}$, and $\cos\gamma=\frac{-6}{7}$, while $\alpha=\cos^{-1}\left(\frac{2}{7}\right)\approx 73°$, $\beta=\cos^{-1}\left(\frac{3}{7}\right)\approx 65°$, and $\gamma=\cos^{-1}\left(-\frac{6}{7}\right)\approx 149°$.

32. Since $|2\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}|=\sqrt{4+1+4}=\sqrt{9}=3$, Equations 8 and 9 give $\cos\alpha=\frac{2}{3}$, $\cos\beta=\frac{-1}{3}$, and $\cos\gamma=\frac{2}{3}$, while $\alpha=\gamma=\cos^{-1}\left(\frac{2}{3}\right)\approx 48°$ and $\beta=\cos^{-1}\left(-\frac{1}{3}\right)\approx 109°$.

33. $|\langle c,c,c\rangle|=\sqrt{c^2+c^2+c^2}=\sqrt{3}\,c$ (since $c>0$), so $\cos\alpha=\cos\beta=\cos\gamma=\dfrac{c}{\sqrt{3}\,c}=\dfrac{1}{\sqrt{3}}$ and $\alpha=\beta=\gamma=\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)\approx 55^\circ$.

34. Since $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$, $\cos^2\gamma=1-\cos^2\alpha-\cos^2\beta=1-\cos^2\left(\frac{\pi}{4}\right)-\cos^2\left(\frac{\pi}{3}\right)=1-\left(\frac{1}{\sqrt{2}}\right)^2-\left(\frac{1}{2}\right)^2=\frac{1}{4}$. Thus $\cos\gamma=\pm\dfrac{1}{2}$ and $\gamma=\dfrac{\pi}{3}$ or $\gamma=\dfrac{2\pi}{3}$.

35. $|\boldsymbol{a}|=\sqrt{3^2+(-4)^2}=5$. The scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\text{comp}_{\boldsymbol{a}}\boldsymbol{b}=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}=\dfrac{3\cdot 5+(-4)\cdot 0}{5}=3$ and the vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\text{proj}_{\boldsymbol{a}}\boldsymbol{b}=\left(\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}\right)\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}=3\cdot\frac{1}{5}\langle 3,-4\rangle=\left\langle\frac{9}{5},-\frac{12}{5}\right\rangle$.

36. $|\boldsymbol{a}|=\sqrt{1^2+2^2}=\sqrt{5}$, so the scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\text{comp}_{\boldsymbol{a}}\boldsymbol{b}=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}=\dfrac{1(-4)+2\cdot 1}{\sqrt{5}}=-\dfrac{2}{\sqrt{5}}$ and the vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\text{proj}_{\boldsymbol{a}}\boldsymbol{b}=\left(\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}\right)\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}=-\dfrac{2}{\sqrt{5}}\cdot\dfrac{1}{\sqrt{5}}\langle 1,2\rangle=\left\langle-\frac{2}{5},-\frac{4}{5}\right\rangle$.

37. $|\boldsymbol{a}|=\sqrt{16+4+0}=2\sqrt{5}$ so the scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\text{comp}_{\boldsymbol{a}}\boldsymbol{b}=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}=\dfrac{1}{2\sqrt{5}}(4+2+0)=\dfrac{3}{\sqrt{5}}$. The vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\text{proj}_{\boldsymbol{a}}\boldsymbol{b}=\dfrac{3}{\sqrt{5}}\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}=\dfrac{3}{\sqrt{5}}\cdot\dfrac{1}{2\sqrt{5}}\langle 4,2,0\rangle=\frac{1}{5}\langle 6,3,0\rangle=\left\langle\frac{6}{5},\frac{3}{5},0\right\rangle$.

38. $|\boldsymbol{a}|=\sqrt{1+4+4}=3$ so the scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\text{comp}_{\boldsymbol{a}}\boldsymbol{b}=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}=\dfrac{-3+(-6)+8}{3}=-\dfrac{1}{3}$, while the vector projection is $\text{proj}_{\boldsymbol{a}}\boldsymbol{b}=-\dfrac{1}{3}\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}=-\dfrac{1}{3}\cdot\dfrac{\langle -1,-2,2\rangle}{3}=\left\langle\frac{1}{9},\frac{2}{9},-\frac{2}{9}\right\rangle$.

39. $|\boldsymbol{a}|=\sqrt{1+0+1}=\sqrt{2}$ so the scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\text{comp}_{\boldsymbol{a}}\boldsymbol{b}=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}=\dfrac{1}{\sqrt{2}}(1+0+0)=\dfrac{1}{\sqrt{2}}$ while the vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\text{proj}_{\boldsymbol{a}}\boldsymbol{b}=\dfrac{1}{\sqrt{2}}\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}=\dfrac{1}{\sqrt{2}}\cdot\dfrac{1}{\sqrt{2}}(\boldsymbol{i}+\boldsymbol{k})=\dfrac{1}{2}(\boldsymbol{i}+\boldsymbol{k})$.

40. $|\boldsymbol{a}|=\sqrt{4+9+1}=\sqrt{14}$, so the scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is

$\text{comp}_{\boldsymbol{a}}\boldsymbol{b}=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}=\dfrac{2-18-2}{\sqrt{14}}=-\dfrac{18}{\sqrt{14}}$ while the vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is

$\text{proj}_{\boldsymbol{a}}\boldsymbol{b}=-\dfrac{18}{\sqrt{14}}\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}=-\dfrac{18}{\sqrt{14}}\cdot\dfrac{2\boldsymbol{i}-3\boldsymbol{j}+\boldsymbol{k}}{\sqrt{14}}=-\dfrac{9}{7}(2\boldsymbol{i}-3\boldsymbol{j}+\boldsymbol{k})$.

41.

$$\left(\text{orth}_{\boldsymbol{a}}\boldsymbol{b}\right)\cdot\boldsymbol{a}=\left(\boldsymbol{b}-\text{proj}_{\boldsymbol{a}}\boldsymbol{b}\right)\cdot\boldsymbol{a}=\boldsymbol{b}\cdot\boldsymbol{a}-\left(\text{proj}_{\boldsymbol{a}}\boldsymbol{b}\right)\cdot\boldsymbol{a}=\boldsymbol{b}\cdot\boldsymbol{a}-\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|^2}\boldsymbol{a}\cdot\boldsymbol{a}$$

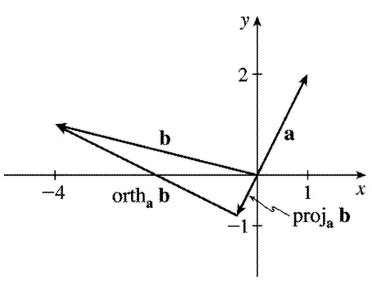
$$=\boldsymbol{b}\cdot\boldsymbol{a}-\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|^2}|\boldsymbol{a}|^2=\boldsymbol{b}\cdot\boldsymbol{a}-\boldsymbol{a}\cdot\boldsymbol{b}=0$$

So they are orthogonal by (7).

42. Using the formula in Exercise 41 and the result of Exercise 36,
we have

$$\text{orth}_{\boldsymbol{a}}\boldsymbol{b}=\boldsymbol{b}-\text{proj}_{\boldsymbol{a}}\boldsymbol{b}=\langle -4,1\rangle-\left\langle -\frac{2}{5},-\frac{4}{5}\right\rangle$$

$$=\left\langle -\frac{18}{5},\frac{9}{5}\right\rangle$$



43. $\text{comp}_{\boldsymbol{a}}\boldsymbol{b}=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}=2\Leftrightarrow\boldsymbol{a}\cdot\boldsymbol{b}=2|\boldsymbol{a}|=2\sqrt{10}$. If $\boldsymbol{b}=\left\langle b_1,b_2,b_3\right\rangle$, then we need $3b_1+0b_2-1b_3=2\sqrt{10}$. One possible solution is obtained by taking $b_1=0$, $b_2=0$, $b_3=-2\sqrt{10}$.
In general, $\boldsymbol{b}=\left\langle s,t,3s-2\sqrt{10}\right\rangle$, s , $t\in R$.

44. **(a)** $\text{comp}_{\boldsymbol{a}}\boldsymbol{b}=\text{comp}_{\boldsymbol{b}}\boldsymbol{a}\Leftrightarrow\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}=\dfrac{\boldsymbol{b}\cdot\boldsymbol{a}}{|\boldsymbol{b}|}\Leftrightarrow\dfrac{1}{|\boldsymbol{a}|}=\dfrac{1}{|\boldsymbol{b}|}$ or $\boldsymbol{a}\cdot\boldsymbol{b}=0\Leftrightarrow|\boldsymbol{b}|=|\boldsymbol{a}|$ or $\boldsymbol{a}\cdot\boldsymbol{b}=0$.

That is, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal or if they have the same length.

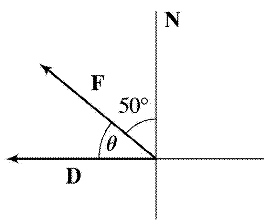
(b) $\text{proj}_{\boldsymbol{a}}\boldsymbol{b}=\text{proj}_{\boldsymbol{b}}\boldsymbol{a} \Leftrightarrow \dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|^2}\boldsymbol{a}=\dfrac{\boldsymbol{b}\cdot\boldsymbol{a}}{|\boldsymbol{b}|^2}\boldsymbol{b} \Leftrightarrow \boldsymbol{a}\cdot\boldsymbol{b}=0$ or $\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|^2}=\dfrac{\boldsymbol{b}}{|\boldsymbol{b}|^2}$. But $\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|^2}=\dfrac{\boldsymbol{b}}{|\boldsymbol{b}|^2} \Rightarrow \dfrac{|\boldsymbol{a}|}{|\boldsymbol{a}|^2}=\dfrac{|\boldsymbol{b}|}{|\boldsymbol{b}|^2} \Rightarrow$ $|\boldsymbol{a}|=|\boldsymbol{b}|$. Substituting this into the previous equation gives $\boldsymbol{a}=\boldsymbol{b}$.

So $\text{proj}_{\boldsymbol{a}}\boldsymbol{b}=\text{proj}_{\boldsymbol{b}}\boldsymbol{a} \Leftrightarrow$ $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal, or they are equal.

45. Here $\boldsymbol{D}=(4-2)\boldsymbol{i}+(9-3)\boldsymbol{j}+(15-0)\boldsymbol{k}=2\boldsymbol{i}+6\boldsymbol{j}+15\boldsymbol{k}$ so by Equation 12 we have $W=\boldsymbol{F}\cdot\boldsymbol{D}=20+108-90=38$ joules.

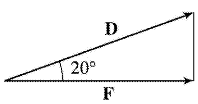
46.

$$\begin{aligned} W &= |\boldsymbol{F}|\,|\boldsymbol{D}|\cos\theta=(20)(4)\cos 40^\circ \\ &\approx 61 \text{ ft–lb} \end{aligned}$$



47.

$$\begin{aligned} W &= |\boldsymbol{F}|\,|\boldsymbol{D}|\cos\theta=(25)(10)\cos 20^\circ \\ &\approx 235 \text{ ft–lb} \end{aligned}$$

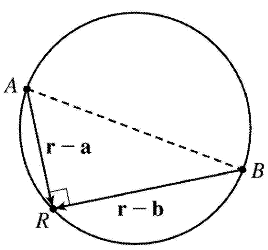


48. Here $|\boldsymbol{D}|=100$ m, $|\boldsymbol{F}|=50$ N, and $\theta=30^\circ$. Thus $W=|\boldsymbol{F}|\,|\boldsymbol{D}|\cos\theta=(50)(100)\left(\dfrac{\sqrt{3}}{2}\right)=2500\sqrt{3}$ joules.

49. First note that $\boldsymbol{n}=\langle a,b\rangle$ is perpendicular to the line, because if $Q_1=(a_1,b_1)$ and $Q_2=(a_2,b_2)$ lie on the line, then $\boldsymbol{n}\cdot\overrightarrow{Q_1Q_2}=aa_2-aa_1+bb_2-bb_1=0$, since $aa_2+bb_2=-c=aa_1+bb_1$ from the equation of the line. Let $P_2=(x_2,y_2)$ lie on the line. Then the distance from P_1 to the line is the absolute value of the scalar projection
of

$\overrightarrow{P_1P_2}$ onto $\boldsymbol{n}$. $\text{comp}_{\boldsymbol{n}}\left(\overrightarrow{P_1P_2}\right)=\dfrac{\left|\boldsymbol{n}\cdot\left\langle x_2-x_1,y_2-y_1\right\rangle\right|}{|\boldsymbol{n}|}=\dfrac{\left|ax_2-ax_1+by_2-by_1\right|}{\sqrt{a^2+b^2}}=\dfrac{\left|ax_1+by_1+c\right|}{\sqrt{a^2+b^2}}$ since $ax_2+by_2=-c$. The required distance is $\dfrac{|3\cdot-2+-4\cdot 3+5|}{\sqrt{3^2+4^2}}=\dfrac{13}{5}$.

50. $(\boldsymbol{r}-\boldsymbol{a})\cdot(\boldsymbol{r}-\boldsymbol{b})=0$ implies that the vectors $\boldsymbol{r}-\boldsymbol{a}$ and $\boldsymbol{r}-\boldsymbol{b}$ are orthogonal. From the diagram (in which A , B and R are the terminal points of the vectors), we see that this implies that R lies on a sphere whose diameter is the line from A to B . The center of this circle is the midpoint of AB , that is, $\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})=\left\langle\frac{1}{2}\left(a_1+b_1\right),\frac{1}{2}\left(a_2+b_2\right),\frac{1}{2}\left(a_3+b_3\right)\right\rangle$, and its radius is $\frac{1}{2}|\boldsymbol{a}-\boldsymbol{b}|=\frac{1}{2}\sqrt{\left(a_1-b_1\right)^2+\left(a_2-b_2\right)^2+\left(a_3-b_3\right)^2}$.



Or: Expand the given equation, substitute $\boldsymbol{r}\cdot\boldsymbol{r}=x^2+y^2+z^2$ and complete the squares.

51. For convenience, consider the unit cube positioned so that its back left corner is at the origin, and its edges lie along the coordinate axes. The diagonal of the cube that begins at the origin and ends at $(1,1,1)$ has vector representation $\langle 1,1,1\rangle$. The angle θ between this vector and the vector of the edge which also begins at the origin and runs along the x –axis is given by $\cos\theta=\dfrac{<1,1,1>\cdot<1,0,0>}{|<1,1,1>|\,|<1,0,0>|}=\dfrac{1}{\sqrt{3}}\Rightarrow\theta=\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)\approx 55^\circ$.

52. Consider a cube with sides of unit length, wholly within the first octant and with edges along each of the three coordinate axes. $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{i}+\boldsymbol{j}$ are vector representations of a diagonal of the cube and a diagonal of one of its faces. If θ is the angle between these diagonals, then $\cos\theta=\dfrac{(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})\cdot(\boldsymbol{i}+\boldsymbol{j})}{|\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}|\,|\boldsymbol{i}+\boldsymbol{j}|}=\dfrac{1+1}{\sqrt{3}\sqrt{2}}=\sqrt{\dfrac{2}{3}}\Rightarrow\theta=\cos^{-1}\sqrt{\dfrac{2}{3}}\approx 35^\circ$.

53. Consider the H–C–H combination consisting of the sole carbon atom and the two hydrogen atoms that are at $(1,0,0)$ and $(0,1,0)$ (or any H–C–H combination, for that matter). Vector representations of

the line segments emanating from the carbon atom and extending to these two hydrogen atoms are $\left\langle 1-\frac{1}{2},0-\frac{1}{2},0-\frac{1}{2}\right\rangle=\left\langle \frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle$ and $\left\langle 0-\frac{1}{2},1-\frac{1}{2},0-\frac{1}{2}\right\rangle=\left\langle -\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle$. The bond angle, θ , is therefore given by

$$\cos\theta=\frac{\left\langle \frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle\cdot\left\langle -\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle}{\left|\left\langle \frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle\right|\left|\left\langle -\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle\right|}=\frac{-\frac{1}{4}-\frac{1}{4}+\frac{1}{4}}{\sqrt{\frac{3}{4}}\sqrt{\frac{3}{4}}}=-\frac{1}{3}\Rightarrow$$

$$\theta=\cos^{-1}\left(-\frac{1}{3}\right)\approx 109.5^\circ .$$

54. Let α be the angle between $\boldsymbol{a}$ and $\boldsymbol{c}$ and β be the angle between $\boldsymbol{c}$ and $\boldsymbol{b}$. We need to show that $\alpha=\beta$. Now $\cos\alpha=\frac{\boldsymbol{a}\cdot\boldsymbol{c}}{|\boldsymbol{a}|\,|\boldsymbol{c}|}=\frac{\boldsymbol{a}\cdot|\boldsymbol{a}|\,\boldsymbol{b}+\boldsymbol{a}\cdot|\boldsymbol{b}|\,\boldsymbol{a}}{|\boldsymbol{a}|\,|\boldsymbol{c}|}=\frac{|\boldsymbol{a}|\,\boldsymbol{a}\cdot\boldsymbol{b}+|\boldsymbol{a}|^2|\boldsymbol{b}|}{|\boldsymbol{a}|\,|\boldsymbol{c}|}=\frac{\boldsymbol{a}\cdot\boldsymbol{b}+|\boldsymbol{a}|\,|\boldsymbol{b}|}{|\boldsymbol{c}|}$. Similarly, $\cos\beta=\frac{\boldsymbol{b}\cdot\boldsymbol{c}}{|\boldsymbol{b}|\,|\boldsymbol{c}|}=\frac{|\boldsymbol{a}|\,|\boldsymbol{b}|+\boldsymbol{b}\cdot\boldsymbol{a}}{|\boldsymbol{c}|}$. Thus $\cos\alpha=\cos\beta$. However $0^\circ\le\alpha\le 180^\circ$ and $0^\circ\le\beta\le 180^\circ$, so $\alpha=\beta$ and $\boldsymbol{c}$ bisects the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

55. Let $\boldsymbol{a}=\langle a_1,a_2,a_3\rangle$ and $\boldsymbol{b}=\langle b_1,b_2,b_3\rangle$.

Property 2:

$$\begin{aligned}\boldsymbol{a}\cdot\boldsymbol{b}&=\langle a_1,a_2,a_3\rangle\cdot\langle b_1,b_2,b_3\rangle=a_1b_1+a_2b_2+a_3b_3\\&=b_1a_1+b_2a_2+b_3a_3=\langle b_1,b_2,b_3\rangle\cdot\langle a_1,a_2,a_3\rangle=\boldsymbol{b}\cdot\boldsymbol{a}\end{aligned}$$

Property 4:

$$\begin{aligned}(c\,\boldsymbol{a})\cdot\boldsymbol{b}&=\langle ca_1,ca_2,ca_3\rangle\cdot\langle b_1,b_2,b_3\rangle=(ca_1)\,b_1+(ca_2)\,b_2+(ca_3)\,b_3\\&=c\left(a_1b_1+a_2b_2+a_3b_3\right)=c\,(\boldsymbol{a}\cdot\boldsymbol{b})=a_1\,(cb_1)+a_2\,(cb_2)+a_3\,(cb_3)\\&=\langle a_1,a_2,a_3\rangle\cdot\langle cb_1,cb_2,cb_3\rangle=\boldsymbol{a}\cdot(c\,\boldsymbol{b})\end{aligned}$$

Property 5: $\boldsymbol{0}\cdot\boldsymbol{a}=\langle 0,0,0\rangle\cdot\langle a_1,a_2,a_3\rangle=(0)\,(a_1)+(0)\,(a_2)+(0)\,(a_3)=0$

56. Let the figure be called quadrilateral $ABCD$. The diagonals can be represented by $\overrightarrow{AC}$ and $\overrightarrow{BD}$. $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}$ and $\overrightarrow{BD}=\overrightarrow{BC}+\overrightarrow{CD}=\overrightarrow{BC}-\overrightarrow{DC}=\overrightarrow{BC}-\overrightarrow{AB}$ (Since opposite sides of the object are of the same

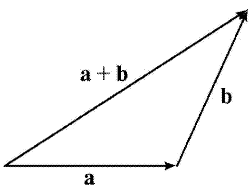
length and parallel, $\overrightarrow{AB}=\overrightarrow{DC}$.) Thus

$$\begin{aligned}\overrightarrow{AC}\cdot\overrightarrow{BD} &= \left(\overrightarrow{AB}+\overrightarrow{BC}\right)\cdot\left(\overrightarrow{BC}-\overrightarrow{AB}\right)=\overrightarrow{AB}\cdot\left(\overrightarrow{BC}-\overrightarrow{AB}\right)+\overrightarrow{BC}\cdot\left(\overrightarrow{BC}-\overrightarrow{AB}\right)\\ &= \overrightarrow{AB}\cdot\overrightarrow{BC}-\left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{BC}\right|^2-\overrightarrow{AB}\cdot\overrightarrow{BC}=\left|\overrightarrow{BC}\right|^2-\left|\overrightarrow{AB}\right|^2\end{aligned}$$

But $\left|\overrightarrow{AB}\right|^2=\left|\overrightarrow{BC}\right|^2$ because all sides of the quadrilateral are equal in length. Therefore $\overrightarrow{AC}\cdot\overrightarrow{BD}=0$, and since both of these vectors are nonzero this tells us that the diagonals of the quadrilateral are perpendicular.

57. $|\boldsymbol{a}\cdot\boldsymbol{b}|=\left||\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta\right|=|\boldsymbol{a}|\,|\boldsymbol{b}|\,|\cos\theta|$. Since $|\cos\theta|\le 1$, $|\boldsymbol{a}\cdot\boldsymbol{b}|=|\boldsymbol{a}|\,|\boldsymbol{b}|\,|\cos\theta|\le|\boldsymbol{a}|\,|\boldsymbol{b}|$. *Note:* We have equality in the case of $\cos\theta=\pm 1$, so $\theta=0$ or $\theta=\pi$, thus equality when ***a*** and ***b*** are parallel.

58. **(a)**



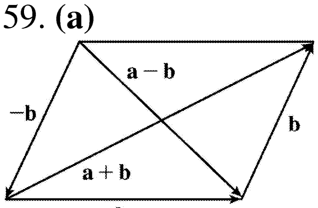
The Triangle Inequality states that the length of the longest side of a triangle is less than or equal to the sum of the lengths of the two shortest sides.

(b)

$$\begin{aligned}|\boldsymbol{a}+\boldsymbol{b}|^2 &=(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}+\boldsymbol{b})=(\boldsymbol{a}\cdot\boldsymbol{a})+2(\boldsymbol{a}\cdot\boldsymbol{b})+(\boldsymbol{b}\cdot\boldsymbol{b})=|\boldsymbol{a}|^2+2(\boldsymbol{a}\cdot\boldsymbol{b})+|\boldsymbol{b}|^2\\ &\le|\boldsymbol{a}|^2+2|\boldsymbol{a}|\,|\boldsymbol{b}|+|\boldsymbol{b}|^2\\ &=(|\boldsymbol{a}|+|\boldsymbol{b}|)^2\end{aligned}$$

Thus, taking the square root of both sides, $|\boldsymbol{a}+\boldsymbol{b}|\le|\boldsymbol{a}|+|\boldsymbol{b}|$.

59. **(a)**



The Parallelogram Law states that the sum of the squares of the lengths of the diagonals of a parallelogram equals the sum of the squares of its (four) sides.

(b) $|\mathbf{a}+\mathbf{b}|^2=(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})=|\mathbf{a}|^2+2(\mathbf{a}\cdot\mathbf{b})+|\mathbf{b}|^2$ and $|\mathbf{a}-\mathbf{b}|^2=(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=|\mathbf{a}|^2-2(\mathbf{a}\cdot\mathbf{b})+|\mathbf{b}|^2$.
Adding these two equations gives $|\mathbf{a}+\mathbf{b}|^2+|\mathbf{a}-\mathbf{b}|^2=2|\mathbf{a}|^2+2|\mathbf{b}|^2$.

1. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&0\\0&3&1\end{vmatrix}=\begin{vmatrix}2&0\\3&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&0\\0&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&2\\0&3\end{vmatrix}\mathbf{k}$

$=(2-0)\mathbf{i}-(1-0)\mathbf{j}+(3-0)\mathbf{k}=2\mathbf{i}-\mathbf{j}+3\mathbf{k}$

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=\langle 2,-1,3\rangle\cdot\langle 1,2,0\rangle=2-2+0=0$ and $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=\langle 2,-1,3\rangle\cdot\langle 0,3,1\rangle=0-3+3=0$, so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

2.

$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\5&1&4\\-1&0&2\end{vmatrix}=\begin{vmatrix}1&4\\0&2\end{vmatrix}\mathbf{i}-\begin{vmatrix}5&4\\-1&2\end{vmatrix}\mathbf{j}+\begin{vmatrix}5&1\\-1&0\end{vmatrix}\mathbf{k}$

$=(2-0)\mathbf{i}-[10-(-4)]\mathbf{j}+[0-(-1)]\mathbf{k}=2\mathbf{i}-14\mathbf{j}+\mathbf{k}$

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=\langle 2,-14,1\rangle\cdot\langle 5,1,4\rangle=10-14+4=0$ and $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=\langle 2,-14,1\rangle\cdot\langle -1,0,2\rangle=-2+0+2=0$, so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

3.

$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&1&-1\\0&1&2\end{vmatrix}=\begin{vmatrix}1&-1\\1&2\end{vmatrix}\mathbf{i}-\begin{vmatrix}2&-1\\0&2\end{vmatrix}\mathbf{j}+\begin{vmatrix}2&1\\0&1\end{vmatrix}\mathbf{k}$

$=[2-(-1)]\mathbf{i}-(4-0)\mathbf{j}+(2-0)\mathbf{k}=3\mathbf{i}-4\mathbf{j}+2\mathbf{k}$

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}-\mathbf{k})=6-4-2=0$ and $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})\cdot(\mathbf{j}+2\mathbf{k})=0-4+4=0$, so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

4.

$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&1\\1&1&1\end{vmatrix}=\begin{vmatrix}-1&1\\1&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&1\\1&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&-1\\1&1\end{vmatrix}\mathbf{k}$

$=(-1-1)\mathbf{i}-(1-1)\mathbf{j}+[1-(-1)]\mathbf{k}=-2\mathbf{i}+2\mathbf{k}$

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=(-2\mathbf{i}+2\mathbf{k})\cdot(\mathbf{i}-\mathbf{j}+\mathbf{k})=-2+0+2=0$ and $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=(-2\mathbf{i}+2\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})=-2+0+2=0$, so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

5.

$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&2&4\\1&-2&-3\end{vmatrix}=\begin{vmatrix}2&4\\-2&-3\end{vmatrix}\mathbf{i}-\begin{vmatrix}3&4\\1&-3\end{vmatrix}\mathbf{j}+\begin{vmatrix}3&2\\1&-2\end{vmatrix}\mathbf{k}$

$=[-6-(-8)]\mathbf{i}-(-9-4)\mathbf{j}+(-6-2)\mathbf{k}=2\mathbf{i}+13\mathbf{j}-8\mathbf{k}$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=(2\mathbf{i}+13\mathbf{j}-8\mathbf{k})\cdot(3\mathbf{i}+2\mathbf{j}+4\mathbf{k})=6+26-32=0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.
Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=(2\mathbf{i}+13\mathbf{j}-8\mathbf{k})\cdot(\mathbf{i}-2\mathbf{j}-3\mathbf{k})=2-26+24=0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

6.

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & e^{t} & e^{-t}\\ 2 & e^{t} & -e^{-t}\end{vmatrix}=\begin{vmatrix} e^{t} & e^{-t}\\ e^{t} & -e^{-t}\end{vmatrix}\mathbf{i}-\begin{vmatrix}1 & e^{-t}\\ 2 & -e^{-t}\end{vmatrix}\mathbf{j}+\begin{vmatrix}1 & e^{t}\\ 2 & e^{t}\end{vmatrix}\mathbf{k}$$

$$=(-1-1)\mathbf{i}-(-e^{-t}-2e^{-t})\mathbf{j}+(e^{t}-2e^{t})\mathbf{k}=-2\mathbf{i}+3e^{-t}\mathbf{j}-e^{t}\mathbf{k}$$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=(-2\mathbf{i}+3e^{-t}\mathbf{j}-e^{t}\mathbf{k})\cdot(\mathbf{i}+e^{t}\mathbf{j}+e^{-t}\mathbf{k})=-2+3-1=0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.
Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=(-2\mathbf{i}+3e^{-t}\mathbf{j}-e^{t}\mathbf{k})\cdot(2\mathbf{i}+e^{t}\mathbf{j}-e^{-t}\mathbf{k})=-4+3+1=0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

7.

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ t & t^{2} & t^{3}\\ 1 & 2t & 3t^{2}\end{vmatrix}=\begin{vmatrix} t^{2} & t^{3}\\ 2t & 3t^{2}\end{vmatrix}\mathbf{i}-\begin{vmatrix} t & t^{3}\\ 1 & 3t^{2}\end{vmatrix}\mathbf{j}+\begin{vmatrix} t & t^{2}\\ 1 & 2t\end{vmatrix}\mathbf{k}$$

$$=(3t^{4}-2t^{4})\mathbf{i}-(3t^{3}-t^{3})\mathbf{j}+(2t^{2}-t^{2})\mathbf{k}=t^{4}\mathbf{i}-2t^{3}\mathbf{j}+t^{2}\mathbf{k}$$

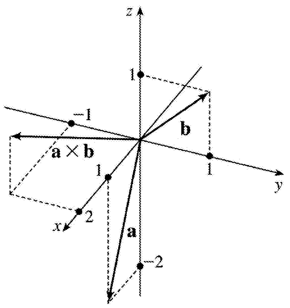
Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=\left\langle t^{4},-2t^{3},t^{2}\right\rangle\cdot\left\langle t,t^{2},t^{3}\right\rangle=t^{5}-2t^{5}+t^{5}=0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.
Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=\left\langle t^{4},-2t^{3},t^{2}\right\rangle\cdot\left\langle 1,2t,3t^{2}\right\rangle=t^{4}-4t^{4}+3t^{4}=0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

8.

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & -2\\ 0 & 1 & 1\end{vmatrix}$$

$$=\begin{vmatrix}0 & -2\\ 1 & 1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1 & -2\\ 0 & 1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}\mathbf{k}$$

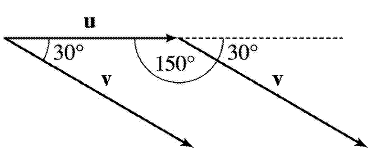
$$=2\mathbf{i}-\mathbf{j}+\mathbf{k}$$



9. **(a)** Since $\boldsymbol{b}\times\boldsymbol{c}$ is a vector, the dot product $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ is meaningful and is a scalar.
(b) $\boldsymbol{b}\cdot\boldsymbol{c}$ is a scalar, so $\boldsymbol{a}\times(\boldsymbol{b}\cdot\boldsymbol{c})$ is meaningless, as the cross product is defined only for two *vectors* .
(c) Since $\boldsymbol{b}\times\boldsymbol{c}$ is a vector, the cross product $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$ is meaningful and results in another vector.
(d) $\boldsymbol{a}\cdot\boldsymbol{b}$ is a scalar, so the cross product $(\boldsymbol{a}\cdot\boldsymbol{b})\times\boldsymbol{c}$ is meaningless.
(e) Since $(\boldsymbol{a}\cdot\boldsymbol{b})$ and $(\boldsymbol{c}\cdot\boldsymbol{d})$ are both scalars, the cross product $(\boldsymbol{a}\cdot\boldsymbol{b})\times(\boldsymbol{c}\cdot\boldsymbol{d})$ is meaningless.
(f) $\boldsymbol{a}\times\boldsymbol{b}$ and $\boldsymbol{c}\times\boldsymbol{d}$ are both vectors, so the dot product $(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d})$ is meaningful and is a scalar.

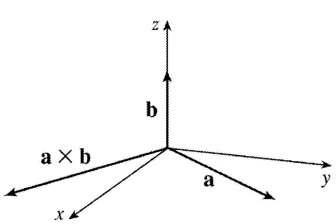
10. Using Theorem 6, we have $|\boldsymbol{u}\times\boldsymbol{v}|=|\boldsymbol{u}|\,|\boldsymbol{v}|\sin\theta=(5)(10)\sin 60°=25\sqrt{3}$. By the right–hand rule, $\boldsymbol{u}\times\boldsymbol{v}$ is directed into the page.

11. If we sketch $\boldsymbol{u}$ and $\boldsymbol{v}$ starting from the same initial point,
we see that the angle between them is 30° . Using Theorem 6, we have
$|\boldsymbol{u}\times\boldsymbol{v}|=|\boldsymbol{u}|\,|\boldsymbol{v}|\sin 30°=(6)(8)\left(\frac{1}{2}\right)=24$
By the right–hand rule, $\boldsymbol{u}\times\boldsymbol{v}$ is directed into the page.



12. **(a)** $|\boldsymbol{a}\times\boldsymbol{b}|=|\boldsymbol{a}|\,|\boldsymbol{b}|\sin\theta=3\cdot 2\cdot\sin\frac{\pi}{2}=6$

(b) $\boldsymbol{a}\times\boldsymbol{b}$ is orthogonal to $\boldsymbol{k}$, so it lies in the xy –plane, and its z –coordinate is 0 . By the right–hand rule, its y –component is negative and its x –component is positive.



13. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 1\\ 0 & 1 & 3\end{vmatrix}=\begin{vmatrix}2 & 1\\ 1 & 3\end{vmatrix}\mathbf{i}-\begin{vmatrix}1 & 1\\ 0 & 3\end{vmatrix}\mathbf{j}+\begin{vmatrix}1 & 2\\ 0 & 1\end{vmatrix}\mathbf{k}$
$=(6-1)\mathbf{i}-(3-0)\mathbf{j}+(1-0)\mathbf{k}=5\mathbf{i}-3\mathbf{j}+\mathbf{k}$
$\mathbf{b}\times\mathbf{a}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 1 & 3\\ 1 & 2 & 1\end{vmatrix}=\begin{vmatrix}1 & 3\\ 2 & 1\end{vmatrix}\mathbf{i}-\begin{vmatrix}0 & 3\\ 1 & 1\end{vmatrix}\mathbf{j}+\begin{vmatrix}0 & 1\\ 1 & 2\end{vmatrix}\mathbf{k}$
$=(1-6)\mathbf{i}-(0-3)\mathbf{j}+(0-1)\mathbf{k}=-5\mathbf{i}+3\mathbf{j}-\mathbf{k}$
Notice $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$ here, as we know is always true by Theorem 8.

14. $\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 1 & 0\\ 0 & 0 & -4\end{vmatrix}=\begin{vmatrix}1 & 0\\ 0 & -4\end{vmatrix}\mathbf{i}-\begin{vmatrix}-1 & 0\\ 0 & -4\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1 & 1\\ 0 & 0\end{vmatrix}\mathbf{k}=-4\mathbf{i}-4\mathbf{j}$ so

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 1 & 2\\ -4 & -4 & 0\end{vmatrix}=\begin{vmatrix}1 & 2\\ -4 & 0\end{vmatrix}\mathbf{i}-\begin{vmatrix}3 & 2\\ -4 & 0\end{vmatrix}\mathbf{j}+\begin{vmatrix}3 & 1\\ -4 & -4\end{vmatrix}\mathbf{k}=8\mathbf{i}-8\mathbf{j}-8\mathbf{k}$.

$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 1 & 2\\ -1 & 1 & 0\end{vmatrix}=\begin{vmatrix}1 & 2\\ 1 & 0\end{vmatrix}\mathbf{i}-\begin{vmatrix}3 & 2\\ -1 & 0\end{vmatrix}\mathbf{j}+\begin{vmatrix}3 & 1\\ -1 & 1\end{vmatrix}\mathbf{k}=-2\mathbf{i}-2\mathbf{j}+4\mathbf{k}$ so

$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -2 & -2 & 4\\ 0 & 0 & -4\end{vmatrix}=\begin{vmatrix}-2 & 4\\ 0 & -4\end{vmatrix}\mathbf{i}-\begin{vmatrix}-2 & 4\\ 0 & -4\end{vmatrix}\mathbf{j}+\begin{vmatrix}-2 & -2\\ 0 & 0\end{vmatrix}\mathbf{k}=8\mathbf{i}-8\mathbf{j}$.

Thus $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\neq(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.

15. We know that the cross product of two vectors is orthogonal to both. So we calculate

$\langle 1,-1,1\rangle\times\langle 0,4,4\rangle=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -1 & 1\\ 0 & 4 & 4\end{vmatrix}=\begin{vmatrix}-1 & 1\\ 4 & 4\end{vmatrix}\mathbf{i}-\begin{vmatrix}1 & 1\\ 0 & 4\end{vmatrix}\mathbf{j}+\begin{vmatrix}1 & -1\\ 0 & 4\end{vmatrix}\mathbf{k}=-8\mathbf{i}-4\mathbf{j}+4\mathbf{k}$.

So two unit vectors orthogonal to both are $\pm\dfrac{\langle -8,-4,4\rangle}{\sqrt{64+16+16}}=\pm\dfrac{\langle -8,-4,4\rangle}{4\sqrt{6}}$, that is,

$\left\langle -\dfrac{2}{\sqrt{6}},-\dfrac{1}{\sqrt{6}},\dfrac{1}{\sqrt{6}}\right\rangle$ and $\left\langle \dfrac{2}{\sqrt{6}},\dfrac{1}{\sqrt{6}},-\dfrac{1}{\sqrt{6}}\right\rangle$.

16. We know that the cross product of two vectors is orthogonal to both. So we calculate

$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 1\\ 2 & 0 & 1\end{vmatrix}=\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1 & 1\\ 2 & 1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1 & 1\\ 2 & 0\end{vmatrix}\mathbf{k}=\mathbf{i}+\mathbf{j}-2\mathbf{k}$. Thus, two unit vectors orthogonal to both are $\pm\dfrac{1}{\sqrt{6}}\langle 1,1,-2\rangle$, that is, $\left\langle \dfrac{1}{\sqrt{6}},\dfrac{1}{\sqrt{6}},-\dfrac{2}{\sqrt{6}}\right\rangle$ and $\left\langle -\dfrac{1}{\sqrt{6}},-\dfrac{1}{\sqrt{6}},\dfrac{2}{\sqrt{6}}\right\rangle$.

17. Let $\boldsymbol{a}=\langle a_1,a_2,a_3\rangle$. Then

$$\mathbf{0}\times \boldsymbol{a}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ a_2 & a_3 \end{vmatrix}\boldsymbol{i}-\begin{vmatrix} 0 & 0 \\ a_1 & a_3 \end{vmatrix}\boldsymbol{j}+\begin{vmatrix} 0 & 0 \\ a_1 & a_2 \end{vmatrix}\boldsymbol{k}=\mathbf{0},$$

$$\boldsymbol{a}\times \mathbf{0}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ 0 & 0 \end{vmatrix}\boldsymbol{i}-\begin{vmatrix} a_1 & a_3 \\ 0 & 0 \end{vmatrix}\boldsymbol{j}+\begin{vmatrix} a_1 & a_2 \\ 0 & 0 \end{vmatrix}\boldsymbol{k}=\mathbf{0}.$$

18. Let $\boldsymbol{a}=\langle a_1,a_2,a_3\rangle$ and $\boldsymbol{b}=\langle b_1,b_2,b_3\rangle$.

$$(\boldsymbol{a}\times \boldsymbol{b})\cdot \boldsymbol{b} = \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle \cdot \langle b_1,b_2,b_3\rangle$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} b_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} b_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} b_3$$

$$=\left(a_2b_3b_1-a_3b_2b_1\right)-\left(a_1b_3b_2-a_3b_1b_2\right)+\left(a_1b_2b_3-a_2b_1b_3\right)=0$$

19.

$$\boldsymbol{a}\times \boldsymbol{b} =\langle a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1\rangle$$

$$=\left\langle (-1)\left(b_2a_3-b_3a_2\right),(-1)\left(b_3a_1-b_1a_3\right),(-1)\left(b_1a_2-b_2a_1\right)\right\rangle$$

$$=-\langle b_2a_3-b_3a_2,b_3a_1-b_1a_3,b_1a_2-b_2a_1\rangle=-\boldsymbol{b}\times \boldsymbol{a}$$

20. $c\boldsymbol{a}=\langle ca_1,ca_2,ca_3\rangle$, so

$$(c\,\boldsymbol{a})\times \boldsymbol{b} =\langle ca_2b_3-ca_3b_2,ca_3b_1-ca_1b_3,ca_1b_2-ca_2b_1\rangle$$

$$=\langle a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1\rangle=c(\boldsymbol{a}\times \boldsymbol{b})$$

$$=\langle ca_2b_3-ca_3b_2,ca_3b_1-ca_1b_3,ca_1b_2-ca_2b_1\rangle$$

$$=\langle a_2(cb_3)-a_3(cb_2),a_3(cb_1)-a_1(cb_3),a_1(cb_2)-a_2(cb_1)\rangle$$
$$=\mathbf{a}\times c\,\mathbf{b}$$

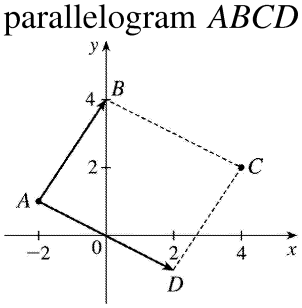
21.

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\langle b_1+c_1,b_2+c_2,b_3+c_3\rangle$$
$$=\langle a_2(b_3+c_3)-a_3(b_2+c_2),a_3(b_1+c_1)-a_1(b_3+c_3),a_1(b_2+c_2)-a_2(b_1+c_1)\rangle$$
$$=\langle a_2b_3+a_2c_3-a_3b_2-a_3c_2,a_3b_1+a_3c_1-a_1b_3-a_1c_3,a_1b_2+a_1c_2-a_2b_1-a_2c_1\rangle$$
$$=\langle (a_2b_3-a_3b_2)+(a_2c_3-a_3c_2),(a_3b_1-a_1b_3)+(a_3c_1-a_1c_3),$$
$$(a_1b_2-a_2b_1)+(a_1c_2-a_2c_1)\rangle$$
$$=\langle a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1\rangle+\langle a_2c_3-a_3c_2,a_3c_1-a_1c_3,a_1c_2-a_2c_1\rangle$$
$$=(\mathbf{a}\times\mathbf{b})+(\mathbf{a}\times\mathbf{c})$$

22.

$(\mathbf{a}+\mathbf{b})\times\mathbf{c}=-\mathbf{c}\times(\mathbf{a}+\mathbf{b})$	by Property 1 of Theorem 8
$=-(\mathbf{c}\times\mathbf{a}+\mathbf{c}\times\mathbf{b})$	by Property 3 of Theorem 8
$=-(-\mathbf{a}\times\mathbf{c}+(-\mathbf{b}\times\mathbf{c}))$	by Property 1 of Theorem 8
$=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$	by Property 2 of Theorem 8

23. By plotting the vertices, we can see that the parallelogram is determined by the vectors $\overrightarrow{AB}=\langle 2,3\rangle$ and $\overrightarrow{AD}=\langle 4,-2\rangle$. We know that the area of the parallelogram determined by two vectors is equal to the length of the cross product of these vectors. In order to compute the cross product, we consider the vector $\overrightarrow{AB}$ as the three–dimensional vector $\langle 2,3,0\rangle$ (and similarly for $\overrightarrow{AD}$) , and then the area of parallelogram $ABCD$ is



$$\left|\overrightarrow{AB}\times\overrightarrow{AD}\right| = \left| \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 3 & 0 \\ 4 & -2 & 0 \end{vmatrix} \right| = |(0)\boldsymbol{i}-(0)\boldsymbol{j}+(-4-12)\boldsymbol{k}| = |-16\boldsymbol{k}| = 16$$

24. The parallelogram is determined by the vectors $\overrightarrow{KL}=\langle 0,1,3\rangle$ and $\overrightarrow{KN}=\langle 2,5,0\rangle$, so the area of parallelogram *KLMN* is

$$\left|\overrightarrow{KL}\times\overrightarrow{KN}\right| = \left| \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 1 & 3 \\ 2 & 5 & 0 \end{vmatrix} \right| = |(-15)\boldsymbol{i}-(-6)\boldsymbol{j}+(-2)\boldsymbol{k}| = |-15\boldsymbol{i}+6\boldsymbol{j}-2\boldsymbol{k}| = \sqrt{265} \approx 16.28$$

25. **(a)** Because the plane through P , Q , and R contains the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, a vector orthogonal to both of these vectors (such as their cross product) is also orthogonal to the plane. Here $\overrightarrow{PQ}=\langle -1,2,0\rangle$ and $\overrightarrow{PR}=\langle -1,0,3\rangle$, so

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\langle (2)(3)-(0)(0),(0)(-1)-(-1)(3),(-1)(0)-(2)(-1)\rangle=\langle 6,3,2\rangle$$

Therefore, $\langle 6,3,2\rangle$ (or any scalar multiple thereof) is orthogonal to the plane through P , Q , and R .
(b) Note that the area of the triangle determined by P , Q , and R is equal to half of the area of the parallelogram determined by the three points. From part (a), the area of the parallelogram is $\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = |\langle 6,3,2\rangle| = \sqrt{36+9+4} = 7$, so the area of the triangle is $\frac{1}{2}(7)=\frac{7}{2}$.

26. **(a)** $\overrightarrow{PQ}=\langle -3,2,-1\rangle$ and $\overrightarrow{PR}=\langle 1,-1,1\rangle$, so a vector orthogonal to the plane through P , Q , and R is $\overrightarrow{PQ}\times\overrightarrow{PR}=\langle (2)(1)-(-1)(-1),(-1)(1)-(-3)(1),(-3)(-1)-(2)(1)\rangle=\langle 1,2,1\rangle$
(or any scalar mutiple thereof).

(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = |\langle 1,2,1\rangle| = \sqrt{1^2+2^2+1^2} = \sqrt{6}$, so the area of triangle *PQR* is $\frac{1}{2}\sqrt{6}$.

27. **(a)** $\overrightarrow{PQ}=\langle 4,3,-2\rangle$ and $\overrightarrow{PR}=\langle 5,5,1\rangle$, so a vector orthogonal to the plane through P , Q , and R is $\overrightarrow{PQ}\times\overrightarrow{PR}=\langle (3)(1)-(-2)(5),(-2)(5)-(4)(1),(4)(5)-(3)(5)\rangle=\langle 13,-14,5\rangle$
(or any scalar mutiple thereof).
(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is
$\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = |\langle 13,-14,5\rangle| = \sqrt{13^2+(-14)^2+5^2} = \sqrt{390}$, so the area of triangle *PQR* is $\frac{1}{2}\sqrt{390}$.

28. **(a)** $\overrightarrow{PQ}=\langle 1,1,3\rangle$ and $\overrightarrow{PR}=\langle 3,2,5\rangle$, so a vector orthogonal to the plane through P , Q , and R is

$\overrightarrow{PQ}\times\overrightarrow{PR}=\langle(1)(5)-(3)(2),(3)(3)-(1)(5),(1)(2)-(1)(3)\rangle=\langle-1,4,-1\rangle$ (or any scalar multiple thereof).

(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is

$\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|=\left|\langle-1,4,-1\rangle\right|=\sqrt{1+16+1}=\sqrt{18}=3\sqrt{2}$, so the area of triangle PQR is $\frac{1}{2}\cdot3\sqrt{2}=\frac{3}{2}\sqrt{2}$.

29. We know that the volume of the parallelepiped determined by $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ is the magnitude of their scalar triple product, which is

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix}6&3&-1\\0&1&2\\4&-2&5\end{vmatrix}=6\begin{vmatrix}1&2\\-2&5\end{vmatrix}-3\begin{vmatrix}0&2\\4&5\end{vmatrix}+(-1)\begin{vmatrix}0&1\\4&-2\end{vmatrix}$$

$$=6(5+4)-3(0-8)-(0-4)=82$$

Thus the volume of the parallelepiped is 82 cubic units.

30.

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix}1&1&-1\\1&-1&1\\-1&1&1\end{vmatrix}=1\begin{vmatrix}-1&1\\1&1\end{vmatrix}-1\begin{vmatrix}1&1\\-1&1\end{vmatrix}+(-1)\begin{vmatrix}1&-1\\-1&1\end{vmatrix}=-2-2+0=-4$$

.

So the volume of the parallelepiped determined by $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ is $|-4|=4$ cubic units.

31. $\boldsymbol{a}=\overrightarrow{PQ}=\langle2,1,1\rangle$, $\boldsymbol{b}=\overrightarrow{PR}=\langle1,-1,2\rangle$, and $\boldsymbol{c}=\overrightarrow{PS}=\langle0,-2,3\rangle$.

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix}2&1&1\\1&-1&2\\0&-2&3\end{vmatrix}=2\begin{vmatrix}-1&2\\-2&3\end{vmatrix}-1\begin{vmatrix}1&2\\0&3\end{vmatrix}+1\begin{vmatrix}1&-1\\0&-2\end{vmatrix}=2-3-2=-3\ ,$$

so the volume of the parallelepiped is 3 cubic units.

32. $\boldsymbol{a}=\overrightarrow{PQ}=\langle2,3,3\rangle$, $\boldsymbol{b}=\overrightarrow{PR}=\langle-1,-1,-1\rangle$ and $\boldsymbol{c}=\overrightarrow{PS}=\langle6,-2,2\rangle$.

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix}2&3&3\\-1&-1&-1\\6&-2&2\end{vmatrix}=2\begin{vmatrix}-1&-1\\-2&2\end{vmatrix}-3\begin{vmatrix}-1&-1\\6&2\end{vmatrix}+3\begin{vmatrix}-1&-1\\6&-2\end{vmatrix}=-8-12+24=4$$

,

so the volume of the parallelepiped is 4 cubic units.

33. $$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix}2&3&1\\1&-1&0\\7&3&2\end{vmatrix}=2\begin{vmatrix}-1&0\\3&2\end{vmatrix}-3\begin{vmatrix}1&0\\7&2\end{vmatrix}+1\begin{vmatrix}1&-1\\7&3\end{vmatrix}=-4-6+10=0\ ,$$

which says that the volume of the parallelepiped determined by $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is 0 , and thus these three vectors are coplanar.

34. $\boldsymbol{a}=\overrightarrow{PQ}=\langle1,4,5\rangle$, $\boldsymbol{b}=\overrightarrow{PR}=\langle2,-1,1\rangle$ and $\boldsymbol{c}=\overrightarrow{PS}=\langle5,2,7\rangle$.

$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix}1&4&5\\2&-1&1\\5&2&7\end{vmatrix}=1\begin{vmatrix}-1&1\\2&7\end{vmatrix}-4\begin{vmatrix}2&1\\5&7\end{vmatrix}+5\begin{vmatrix}2&-1\\5&2\end{vmatrix}=-9-36+45=0$,
so the volume of the parallelepiped determined by $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is 0 , which says that these vectors lie in the same plane. Therefore, their initial and terminal points P , Q , R and S also lie in the same plane.

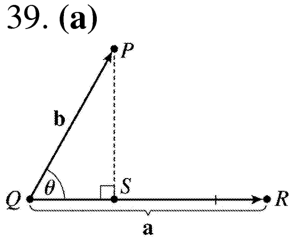
35. The magnitude of the torque is
$|\boldsymbol{\tau}|=|\boldsymbol{r}\times\boldsymbol{F}|=|\boldsymbol{r}|\,|\boldsymbol{F}|\sin\theta=(0.18\text{m})(60\text{N})\sin(70+10)^\circ=10.8\sin 80^\circ\approx 10.6$ J.

36. $|\boldsymbol{r}|=\sqrt{4^2+4^2}=4\sqrt{2}$ ft. A line drawn from the point P to the point of application of the force makes an angle of $180^\circ-(45+30)^\circ=105^\circ$ with the force vector. Therefore,
$|\boldsymbol{\tau}|=|\boldsymbol{r}\times\boldsymbol{F}|=|\boldsymbol{r}|\,|\boldsymbol{F}|\sin\theta=(4\sqrt{2})(36)\sin 105^\circ\approx 197$ ft–lb.

37. Using the notation of the text, $\boldsymbol{r}=\langle 0,0.3,0\rangle$ and $\boldsymbol{F}$ has direction $\langle 0,3,-4\rangle$. The angle θ between them can be determined by $\cos\theta=\dfrac{\langle 0,0.3,0\rangle\cdot\langle 0,3,-4\rangle}{|\langle 0,0.3,0\rangle|\,|\langle 0,3,-4\rangle|}\Rightarrow\cos\theta=\dfrac{0.9}{(0.3)(5)}\Rightarrow\cos\theta=0.6\Rightarrow$
$\theta\approx 53.1^\circ$. Then $|\boldsymbol{\tau}|=|\boldsymbol{r}|\,|\boldsymbol{F}|\sin\theta\Rightarrow 100=0.3|\boldsymbol{F}|\sin 53.1^\circ\Rightarrow|\boldsymbol{F}|\approx 417$ N.

38. Since $|\boldsymbol{u}\times\boldsymbol{v}|=|\boldsymbol{u}|\,|\boldsymbol{v}|\sin\theta$, $0\le\theta\le\pi$, $|\boldsymbol{u}\times\boldsymbol{v}|$ achieves its maximum value for $\sin\theta=1\Rightarrow\theta=\dfrac{\pi}{2}$, in which case $|\boldsymbol{u}\times\boldsymbol{v}|=|\boldsymbol{u}|\,|\boldsymbol{v}|=15$. The minimum value is zero, which occurs when $\sin\theta=0\Rightarrow\theta=0$ or π , so when $\boldsymbol{u}$, $\boldsymbol{v}$ are parallel. Thus, when $\boldsymbol{u}$ points in the same direction as $\boldsymbol{v}$, so $\boldsymbol{u}=3\boldsymbol{j}$, $|\boldsymbol{u}\times\boldsymbol{v}|=0$. As $\boldsymbol{u}$ rotates counterclockwise, $\boldsymbol{u}\times\boldsymbol{v}$ is directed in the negative z –direction (by the right–hand rule) and the length increases until $\theta=\dfrac{\pi}{2}$, in which case $\boldsymbol{u}=-3\boldsymbol{i}$ and $|\boldsymbol{u}\times\boldsymbol{v}|=15$. As $\boldsymbol{u}$ rotates to the negative y –axis, $\boldsymbol{u}\times\boldsymbol{v}$ remains pointed in the negative z –direction and the length of $\boldsymbol{u}\times\boldsymbol{v}$ decreases to 0, after which the direction of $\boldsymbol{u}\times\boldsymbol{v}$ reverses to point in the positive z –direction and $|\boldsymbol{u}\times\boldsymbol{v}|$ increases. When $\boldsymbol{u}=3\boldsymbol{i}$ (so $\theta=\dfrac{\pi}{2}$), $|\boldsymbol{u}\times\boldsymbol{v}|$ again reaches its maximum of 15 , after which $|\boldsymbol{u}\times\boldsymbol{v}|$ decreases to 0 as $\boldsymbol{u}$ rotates to the positive y –axis.

39. **(a)**

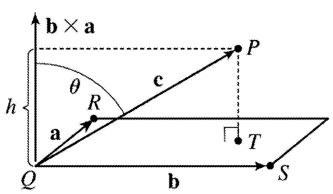


The distance between a point and a line is the length of the perpendicular from the point to the line, here $\left|\overrightarrow{PS}\right|=d$. But referring to triangle PQS , $d=\left|\overrightarrow{PS}\right|=\left|\overrightarrow{QP}\right|\sin\theta=|\boldsymbol{b}|\sin\theta$. But θ is the angle between $\overrightarrow{QP}=\boldsymbol{b}$ and $\overrightarrow{QR}=\boldsymbol{a}$. Thus by Theorem 6, $\sin\theta=\dfrac{|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$ and so

$d=|\boldsymbol{b}|\sin\theta=\dfrac{|\boldsymbol{b}|\,|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{a}|\,|\boldsymbol{b}|}=\dfrac{|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{a}|}$.

(b) $\boldsymbol{a}=\overrightarrow{QR}=\langle -1,-2,-1\rangle$ and $\boldsymbol{b}=\overrightarrow{QP}=\langle 1,-5,-7\rangle$. Then
$\boldsymbol{a}\times\boldsymbol{b}=\langle(-2)(-7)-(-1)(-5),(-1)(1)-(-1)(-7),(-1)(-5)-(-2)(1)\rangle=\langle 9,-8,7\rangle$. Thus the distance is
$d=\dfrac{|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{a}|}=\dfrac{1}{\sqrt{6}}\sqrt{81+64+49}=\sqrt{\dfrac{194}{6}}=\sqrt{\dfrac{97}{3}}$.

40. **(a)** The distance between a point and a plane is the length of the perpendicular from the point to the plane, here $\left|\overrightarrow{TP}\right|=d$. But $\overrightarrow{TP}$ is parallel to $\boldsymbol{b}\times\boldsymbol{a}$ (because $\boldsymbol{b}\times\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$ and $\boldsymbol{a}$) and $d=\left|\overrightarrow{TP}\right|=$ the absolute value of the scalar projection of $\boldsymbol{c}$ along



$\boldsymbol{b}\times\boldsymbol{a}$, which is $|\boldsymbol{c}|\,|\cos\theta|$. (Notice that this is the same setup as the development of the volume of a parallelepiped with $h=|\boldsymbol{c}|\,|\cos\theta|$). Thus $d=|\boldsymbol{c}|\,|\cos\theta|=h=V/A$ where $A=|\boldsymbol{a}\times\boldsymbol{b}|$, the area of the base. So finally $d=\dfrac{V}{A}=\dfrac{|\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})|}{|\boldsymbol{a}\times\boldsymbol{b}|}=\dfrac{|(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}|}{|\boldsymbol{a}\times\boldsymbol{b}|}$ by Theorem 8 # 5.
(b) $\boldsymbol{a}=\overrightarrow{QR}=\langle -1,2,0\rangle$, $\boldsymbol{b}=\overrightarrow{QS}=\langle -1,0,3\rangle$ and $\boldsymbol{c}=\overrightarrow{QP}=\langle 1,1,4\rangle$. Then

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}=\begin{vmatrix}-1&2&0\\-1&0&3\\1&1&4\end{vmatrix}=(-1)\begin{vmatrix}0&3\\1&4\end{vmatrix}-2\begin{vmatrix}-1&3\\1&4\end{vmatrix}+0=17$$

and $\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\-1&2&0\\-1&0&3\end{vmatrix}=\begin{vmatrix}2&0\\0&3\end{vmatrix}\boldsymbol{i}-\begin{vmatrix}-1&0\\-1&3\end{vmatrix}\boldsymbol{j}+\begin{vmatrix}-1&2\\-1&0\end{vmatrix}\boldsymbol{k}=6\boldsymbol{i}+3\boldsymbol{j}+2\boldsymbol{k}$

Thus $d=\dfrac{|(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}|}{|\boldsymbol{a}\times\boldsymbol{b}|}=\dfrac{17}{\sqrt{36+9+4}}=\dfrac{17}{7}$.

41.

$(\boldsymbol{a}-\boldsymbol{b})\times(\boldsymbol{a}+\boldsymbol{b})$	$=(\boldsymbol{a}-\boldsymbol{b})\times\boldsymbol{a}+(\boldsymbol{a}-\boldsymbol{b})\times\boldsymbol{b}$	by Theorem 8 # 3
	$=\boldsymbol{a}\times\boldsymbol{a}+(-\boldsymbol{b})\times\boldsymbol{a}+\boldsymbol{a}\times\boldsymbol{b}+(-\boldsymbol{b})\times\boldsymbol{b}$	by Theorem 8 # 4
	$=(\boldsymbol{a}\times\boldsymbol{a})-(\boldsymbol{b}\times\boldsymbol{a})+(\boldsymbol{a}\times\boldsymbol{b})-(\boldsymbol{b}\times\boldsymbol{b})$	by Theorem 8 # 2 (with $c=-1$)
	$=\mathbf{0}-(\boldsymbol{b}\times\boldsymbol{a})+(\boldsymbol{a}\times\boldsymbol{b})-\mathbf{0}$	by Example 2
	$=(\boldsymbol{a}\times\boldsymbol{b})+(\boldsymbol{a}\times\boldsymbol{b})$	by Theorem 8 # 1
	$=2(\boldsymbol{a}\times\boldsymbol{b})$	

42. Let $\boldsymbol{a}=\left\langle a_1,a_2,a_3\right\rangle$, $\boldsymbol{b}=\left\langle b_1,b_2,b_3\right\rangle$ and $\boldsymbol{c}=\left\langle c_1,c_2,c_3\right\rangle$, so $\boldsymbol{b}\times\boldsymbol{c}=\left\langle b_2c_3-b_3c_2,b_3c_1-b_1c_3,b_1c_2-b_2c_1\right\rangle$ and

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) =\langle\ a_2\left(b_1c_2-b_2c_1\right)-a_3\left(b_3c_1-b_1c_3\right),a_3\left(b_2c_3-b_3c_2\right)-a_1\left(b_1c_2-b_2c_1\right),$$
$$a_1\left(b_3c_1-b_1c_3\right)-a_2\left(b_2c_3-b_3c_2\right)\rangle$$
$$=\langle\ a_2b_1c_2-a_2b_2c_1-a_3b_3c_1+a_3b_1c_3,a_3b_2c_3-a_3b_3c_2-a_1b_1c_2+a_1b_2c_1,$$
$$a_1b_3c_1-a_1b_1c_3-a_2b_2c_3+a_2b_3c_2\rangle$$
$$=\langle\ \left(a_2c_2+a_3c_3\right)b_1-\left(a_2b_2+a_3b_3\right)c_1,\left(a_1c_1+a_3c_3\right)b_2-\left(a_1b_1+a_3b_3\right)c_2,$$
$$\left(a_1c_1+a_2c_2\right)b_3-\left(a_1b_1+a_2b_2\right)c_3\rangle$$

$$(\ *\ )\quad =\langle\ \left(a_2c_2+a_3c_3\right)b_1-\left(a_2b_2+a_3b_3\right)c_1+a_1b_1c_1-a_1b_1c_1,$$
$$\left(a_1c_1+a_3c_3\right)b_2-\left(a_1b_1+a_3b_3\right)c_2+a_2b_2c_2-a_2b_2c_2,$$
$$\left(a_1c_1+a_2c_2\right)b_3-\left(a_1b_1+a_2b_2\right)c_3+a_3b_3c_3-a_3b_3c_3\rangle$$
$$=\langle\ \left(a_1c_1+a_2c_2+a_3c_3\right)b_1-\left(a_1b_1+a_2b_2+a_3b_3\right)c_1,$$
$$\left(a_1c_1+a_2c_2+a_3c_3\right)b_2-\left(a_1b_1+a_2b_2+a_3b_3\right)c_2,$$
$$\left(a_1c_1+a_2c_2+a_3c_3\right)b_3-\left(a_1b_1+a_2b_2+a_3b_3\right)c_3\rangle$$
$$=\left(a_1c_1+a_2c_2+a_3c_3\right)\left\langle b_1,b_2,b_3\right\rangle-\left(a_1b_1+a_2b_2+a_3b_3\right)\left\langle c_1,c_2,c_3\right\rangle$$
$$=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$$

(*) Here we look ahead to see what terms are still needed to arrive at the desired equation. By adding and subtracting the same terms, we don’t change the value of the component.

43. $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})+\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a})+\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})$
$=[(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}]+[(\boldsymbol{b}\cdot\boldsymbol{a})\boldsymbol{c}-(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}]+[(\boldsymbol{c}\cdot\boldsymbol{b})\boldsymbol{a}-(\boldsymbol{c}\cdot\boldsymbol{a})\boldsymbol{b}]$ by Exercise 42
$=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}+(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}-(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}+(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}-(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}=\boldsymbol{0}$

44. Let $\boldsymbol{c}\times\boldsymbol{d}=\boldsymbol{v}$. Then

$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d})=(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{v}=\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{v})$ by Theorem 8 # 5

$=\boldsymbol{a}\cdot[\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{d})]$

$=\boldsymbol{a}\cdot[(\boldsymbol{b}\cdot\boldsymbol{d})\boldsymbol{c}-(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{d}]$ by Exercise 42

$=(\boldsymbol{b}\cdot\boldsymbol{d})(\boldsymbol{a}\cdot\boldsymbol{c})-(\boldsymbol{b}\cdot\boldsymbol{c})(\boldsymbol{a}\cdot\boldsymbol{d})$ by Properties 3 and 4 of the dot product

$$=\begin{vmatrix} \boldsymbol{a}\cdot\boldsymbol{c} & \boldsymbol{b}\cdot\boldsymbol{c} \\ \boldsymbol{a}\cdot\boldsymbol{d} & \boldsymbol{b}\cdot\boldsymbol{d} \end{vmatrix}$$

45. **(a)** No. If $\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{a}\cdot\boldsymbol{c}$, then $\boldsymbol{a}\cdot(\boldsymbol{b}-\boldsymbol{c})=0$, so $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}-\boldsymbol{c}$, which can happen if $\boldsymbol{b}\neq\boldsymbol{c}$. For example, let $\boldsymbol{a}=\langle 1,1,1\rangle$, $\boldsymbol{b}=\langle 1,0,0\rangle$ and $\boldsymbol{c}=\langle 0,1,0\rangle$.
(b) No. If $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{a}\times\boldsymbol{c}$ then $\boldsymbol{a}\times(\boldsymbol{b}-\boldsymbol{c})=\boldsymbol{0}$, which implies that $\boldsymbol{a}$ is parallel to $\boldsymbol{b}-\boldsymbol{c}$, which of course can happen if $\boldsymbol{b}\neq\boldsymbol{c}$.
(c) Yes. Since $\boldsymbol{a}\cdot\boldsymbol{c}=\boldsymbol{a}\cdot\boldsymbol{b}$, $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}-\boldsymbol{c}$, by part (a). From part (b), $\boldsymbol{a}$ is also parallel to $\boldsymbol{b}-\boldsymbol{c}$. Thus since $\boldsymbol{a}\neq\boldsymbol{0}$ but is both parallel and perpendicular to $\boldsymbol{b}-\boldsymbol{c}$, we have $\boldsymbol{b}-\boldsymbol{c}=\boldsymbol{0}$, so $\boldsymbol{b}=\boldsymbol{c}$.

46. **(a)** $\boldsymbol{k}_i$ is perpendicular to $\boldsymbol{v}_i$ if $i\neq j$ by the definition of $\boldsymbol{k}_i$ and Theorem 5.

(b) $\boldsymbol{k}_1\cdot\boldsymbol{v}_1=\dfrac{\boldsymbol{v}_2\times\boldsymbol{v}_3}{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}\cdot\boldsymbol{v}_1=\dfrac{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}=1$

$\boldsymbol{k}_2\cdot\boldsymbol{v}_2=\dfrac{\boldsymbol{v}_3\times\boldsymbol{v}_1}{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}\cdot\boldsymbol{v}_2=\dfrac{\boldsymbol{v}_2\cdot(\boldsymbol{v}_3\times\boldsymbol{v}_1)}{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}=\dfrac{(\boldsymbol{v}_2\times\boldsymbol{v}_3)\cdot\boldsymbol{v}_1}{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}=1$ [by Theorem 8 # 5]

$\boldsymbol{k}_3\cdot\boldsymbol{v}_3=\dfrac{(\boldsymbol{v}_1\times\boldsymbol{v}_2)\cdot\boldsymbol{v}_3}{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}=\dfrac{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}=1$ [by Theorem 8 # 5]

(c)

$$\boldsymbol{k}_1\cdot(\boldsymbol{k}_2\times\boldsymbol{k}_3)=\boldsymbol{k}_1\cdot\left(\frac{\boldsymbol{v}_3\times\boldsymbol{v}_1}{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}\times\frac{\boldsymbol{v}_1\times\boldsymbol{v}_2}{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}\right)=\frac{\boldsymbol{k}_1}{[\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)]^2}\cdot[(\boldsymbol{v}_3\times\boldsymbol{v}_1)\times(\boldsymbol{v}_1\times\boldsymbol{v}_2)]$$

$$=\frac{\boldsymbol{k}_1}{[\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)]^2}\cdot\left([(\boldsymbol{v}_3\times\boldsymbol{v}_1)\cdot\boldsymbol{v}_2]\boldsymbol{v}_1-[(\boldsymbol{v}_3\times\boldsymbol{v}_1)\cdot\boldsymbol{v}_1]\boldsymbol{v}_2\right)\quad\text{[by Exercise 42]}$$

But $(\boldsymbol{v}_3\times\boldsymbol{v}_1)\cdot\boldsymbol{v}_1=0$ since $\boldsymbol{v}_3\times\boldsymbol{v}_1$ is orthogonal to $\boldsymbol{v}_1$, and

$\left(\boldsymbol{v}_3 \times \boldsymbol{v}_1\right) \cdot \boldsymbol{v}_2 = \boldsymbol{v}_2 \cdot \left(\boldsymbol{v}_3 \times \boldsymbol{v}_1\right) = \left(\boldsymbol{v}_2 \times \boldsymbol{v}_3\right) \cdot \boldsymbol{v}_1 = \boldsymbol{v}_1 \cdot \left(\boldsymbol{v}_2 \times \boldsymbol{v}_3\right)$. Thus

$$\boldsymbol{k}_1 \cdot \left(\boldsymbol{k}_2 \times \boldsymbol{k}_3\right) = \frac{\boldsymbol{k}_1}{\left[\boldsymbol{v}_1 \cdot \left(\boldsymbol{v}_2 \times \boldsymbol{v}_3\right)\right]^2} \cdot \left[\boldsymbol{v}_1 \cdot \left(\boldsymbol{v}_2 \times \boldsymbol{v}_3\right)\right] \boldsymbol{v}_1 = \frac{\boldsymbol{k}_1 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \left(\boldsymbol{v}_2 \times \boldsymbol{v}_3\right)}$$

$$= \frac{1}{\boldsymbol{v}_1 \cdot \left(\boldsymbol{v}_2 \times \boldsymbol{v}_3\right)} \qquad \text{[by part (b)]}$$

1. **(a)** True; each of the first two lines has a direction vector parallel to the direction vector of the third line, so these vectors are each scalar multiples of the third direction vector. Then the first two direction vectors are also scalar multiples of each other, so these vectors, and hence the two lines, are parallel.
(b) False; for example, the x – and y –axes are both perpendicular to the z –axis, yet the x – and y – axes are not parallel.
(c) True; each of the first two planes has a normal vector parallel to the normal vector of the third plane, so these two normal vectors are parallel to each other and the planes are parallel.
(d) False; for example, the xy – and yz –planes are not parallel, yet they are both perpendicular to the xz –plane.
(e) False; the x – and y –axes are not parallel, yet they are both parallel to the plane $z=1$.
(f) True; if each line is perpendicular to a plane, then the lines' direction vectors are both parallel to a normal vector for the plane. Thus, the direction vectors are parallel to each other and the lines are parallel.
(g) False; the planes $y=1$ and $z=1$ are not parallel, yet they are both parallel to the x –axis.
(h) True; if each plane is perpendicular to a line, then any normal vector for each plane is parallel to a direction vector for the line. Thus, the normal vectors are parallel to each other and the planes are parallel.
(i) True; see Figure 9 and the accompanying discussion.
(j) False; they can be skew, as in Example 3.
(k) True. Consider any normal vector for the plane and any direction vector for the line. If the normal vector is perpendicular to the direction vector, the line and plane are parallel. Otherwise, the vectors meet at an angle θ , $0° \le \theta < 90°$, and the line will intersect the plane at an angle $90° - \theta$.

2. For this line, we have $\mathbf{r}_0 = \mathbf{i} - 3\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, so a vector equation is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = (\mathbf{i} - 3\mathbf{k}) + t(2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}) = (1+2t)\mathbf{i} - 4t\mathbf{j} + (-3+5t)\mathbf{k}$ and parametric equations are $x=1+2t$, $y=-4t$, $z=-3+5t$.

3. For this line, we have $\mathbf{r}_0 = -2\mathbf{i} + 4\mathbf{j} + 10\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 8\mathbf{k}$, so a vector equation is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = (-2\mathbf{i} + 4\mathbf{j} + 10\mathbf{k}) + t(3\mathbf{i} + \mathbf{j} - 8\mathbf{k}) = (-2+3t)\mathbf{i} + (4+t)\mathbf{j} + (10-8t)\mathbf{k}$ and parametric equations are $x=-2+3t$, $y=4+t$, $z=10-8t$.
4. This line has the same direction as the given line, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Here $\mathbf{r}_0 = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$, so a vector equation is $\mathbf{r} = (0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) + t(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 2t\mathbf{i} - t\mathbf{j} + 3t\mathbf{k}$ and parametric equations are $x=2t$, $y=-t$, $z=3t$.

5. A line perpendicular to the given plane has the same direction as a normal vector to the plane, such as $\mathbf{n} = \langle 1,3,1 \rangle$. So $\mathbf{r}_0 = \mathbf{i} + 6\mathbf{k}$, and we can take $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$. Then a vector equation is $\mathbf{r} = (\mathbf{i} + 6\mathbf{k}) + t(\mathbf{i} + 3\mathbf{j} + \mathbf{k}) = (1+t)\mathbf{i} + 3t\mathbf{j} + (6+t)\mathbf{k}$, and parametric equations are $x=1+t$, $y=3t$, $z=6+t$.
6. The vector $\mathbf{v} = \langle 1-0, 2-0, 3-0 \rangle = \langle 1,2,3 \rangle$ is parallel to the line. Letting $P_0 = (0,0,0)$, parametric

equations are $x=0+1\cdot t=t$, $y=0+2\cdot t=2t$, $z=0+3\cdot t=3t$, while symmetric equations are $x=\frac{y}{2}=\frac{z}{3}$.

7. The vector $\boldsymbol{v}=\langle -4-1,3-3,0-2\rangle=\langle -5,0,-2\rangle$ is parallel to the line. Letting $P_0=(1,3,2)$, parametric equations are $x=1-5t$, $y=3+0t=3$, $z=2-2t$, while symmetric equations are $\frac{x-1}{-5}=\frac{z-2}{-2}$,$y=3$. Notice here that the direction number $b=0$, so rather than writing $\frac{y-3}{0}$ in the symmetric equation we must write the equation $y=3$ separately.

8. $\boldsymbol{v}=\langle 2-6,4-1,5-(-3)\rangle=\langle -4,3,8\rangle$, and letting $P_0=(6,1,-3)$, parametric equations are $x=6-4t$, $y=1+3t$, $z=-3+8t$, while symmetric equations are $\frac{x-6}{-4}=\frac{y-1}{3}=\frac{z+3}{8}$.

9. $\boldsymbol{v}=\left\langle 2-0,1-\frac{1}{2},-3-1\right\rangle=\left\langle 2,\frac{1}{2},-4\right\rangle$, and letting $P_0=(2,1,-3)$, parametric equations are $x=2+2t$, $y=1+\frac{1}{2}t$, $z=-3-4t$, while symmetric equations are $\frac{x-2}{2}=\frac{y-1}{1/2}=\frac{z+3}{-4}$ or $\frac{x-2}{2}=2y-2=\frac{z+3}{-4}$.

10. $\boldsymbol{v}=(\boldsymbol{i}+\boldsymbol{j})\times(\boldsymbol{j}+\boldsymbol{k})=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}=\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}$ is the direction of the line perpendicular to both $\boldsymbol{i}+\boldsymbol{j}$ and $\boldsymbol{j}+\boldsymbol{k}$. With $P_0=(2,1,0)$, parametric equations are $x=2+t$, $y=1-t$, $z=t$ and symmetric equations are $x-2=\frac{y-1}{-1}=z$ or $x-2=1-y=z$.

11. The line has direction $\boldsymbol{v}=\langle 1,2,1\rangle$. Letting $P_0=(1,-1,1)$, parametric equations are $x=1+t$, $y=-1+2t$, $z=1+t$ and symmetric equations are $x-1=\frac{y+1}{2}=z-1$.

12. Setting $x=0$, we see that $(0,1,0)$ satisfies the equations of both planes, so they do in fact have a line of intersection. $\boldsymbol{v}=\boldsymbol{n}_1\times\boldsymbol{n}_2=\langle 1,1,1\rangle\times\langle 1,0,1\rangle=\langle 1,0,-1\rangle$ is the direction of this line. Taking the point $(0,1,0)$ as P_0 , parametric equations are $x=t$, $y=1$, $z=-t$, and symmetric equations are $x=-z$, $y=1$.

13. Direction vectors of the lines are $\boldsymbol{v}_1=\langle -2-(-4),0-(-6),-3-1\rangle=\langle 2,6,-4\rangle$ and $\boldsymbol{v}_2=\langle 5-10,3-18,14-4\rangle=\langle -5,-15,10\rangle$, and since $\boldsymbol{v}_2=-\frac{5}{2}\boldsymbol{v}_1$, the direction vectors and thus the lines are

parallel.

14. Direction vectors of the lines are $\boldsymbol{v}_1=\langle -2,4,4\rangle$ and $\boldsymbol{v}_2=\langle 8,-1,4\rangle$. Since $\boldsymbol{v}_1\cdot \boldsymbol{v}_2=-16-4+16\neq 0$, the vectors and thus the lines are not perpendicular.

15. **(a)** A direction vector of the line with parametric equations $x=1+2t$, $y=3t$, $z=5-7t$ is $\boldsymbol{v}=\langle 2,3,-7\rangle$ and the desired parallel line must also have $\boldsymbol{v}$ as a direction vector. Here $P_0=(0,2,-1)$, so symmetric equations for the line are $\frac{x}{2}=\frac{y-2}{3}=\frac{z+1}{-7}$.

(b) The line intersects the xy –plane when $z=0$, so we need $\frac{x}{2}=\frac{y-2}{3}=\frac{1}{-7}$ or $x=-\frac{2}{7}$, $y=\frac{11}{7}$. Thus the point of intersection with the xy –plane is $\left(-\frac{2}{7},\frac{11}{7},0\right)$. Similarly for the yz –plane, we need $x=0\Leftrightarrow 0=\frac{y-2}{3}=\frac{z+1}{-7}\Leftrightarrow y=2$, $z=-1$. Thus the line intersects the yz –plane at $(0,2,-1)$. For the xz –plane, we need $y=0\Leftrightarrow \frac{x}{2}=-\frac{2}{3}=\frac{z+1}{-7}\Leftrightarrow x=-\frac{4}{3}$, $z=\frac{11}{3}$. So the line intersects the xz –plane at $\left(-\frac{4}{3},0,\frac{11}{3}\right)$.

16. **(a)** A vector normal to the plane $2x-y+z=1$ is $\boldsymbol{n}=\langle 2,-1,1\rangle$, and since the line is to be perpendicular to the plane, $\boldsymbol{n}$ is also a direction vector for the line. Thus parametric equations of the line are $x=5+2t$, $y=1-t$, $z=t$.

(b) On the xy –plane, $z=0$. So $z=t=0$ in the parametric equations of the line, and therefore $x=5$ and $y=1$, giving the point of intersection $(5,1,0)$. For the yz –plane, $x=0$ which implies $t=-\frac{5}{2}$, so $y=\frac{7}{2}$ and $z=-\frac{5}{2}$ and the point is $\left(0,\frac{7}{2},-\frac{5}{2}\right)$. For the xz –plane, $y=0$ which implies $t=1$, so $x=7$ and $z=1$ and the point of intersection is $(7,0,1)$.

17. From Equation 4, the line segment from $\boldsymbol{r}_0=2\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k}$ to $\boldsymbol{r}_1=4\boldsymbol{i}+6\boldsymbol{j}+\boldsymbol{k}$ is

$\boldsymbol{r}(t)=(1-t)\boldsymbol{r}_0+t\boldsymbol{r}_1=(1-t)(2\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k})+t(4\boldsymbol{i}+6\boldsymbol{j}+\boldsymbol{k})=(2\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k})+t(2\boldsymbol{i}+7\boldsymbol{j}-3\boldsymbol{k})$, $0\leq t\leq 1$.

18. From Equation 4, the line segment from $\boldsymbol{r}_0=10\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$ to $\boldsymbol{r}_1=5\boldsymbol{i}+6\boldsymbol{j}-3\boldsymbol{k}$ is

$$\begin{aligned}\boldsymbol{r}(t)&=(1-t)\boldsymbol{r}_0+t\boldsymbol{r}_1=(1-t)(10\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k})+t(5\boldsymbol{i}+6\boldsymbol{j}-3\boldsymbol{k})\\&=(10\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k})+t(-5\boldsymbol{i}+3\boldsymbol{j}-4\boldsymbol{k})\ ,\ 0\leq t\leq 1\end{aligned}$$

The corresponding parametric equations are $x=10-5t$, $y=3+3t$, $z=1-4t$, $0\leq t\leq 1$.

19. Since the direction vectors are $\boldsymbol{v}_1=\langle -6,9,-3\rangle$ and $\boldsymbol{v}_2=\langle 2,-3,1\rangle$, we have $\boldsymbol{v}_1=-3\boldsymbol{v}_2$ so the lines are parallel.

20. The lines aren't parallel since the direction vectors $\langle 2,3,-1\rangle$ and $\langle 1,1,3\rangle$ aren't parallel. For the lines to intersect we must be able to find one value of t and one value of s that produce the same point from the respective parametric equations. Thus we need to satisfy the following three equations: $1+2t=-1+s$, $3t=4+s$, $2-t=1+3s$. Solving the first two equations we get $t=6$, $s=14$ and checking, we see that these values don't satisfy the third equation. Thus L_1 and L_2 aren't parallel and don't intersect, so they must be skew lines.

21. Since the direction vectors $\langle 1,2,3\rangle$ and $\langle -4,-3,2\rangle$ are not scalar multiples of each other, the lines are not parallel, so we check to see if the lines intersect. The parametric equations of the lines are L_1 : $x=t$, $y=1+2t$, $z=2+3t$ and L_2 : $x=3-4s$, $y=2-3s$, $z=1+2s$. For the lines to intersect, we must be able to find one value of t and one value of s that produce the same point from the respective parametric equations. Thus we need to satisfy the following three equations: $t=3-4s$, $1+2t=2-3s$, $2+3t=1+2s$. Solving the first two equations we get $t=-1$, $s=1$ and checking, we see that these values don't satisfy the third equation. Thus the lines aren't parallel and don't intersect, so they must be skew lines.

22. Since the direction vectors $\langle 2,2,-1\rangle$ and $\langle 1,-1,3\rangle$ aren't parallel, the lines aren't parallel. Here the parametric equations are L_1 : $x=1+2t$, $y=3+2t$, $z=2-t$ and L_2 : $x=2+s$, $y=6-s$, $z=-2+3s$. Thus, for the lines to intersect, the three equations $1+2t=2+s$, $3+2t=6-s$, and $2-t=-2+3s$ must be satisfied simultaneously. Solving the first two equations gives $t=1$, $s=1$ and, checking, we see that these values do satisfy the third equation, so the lines intersect when $t=1$ and $s=1$, that is, at the point $(3,5,1)$.

23. Since the plane is perpendicular to the vector $\langle -2,1,5\rangle$, we can take $\langle -2,1,5\rangle$ as a normal vector to the plane. $(6,3,2)$ is a point on the plane, so setting $a=-2$, $b=1$, $c=5$ and $x_0=6$, $y_0=3$, $z_0=2$ in Equation 7 gives $-2(x-6)+1(y-3)+5(z-2)=0$ or $-2x+y+5z=1$ to be an equation of the plane.

24. $\boldsymbol{j}+2\boldsymbol{k}=\langle 0,1,2\rangle$ is a normal vector to the plane and $(4,0,-3)$ is a point on the plane, so setting $a=0$, $b=1$, $c=2, x_0=4$, $y_0=0$, $z_0=-3$ in Equation 7 gives $0(x-4)+1(y-0)+2[z-(-3)]=0$ or $y+2z=-6$ to be an equation of the plane.

25. $\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}=\langle 1,1,-1\rangle$ is a normal vector to the plane and $(1,-1,1)$ is a point on the plane, so setting $a=1$, $b=1$, $c=-1$, $x_0=1$, $y_0=-1$, $z_0=1$ in Equation 7 gives $1(x-1)+1[y-(-1)]-1(z-1)=0$ or $x+y-z=-1$ to be an equation of the plane.

26. Since the line is perpendicular to the plane, its direction vector $\langle 1,2,-3\rangle$ is a normal vector to the plane. An equation of the plane, then, is $1[x-(-2)]+2(y-8)-3(z-10)=0$ or $x+2y-3z=-16$.

27. Since the two planes are parallel, they will have the same normal vectors. So we can take $\boldsymbol{n}=\langle 2,-1,3\rangle$, and an equation of the plane is $2(x-0)-1(y-0)+3(z-0)=0$ or $2x-y+3z=0$.

28. Since the two planes are parallel, they will have the same normal vectors. So we can take $\boldsymbol{n}=\langle 1,1,1\rangle$, and an equation of the plane is $1[x-(-1)]+1(y-6)+1[z-(-5)]=0$ or $x+y+z=0$.

29. Since the two planes are parallel, they will have the same normal vectors. So we can take $\boldsymbol{n}=\langle 3,0,-7\rangle$, and an equation of the plane is $3(x-4)+0[y-(-2)]-7(z-3)=0$ or $3x-7z=-9$.

30. First, a normal vector for the plane $2x+4y+8z=17$ is $\boldsymbol{n}=\langle 2,4,8\rangle$. A direction vector for the line is $\boldsymbol{v}=\langle 2,1,-1\rangle$, and since $\boldsymbol{n}\cdot \boldsymbol{v}=0$ we know the line is perpendicular to $\boldsymbol{n}$ and hence parallel to the plane. Thus, there is a parallel plane which contains the line. By putting $t=0$, we know the point $(3,0,8)$ is on the line and hence the new plane. We can use the same normal vector $\boldsymbol{n}=\langle 2,4,8\rangle$, so an equation of the plane is $2(x-3)+4(y-0)+8(z-8)=0$ or $x+2y+4z=35$.

31. Here the vectors $\boldsymbol{a}=\langle 1-0,0-1,1-1\rangle=\langle 1,-1,0\rangle$ and $\boldsymbol{b}=\langle 1-0,1-1,0-1\rangle=\langle 1,0,-1\rangle$ lie in the plane, so $\boldsymbol{a}\times \boldsymbol{b}$ is a normal vector to the plane. Thus, we can take $\boldsymbol{n}=\boldsymbol{a}\times \boldsymbol{b}=\langle 1-0,0+1,0+1\rangle=\langle 1,1,1\rangle$. If P_0 is the point $(0,1,1)$, an equation of the plane is $1(x-0)+1(y-1)+1(z-1)=0$ or $x+y+z=2$.

32. Here the vectors $\boldsymbol{a}=\langle 2,-4,6\rangle$ and $\boldsymbol{b}=\langle 5,1,3\rangle$ lie in the plane, so $\boldsymbol{n}=\boldsymbol{a}\times \boldsymbol{b}=\langle -12-6,30-6,2+20\rangle=\langle -18,24,22\rangle$ is a normal vector to the plane and an equation of the plane is $-18(x-0)+24(y-0)+22(z-0)=0$ or $-18x+24y+22z=0$.

33. Here the vectors $\boldsymbol{a}=\langle 8-3,2-(-1),4-2\rangle=\langle 5,3,2\rangle$ and $\boldsymbol{b}=\langle -1-3,-2-(-1),-3-2\rangle=\langle -4,-1,-5\rangle$ lie in the plane, so a normal vector to the plane is $\boldsymbol{n}=\boldsymbol{a}\times \boldsymbol{b}=\langle -15+2,-8+25,-5+12\rangle=\langle -13,17,7\rangle$ and an equation of the plane is $-13(x-3)+17[y-(-1)]+7(z-2)=0$ or $-13x+17y+7z=-42$.

34. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\boldsymbol{a}=\langle 3,1,-1\rangle$ is one vector in the plane. We can verify that the given point $(1,2,3)$ does not lie on this line, so to find another nonparallel vector $\boldsymbol{b}$ which lies in the plane, we can pick any point on the line and find a vector connecting the points. If we put $t=0$, we see that $(0,1,2)$ is on the line, so $\boldsymbol{b}=\langle 1-0,2-1,3-2\rangle=\langle 1,1,1\rangle$ and $\boldsymbol{n}=\boldsymbol{a}\times \boldsymbol{b}=\langle 1+1,-1-3,3-1\rangle=\langle 2,-4,2\rangle$. Thus, an equation of the plane is $2(x-1)-4(y-2)+2(z-3)=0$ or $2x-4y+2z=0$. (Equivalently, we can write $x-2y+z=0$.)

35. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to

the plane. Since the given line lies in the plane, its direction vector $\boldsymbol{a}=\langle -2,5,4\rangle$ is one vector in the plane. We can verify that the given point $(6,0,-2)$ does not lie on this line, so to find another nonparallel vector $\boldsymbol{b}$ which lies in the plane, we can pick any point on the line and find a vector connecting the points. If we put $t=0$, we see that $(4,3,7)$ is on the line, so $\boldsymbol{b}=\langle 6-4,0-3,-2-7\rangle=\langle 2,-3,-9\rangle$ and $\boldsymbol{n}=\boldsymbol{a}\times\boldsymbol{b}=\langle -45+12,8-18,6-10\rangle=\langle -33,-10,-4\rangle$. Thus, an equation of the plane is $-33(x-6)-10(y-0)-4[z-(-2)]=0$ or $33x+10y+4z=190$.

36. Since the line $x=2y=3z$, or $x=\dfrac{y}{1/2}=\dfrac{z}{1/3}$, lies in the plane, its direction vector $\boldsymbol{a}=\left\langle 1,\dfrac{1}{2},\dfrac{1}{3}\right\rangle$ is parallel to the plane. The point $(0,0,0)$ is on the line (put $t=0$), and we can verify that the given point $(1,-1,1)$ in the plane is not on the line. The vector connecting these two points, $\boldsymbol{b}=\langle 1,-1,1\rangle$, is therefore parallel to the plane, but not parallel to $\langle 1,2,3\rangle$. Then $\boldsymbol{a}\times\boldsymbol{b}=\left\langle \dfrac{1}{2}+\dfrac{1}{3},\dfrac{1}{3}-1,-1-\dfrac{1}{2}\right\rangle=\left\langle \dfrac{5}{6},-\dfrac{2}{3},-\dfrac{3}{2}\right\rangle$ is a normal vector to the plane, and an equation of the plane is $\dfrac{5}{6}(x-0)-\dfrac{2}{3}(y-0)-\dfrac{3}{2}(z-0)=0$ or $5x-4y-9z=0$.

37. A direction vector for the line of intersection is $\boldsymbol{a}=\boldsymbol{n}_1\times\boldsymbol{n}_2=\langle 1,1,-1\rangle\times\langle 2,-1,3\rangle=\langle 2,-5,-3\rangle$, and $\boldsymbol{a}$ is parallel to the desired plane. Another vector parallel to the plane is the vector connecting any point on the line of intersection to the given point $(-1,2,1)$ in the plane. Setting $x=0$, the equations of the planes reduce to $y-z=2$ and $-y+3z=1$ with simultaneous solution $y=\dfrac{7}{2}$ and $z=\dfrac{3}{2}$. So a point on the line is $\left(0,\dfrac{7}{2},\dfrac{3}{2}\right)$ and another vector parallel to the plane is $\left\langle -1,-\dfrac{3}{2},-\dfrac{1}{2}\right\rangle$. Then a normal vector to the plane is $\boldsymbol{n}=\langle 2,-5,-3\rangle\times\left\langle -1,-\dfrac{3}{2},-\dfrac{1}{2}\right\rangle=\langle -2,4,-8\rangle$ and an equation of the plane is $-2(x+1)+4(y-2)-8(z-1)=0$ or $x-2y+4z=-1$.

38. $\boldsymbol{n}_1=\langle 1,0,-1\rangle$ and $\boldsymbol{n}_2=\langle 0,1,2\rangle$. Setting $z=0$, it is easy to see that $(1,3,0)$ is a point on the line of intersection of $x-z=1$ and $y+2z=3$. The direction of this line is $\boldsymbol{v}_1=\boldsymbol{n}_1\times\boldsymbol{n}_2=\langle 1,-2,1\rangle$. A second vector parallel to the desired plane is $\boldsymbol{v}_2=\langle 1,1,-2\rangle$, since it is perpendicular to $x+y-2z=1$. Therefore, a normal of the plane in question is $\boldsymbol{n}=\boldsymbol{v}_1\times\boldsymbol{v}_2=\langle 4-1,1+2,1+2\rangle=\langle 3,3,3\rangle$, or we can use $\langle 1,1,1\rangle$. Taking $\left(x_0,y_0,z_0\right)=(1,3,0)$, the equation we are looking for is $(x-1)+(y-3)+z=0 \Leftrightarrow x+y+z=4$.

39. Substitute the parametric equations of the line into the equation of the plane: $(3-t)-(2+t)+2(5t)=9$

$\Rightarrow 8t=8 \Rightarrow t=1$. Therefore, the point of intersection of the line and the plane is given by $x=3-1=2$, $y=2+1=3$, and $z=5(1)=5$, that is, the point $(2,3,5)$.

40. Substitute the parametric equations of the line into the equation of the plane: $(1+2t)+2(4t)-(2-3t)+1=0 \Rightarrow 13t=0 \Rightarrow t=0$. Therefore, the point of intersection of the line and the plane is given by $x=1+2(0)=1$, $y=4(0)=0$, and $z=2-3(0)=2$, that is, the point $(1,0,2)$.

41. Parametric equations for the line are $x=t$, $y=1+t$, $z=\frac{1}{2}t$ and substituting into the equation of the plane gives $4(t)-(1+t)+3\left(\frac{1}{2}t\right)=8 \Rightarrow \frac{9}{2}t=9 \Rightarrow t=2$. Thus $x=2$, $y=1+2=3$, $z=\frac{1}{2}(2)=1$ and the point of intersection is $(2,3,1)$.

42. A direction vector for the line through $(1,0,1)$ and $(4,-2,2)$ is $\mathbf{v}=\langle 3,-2,1\rangle$ and, taking $P_0=(1,0,1)$, parametric equations for the line are $x=1+3t$, $y=-2t$, $z=1+t$. Substitution of the parametric equations into the equation of the plane gives $1+3t-2t+1+t=6 \Rightarrow t=2$. Then $x=1+3(2)=7$, $y=-2(2)=-4$, and $z=1+2=3$ so the point of intersection is $(7,-4,3)$.

43. Setting $x=0$, we see that $(0,1,0)$ satisfies the equations of both planes, so that they do in fact have a line of intersection. $\mathbf{v}=\mathbf{n}_1\times\mathbf{n}_2=\langle 1,1,1\rangle\times\langle 1,0,1\rangle=\langle 1,0,-1\rangle$ is the direction of this line. Therefore, direction numbers of the intersecting line are 1 , 0 , -1 .

44. The angle between the two planes is the same as the angle between their normal vectors. The normal vectors of the two planes are $\langle 1,1,1\rangle$ and $\langle 1,2,3\rangle$. The cosine of the angle θ between these two planes is $\cos\theta=\dfrac{\langle 1,1,1\rangle\cdot\langle 1,2,3\rangle}{|\langle 1,1,1\rangle|\,|\langle 1,2,3\rangle|}=\dfrac{1+2+3}{\sqrt{1+1+1}\sqrt{1+4+9}}=\dfrac{6}{\sqrt{42}}=\sqrt{\dfrac{6}{7}}$.

45. Normal vectors for the planes are $\mathbf{n}_1=\langle 1,4,-3\rangle$ and $\mathbf{n}_2=\langle -3,6,7\rangle$, so the normals (and thus the planes) aren't parallel. But $\mathbf{n}_1\cdot\mathbf{n}_2=-3+24-21=0$, so the normals (and thus the planes) are perpendicular.

46. Normal vectors for the planes are $\mathbf{n}_1=\langle -1,4,-2\rangle$ and $\mathbf{n}_2=\langle 3,-12,6\rangle$. Since $\mathbf{n}_2=-3\mathbf{n}_1$, the normals (and thus the planes) are parallel.

47. Normal vectors for the planes are $\mathbf{n}_1=\langle 1,1,1\rangle$ and $\mathbf{n}_2=\langle 1,-1,1\rangle$. The normals are not parallel, so neither are the planes. Furthermore, $\mathbf{n}_1\cdot\mathbf{n}_2=1-1+1=1\neq 0$, so the planes aren't perpendicular. The angle between them is given by

$\cos\theta = \dfrac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1|\,|\boldsymbol{n}_2|} = \dfrac{1}{\sqrt{3}\sqrt{3}} = \dfrac{1}{3} \Rightarrow \theta = \cos^{-1}\left(\dfrac{1}{3}\right) \approx 70.5°$.

48. The normals are $\boldsymbol{n}_1 = \langle 2,-3,4 \rangle$ and $\boldsymbol{n}_2 = \langle 1,6,4 \rangle$ so the planes aren't parallel. Since $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = 2-18+16=0$, the normals (and thus the planes) are perpendicular.

49. The normals are $\boldsymbol{n}_1 = \langle 1,-4,2 \rangle$ and $\boldsymbol{n}_2 = \langle 2,-8,4 \rangle$. Since $\boldsymbol{n}_2 = 2\boldsymbol{n}_1$, the normals (and thus the planes) are parallel.

50. The normal vectors are $\boldsymbol{n}_1 = \langle 1,2,2 \rangle$ and $\boldsymbol{n}_2 = \langle 2,-1,2 \rangle$. The normals are not parallel, so neither are the planes. Furthermore, $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = 2-2+4=4 \neq 0$, so the planes aren't perpendicular. The angle between them is given by $\cos\theta = \dfrac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1|\,|\boldsymbol{n}_2|} = \dfrac{4}{\sqrt{9}\sqrt{9}} = \dfrac{4}{9} \Rightarrow \theta = \cos^{-1}\left(\dfrac{4}{9}\right) \approx 63.6°$.

51. **(a)** To find a point on the line of intersection, set one of the variables equal to a constant, say $z=0$. (This will only work if the line of intersection crosses the xy –plane; otherwise, try setting x or y equal to 0 .) Then the equations of the planes reduce to $x+y=2$ and $3x-4y=6$. Solving these two equations gives $x=2$, $y=0$. So a point on the line of intersection is $(2,0,0)$. The direction of the line is $\boldsymbol{v} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 = \langle 5-4,-3-5,-4-3 \rangle = \langle 1,-8,-7 \rangle$, and symmetric equations for the line are $x-2 = \dfrac{y}{-8} = \dfrac{z}{-7}$.

(b) The angle between the planes satisfies $\cos\theta = \dfrac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1|\,|\boldsymbol{n}_2|} = \dfrac{3-4-5}{\sqrt{3}\sqrt{50}} = -\dfrac{\sqrt{6}}{5}$. Therefore $\theta = \cos^{-1}\left(-\dfrac{\sqrt{6}}{5}\right) \approx 119°$ (or $61°$).

52. **(a)** $x-2y+z=1 \Rightarrow \boldsymbol{n}_1 = \langle 1,-2,1 \rangle$ and $2x+y+z=1 \Rightarrow \boldsymbol{n}_2 = \langle 2,1,1 \rangle$. The vector that gives the direction of the line of intersection of these two planes is $\boldsymbol{v} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 = \langle -2-1,2-1,1+4 \rangle = \langle -3,1,5 \rangle$. Setting $x=y=0$, we see that both planes contain $(0,0,1)$ so that this point must lie on their line of intersection. Then symmetric equations for this line are $\dfrac{x}{-3} = y = \dfrac{z-1}{5}$.

(b) $\cos\theta = \dfrac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1|\,|\boldsymbol{n}_2|} = \dfrac{2-2+1}{\sqrt{1+4+1}\sqrt{4+1+1}} = \dfrac{1}{6} \Rightarrow \theta = \cos^{-1}\left(\dfrac{1}{6}\right) \approx 80°$.

53. Setting $x=0$, the equations of the two planes become $z=y$ and $5y+z=-1$, which intersect at $y=-\frac{1}{6}$ and $z=-\frac{1}{6}$. Thus we can choose $\left(x_0,y_0,z_0\right)=\left(0,-\frac{1}{6},-\frac{1}{6}\right)$. The vector giving the direction of this intersecting line, $\boldsymbol{v}$, is perpendicular to the normal vectors of both planes. So $\boldsymbol{v}=\boldsymbol{n}_1\times\boldsymbol{n}_2=\langle 2,-5,-1\rangle\times\langle 1,1,-1\rangle=\langle 6,1,7\rangle$. Therefore, by Equations 2, parametric equations for this line are $x=6t$, $y=-\frac{1}{6}+t$, $z=-\frac{1}{6}+7t$.

54. Setting $y=0$, the equations of the two planes become $2x+5z=-3$ and $x+z=-2$, which intersect at $x=-\frac{7}{3}$ and $z=\frac{1}{3}$. Thus we can choose $\left(x_0,y_0,z_0\right)=\left(-\frac{7}{3},0,\frac{1}{3}\right)$. The vector giving the direction of this intersecting line, $\boldsymbol{v}$, is perpendicular to the normal vectors of both planes. So $\boldsymbol{v}=\boldsymbol{n}_1\times\boldsymbol{n}_2=\langle 2,0,5\rangle\times\langle 1,-3,1\rangle=\langle 15,5-2,-6\rangle=3\langle 5,1,-2\rangle$. Therefore, by Equations 2, parametric equations of the line of intersection of the two planes are $x=-\frac{7}{3}+5t$, $y=t$, $z=\frac{1}{3}-2t$.

55. The plane contains all perpendicular bisectors of the line segment joining $(1,1,0)$ and $(0,1,1)$. All of these bisectors pass through the midpoint of this segment $\left(\frac{1}{2},\frac{1+1}{2},\frac{1}{2}\right)=\left(\frac{1}{2},1,\frac{1}{2}\right)$. The direction of this line segment $\langle 1-0,1-1,0-1\rangle=\langle 1,0,-1\rangle$ is perpendicular to the plane so that we can choose this to be $\boldsymbol{n}$. Therefore the equation of the plane is $1\left(x-\frac{1}{2}\right)+0(y-1)-1\left(z-\frac{1}{2}\right)=0\Leftrightarrow x=z$.

56. The plane will contain all perpendicular bisectors of the line segment joining the two points. Thus, a point in the plane is $P_0=(-1,-1,2)$, the midpoint of the line segment joining the two given points, and a normal to the plane is $\boldsymbol{n}=\langle 6,-6,2\rangle$, the vector connecting the two points. So an equation of the plane is $6(x+1)-6(y+1)+2(z-2)=0$ or $3x-3y+z=2$.

57. The plane contains the points $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$. Thus the vectors $\boldsymbol{a}=\langle -a,b,0\rangle$ and $\boldsymbol{b}=\langle -a,0,c\rangle$ lie in the plane, and $\boldsymbol{n}=\boldsymbol{a}\times\boldsymbol{b}=\langle bc-0,0+ac,0+ab\rangle=\langle bc,ac,ab\rangle$ is a normal vector to the plane. The equation of the plane is therefore $bcx+acy+abz=abc+0+0$ or $bcx+acy+abz=abc$. Notice that if $a\neq 0$, $b\neq 0$ and $c\neq 0$ then we can rewrite the equation as $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. This is a good equation to remember!

58. **(a)** For the lines to intersect, we must be able to find one value of t and one value of s satisfying the three equations $1+t=2-s$, $1-t=s$ and $2t=2$. From the third we get $t=1$, and putting this in the second gives $s=0$. These values of s and t do satisfy the first equation, so the lines intersect at the

point $P_0=(1+1,1-1,2(1))=(2,0,2)$.

(b) The direction vectors of the lines are $\langle 1,-1,2\rangle$ and $\langle -1,1,0\rangle$, so a normal vector for the plane is $\langle -1,1,0\rangle\times\langle 1,-1,2\rangle=\langle 2,2,0\rangle$ and it contains the point $(2,0,2)$. Then the equation of the plane is $2(x-2)+2(y-0)+0(z-2)=0\Leftrightarrow x+y=2$.

59. Two vectors which are perpendicular to the required line are the normal of the given plane, $\langle 1,1,1\rangle$, and a direction vector for the given line, $\langle 1,-1,2\rangle$. So a direction vector for the required line is $\langle 1,1,1\rangle\times\langle 1,-1,2\rangle=\langle 3,-1,-2\rangle$. Thus L is given by $\langle x,y,z\rangle=\langle 0,1,2\rangle+t\langle 3,-1,-2\rangle$, or in parametric form, $x=3t$, $y=1-t$, $z=2-2t$.

60. Let L be the given line. Then $(1,1,0)$ is the point on L corresponding to $t=0$. L is in the direction of $\boldsymbol{a}=\langle 1,-1,2\rangle$ and $\boldsymbol{b}=\langle -1,0,2\rangle$ is the vector joining $(1,1,0)$ and $(0,1,2)$. Then $\boldsymbol{b}-\text{proj}_{\boldsymbol{a}}\boldsymbol{b}=\langle -1,0,2\rangle-\dfrac{\langle 1,-1,2\rangle\cdot\langle -1,0,2\rangle}{1^2+(-1)^2+2^2}\langle 1,-1,2\rangle=\langle -1,0,2\rangle-\dfrac{1}{2}\langle 1,-1,2\rangle=\left\langle -\dfrac{3}{2},\dfrac{1}{2},1\right\rangle$ is a direction vector for the required line. Thus $2\left\langle -\dfrac{3}{2},\dfrac{1}{2},1\right\rangle=\langle -3,1,2\rangle$ is also a direction vector, and the line has parametric equations $x=-3t$, $y=1+t$, $z=2+2t$. (Notice that this is the same line as in Exercise 59.)

61. Let P_i have normal vector $\boldsymbol{n}_i$. Then $\boldsymbol{n}_1=\langle 4,-2,6\rangle$, $\boldsymbol{n}_2=\langle 4,-2,-2\rangle$, $\boldsymbol{n}_3=\langle -6,3,-9\rangle$, $\boldsymbol{n}_4=\langle 2,-1,-1\rangle$. Now $\boldsymbol{n}_1=-\dfrac{2}{3}\boldsymbol{n}_3$, so $\boldsymbol{n}_1$ and $\boldsymbol{n}_3$ are parallel, and hence P_1 and P_3 are parallel; similarly P_2 and P_4 are parallel because $\boldsymbol{n}_2=2\boldsymbol{n}_4$. However, $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are not parallel. $\left(0,0,\dfrac{1}{2}\right)$ lies on P_1 , but not on P_3 , so they are not the same plane, but both P_2 and P_4 contain the point $(0,0,-3)$, so these two planes are identical.

62. Let L_i have direction vector $\boldsymbol{v}_i$. Then $\boldsymbol{v}_1=\langle 1,1,-5\rangle$, $\boldsymbol{v}_2=\langle 1,1,-1\rangle$, $\boldsymbol{v}_3=\langle 1,1,-1\rangle$, $\boldsymbol{v}_4=\langle 2,2,-10\rangle$. $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are equal so they're parallel. $\boldsymbol{v}_4=2\boldsymbol{v}_1$, so L_4 and L_1 are parallel. L_3 contains the point $(1,4,1)$, but this point does not lie on L_2 , so they're not equal. $(2,1,-3)$ lies on L_4 , and on L_1 , with $t=1$. So L_1 and L_4 are identical.

63. Let $Q=(2,2,0)$ and $R=(3,-1,5)$, points on the line corresponding to $t=0$ and $t=1$. Let $P=(1,2,3)$. Then $\boldsymbol{a}=\overrightarrow{QR}=\langle 1,-3,5\rangle$, $\boldsymbol{b}=\overrightarrow{QP}=\langle -1,0,3\rangle$. The distance is

$$d=\frac{|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{a}|}=\frac{|\langle 1,-3,5\rangle\times\langle -1,0,3\rangle|}{|\langle 1,-3,5\rangle|}=\frac{|\langle -9,-8,-3\rangle|}{|\langle 1,-3,5\rangle|}=\frac{\sqrt{9^2+8^2+3^2}}{\sqrt{1^2+3^2+5^2}}=\frac{\sqrt{154}}{\sqrt{35}}=\sqrt{\frac{22}{5}}\ .$$

64. Let $Q=(5,0,1)$ and $R=(4,3,3)$, points on the line corresponding to $t=0$ and $t=1$. Let $P=(1,0,-1)$. Then $\boldsymbol{a}=\overrightarrow{QR}=\langle -1,3,2\rangle$ and $\boldsymbol{b}=\overrightarrow{QP}=\langle -4,0,-2\rangle$. The distance is

$$d=\frac{|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{a}|}=\frac{|\langle -1,3,2\rangle\times\langle -4,0,-2\rangle|}{|\langle -1,3,2\rangle|}=\frac{|\langle -6,-10,12\rangle|}{|\langle -1,3,2\rangle|}=\frac{2\sqrt{3^2+5^2+6^2}}{\sqrt{1^2+3^2+2^2}}=\frac{2\sqrt{70}}{\sqrt{14}}=2\sqrt{5}\ .$$

65. By Equation 9, the distance is $D=\frac{1}{\sqrt{1+4+4}}\left[(1)(2)+(-2)(8)+(-2)(5)-1\right]=\frac{25}{3}$.

66. By Equation 9, the distance is $D=\frac{1}{\sqrt{16+36+1}}\left[4(3)+(-6)(-2)+1(7)-5\right]=\frac{26}{\sqrt{53}}$.

67. Put $y=z=0$ in the equation of the first plane to get the point $(-1,0,0)$ on the plane. Because the planes are parallel, the distance D between them is the distance from $(-1,0,0)$ to the second plane. By Equation 9, $D=\frac{|3(-1)+6(0)-3(0)-4|}{\sqrt{3^2+6^2+(-3)^2}}=\frac{7}{3\sqrt{6}}$ or $\frac{7\sqrt{6}}{18}$.

68. Put $y=z=0$ in the equation of the first plane to get the point $\left(\frac{4}{3},0,0\right)$ on the plane. Because the planes are parallel the distance D between them is the distance from $\left(\frac{4}{3},0,0\right)$ to the second plane. By Equation 9, $D=\frac{\left|1\left(\frac{4}{3}\right)+2(0)-3(0)-1\right|}{\sqrt{1^2+2^2+(-3)^2}}=\frac{1}{3\sqrt{14}}$.

69. The distance between two parallel planes is the same as the distance between a point on one of the planes and the other plane. Let $P_0=(x_0,y_0,z_0)$ be a point on the plane given by $ax+by+cz+d_1=0$. Then $ax_0+by_0+cz_0+d_1=0$ and the distance between P_0 and the plane given by $ax+by+cz+d_2=0$ is, from Equation 9, $D=\frac{|ax_0+by_0+cz_0+d_2|}{\sqrt{a^2+b^2+c^2}}=\frac{|-d_1+d_2|}{\sqrt{a^2+b^2+c^2}}=\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$.

70. The planes must have parallel normal vectors, so if $ax+by+cz+d=0$ is such a plane, then for some

$t\neq0$, $\langle a,b,c\rangle=t\langle 1,2,-2\rangle=\langle t,2t,-2t\rangle$. So this plane is given by the equation $x+2y-2z+e=0$, where $e=d/t$. By Exercise 69, the distance between the planes is $2=\dfrac{|1-e|}{\sqrt{1^2+2^2+(-2)^2}}\Leftrightarrow 6=|1-e|\Leftrightarrow e=7$ or -5 . So the desired planes have equations $x+2y-2z=7$ and $x+2y-2z=-5$.

71. L_1 : $x=y=z\Rightarrow x=y$ (1). L_2 : $x+1=y/2=z/3\Rightarrow x+1=y/2$ (2). The solution of (1) and (2) is $x=y=-2$. However, when $x=-2$, $x=z\Rightarrow z=-2$, but $x+1=z/3\Rightarrow z=-3$, a contradiction. Hence the lines do not intersect. For L_1 , $\boldsymbol{v}_1=\langle 1,1,1\rangle$, and for L_2 , $\boldsymbol{v}_2=\langle 1,2,3\rangle$, so the lines are not parallel. Thus the lines are skew lines. If two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the planes must be perpendicular to both $\langle 1,1,1\rangle$ and $\langle 1,2,3\rangle$, the direction vectors of the two lines. So set $\boldsymbol{n}=\langle 1,1,1\rangle\times\langle 1,2,3\rangle=\langle 3-2,-3+1,2-1\rangle=\langle 1,-2,1\rangle$. From above, we know that $(-2,-2,-2)$ and $(-2,-2,-3)$ are points of L_1 and L_2 respectively. So in the notation of Equation 8, $1(-2)-2(-2)+1(-2)+d_1=0\Rightarrow d_1=0$ and $1(-2)-2(-2)+1(-3)+d_2=0\Rightarrow d_2=1$.

By Exercise 69, the distance between these two skew lines is $D=\dfrac{|0-1|}{\sqrt{1+4+1}}=\dfrac{1}{\sqrt{6}}$.

Alternate solution (without reference to planes): A vector which is perpendicular to both of the lines is $\boldsymbol{n}=\langle 1,1,1\rangle\times\langle 1,2,3\rangle=\langle 1,-2,1\rangle$. Pick any point on each of the lines, say $(-2,-2,-2)$ and $(-2,-2,-3)$, and form the vector $\boldsymbol{b}=\langle 0,0,1\rangle$ connecting the two points. The distance between the two skew lines is the absolute value of the scalar projection of $\boldsymbol{b}$ along $\boldsymbol{n}$, that is, $D=\dfrac{|\boldsymbol{n}\cdot\boldsymbol{b}|}{|\boldsymbol{n}|}=\dfrac{|1\cdot 0-2\cdot 0+1\cdot 1|}{\sqrt{1+4+1}}=\dfrac{1}{\sqrt{6}}$.

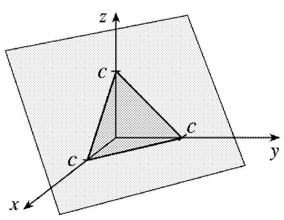
72. First notice that if two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the planes must be perpendicular to both $\boldsymbol{v}_1=\langle 1,6,2\rangle$ and $\boldsymbol{v}_2=\langle 2,15,6\rangle$, the direction vectors of the two lines respectively. Thus set $\boldsymbol{n}=\boldsymbol{v}_1\times\boldsymbol{v}_2=\langle 36-30,4-6,15-12\rangle=\langle 6,-2,3\rangle$. Setting $t=0$ and $s=0$ gives the points $(1,1,0)$ and $(1,5,-2)$. So in the notation of Equation 8, $6-2+0+d_1=0\Rightarrow d_1=-4$ and $6-10-6+d_2=0\Rightarrow d_2=10$. Then by Exercise 69 , the distance between the two skew lines is given by $D=\dfrac{|-4-10|}{\sqrt{36+4+9}}=\dfrac{14}{7}=2$.

Alternate solution (without reference to planes): We already know that the direction vectors of the two lines are $\boldsymbol{v}_1=\langle 1,6,2\rangle$ and $\boldsymbol{v}_2=\langle 2,15,6\rangle$. Then $\boldsymbol{n}=\boldsymbol{v}_1\times\boldsymbol{v}_2=\langle 6,-2,3\rangle$ is perpendicular to both lines. Pick any point on each of the lines, say $(1,1,0)$ and $(1,5,-2)$, and form the vector $\boldsymbol{b}=\langle 0,4,-2\rangle$ connecting the two points. Then the distance between the two skew lines is the absolute value of the scalar projection of $\boldsymbol{b}$ along $\boldsymbol{n}$, that is,

$D=\dfrac{|\boldsymbol{n}\cdot\boldsymbol{b}|}{|\boldsymbol{n}|}=\dfrac{1}{\sqrt{36+4+9}}\ |0-8-6|=\dfrac{14}{7}=2$.

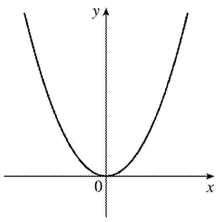
73. If $a\neq 0$, then $ax+by+cz+d=0\Rightarrow a\left(x+d/a\right)+b(y-0)+c(z-0)=0$ which by (7) is the scalar equation of the plane through the point $(-d/a,0,0)$ with normal vector $\langle a,b,c\rangle$. Similarly, if $b\neq 0$ (or if $c\neq 0$) the equation of the plane can be rewritten as $a(x-0)+b\left(y+d/b\right)+c(z-0)=0$ which by (7) is the scalar equation of a plane through the point $(0,-d/b,0)$ with normal vector $\langle a,b,c\rangle$.

74. **(a)** The planes $x+y+z=c$ have normal vector $\langle 1,1,1\rangle$, so they are all parallel. Their x –, y –, and z –intercepts are all c . When $c>0$ their intersection with the first octant is an equilateral triangle and when $c<0$ their intersection with the octant diagonally opposite the first is an equilateral triangle.

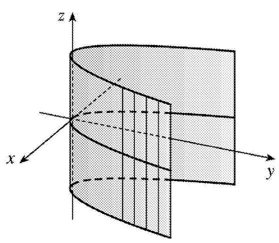


(b) The planes $x+y+cz=1$ have x –intercept 1 , y – intercept 1 , and z –intercept $1/c$. The plane with $c=0$ is parallel to the z –axis. As c gets larger, the planes get closer to the xy –plane.
(c) The planes $y\cos\theta+z\cos\theta=1$ have normal vectors $\langle 0,\cos\theta,\sin\theta\rangle$, which are perpendicular to the x –axis, and so the planes are parallel to the x –axis. We look at their intersection with the yz –plane. These are lines that are perpendicular to $\langle\cos\theta,\sin\theta\rangle$ and pass through $(\cos\theta,\sin\theta)$, since $\cos^2\theta+\sin^2\theta=1$. So these are the tangent lines to the unit circle. Thus the family consists of all planes tangent to the circular cylinder with radius 1 and axis the x –axis.

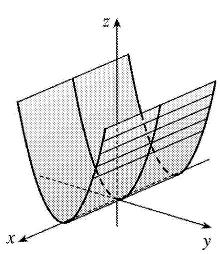
1. **(a)** In R^2, the equation $y=x^2$ represents a parabola.



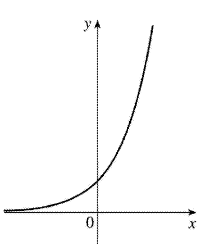
(b) In R^3, the equation $y=x^2$ doesn't involve z, so any horizontal plane with equation $z=k$ intersects the graph in a curve with equation $y=x^2$. Thus, the surface is a parabolic cylinder, made up of infinitely many shifted copies of the same parabola. The rulings are parallel to the z –axis.



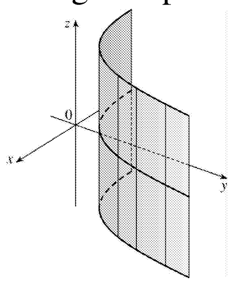
(c) In R^3, the equation $z=y^2$ also represents a parabolic cylinder. Since x doesn't appear, the graph is formed by moving the parabola $z=y^2$ in the direction of the x –axis. Thus, the rulings of the cylinder are parallel to the x –axis.



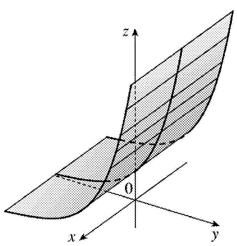
2. **(a)**



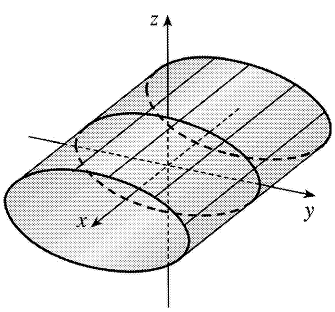
(b) Since the equation $y=e^{x}$ doesn't involve z , horizontal traces are copies of the curve $y=e^{x}$. The rulings are parallel to the z –axis.



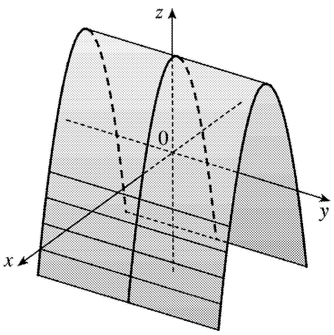
(c) The equation $z=e^{y}$ doesn't involve x , so vertical traces in $x=k$ (parallel to the yz –plane) are copies of the curve $z=e^{y}$. The rulings are parallel to the x –axis.



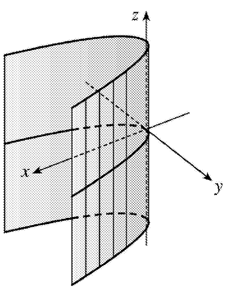
3. Since x is missing from the equation, the vertical traces $y^{2}+4z^{2}=4$, $x=k$, are copies of the same ellipse in the plane $x=k$. Thus, the surface $y^{2}+4z^{2}=4$ is an elliptic cylinder with rulings parallel to the x –axis.



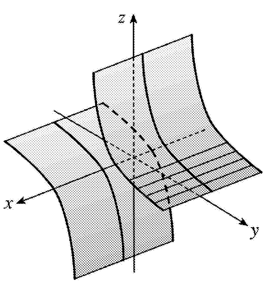
4. Since y is missing from the equation, each vertical trace $z=4-x^{2}$, $y=k$, is a copy of the same parabola in the plane $y=k$. Thus, the surface $z=4-x^{2}$ is a parabolic cylinder with rulings parallel to the y –axis.



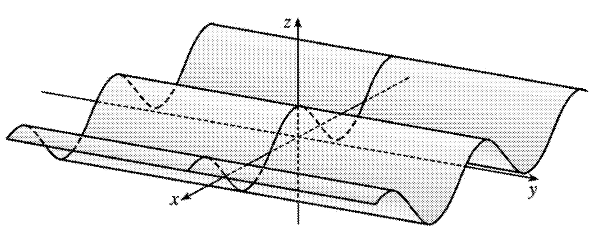
5. Since z is missing, each horizontal trace $x=y^2$, $z=k$, is a copy of the same parabola in the plane $z=k$. Thus, the surface $x-y^2=0$ is a parabolic cylinder with rulings parallel to the z –axis.



6. Since x is missing, each vertical trace $yz=4$, $x=k$ is a copy of the same hyperbola in the plane $x=k$. Thus, the surface $yz=4$ is a hyperbolic cylinder with rulings parallel to the x –axis.

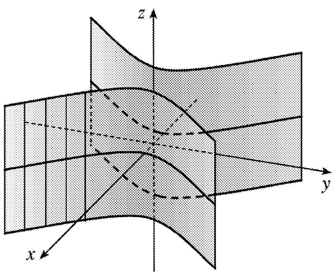


7. Since y is missing, each vertical trace $z=\cos x$, $y=k$ is a copy of a cosine curve in the plane $y=k$. Thus, the surface $z=\cos x$ is a cylindrical surface with rulings parallel to the y –axis.

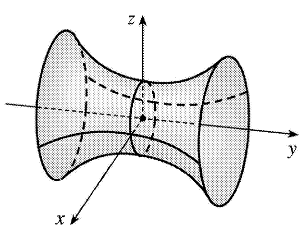


8. Since z is missing, each horizontal trace $x^2-y^2=1$, $z=k$ is a copy of the same hyperbola in the plane

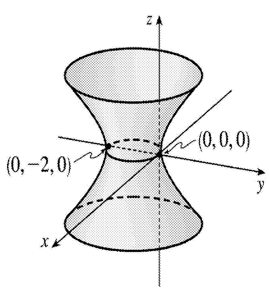
$z=k$. Thus, the surface $x^2-y^2=1$ is a hyperbolic cylinder with rulings parallel to the z –axis.



9. **(a)** The traces of $x^2+y^2-z^2=1$ in $x=k$ are $y^2-z^2=1-k^2$, a family of hyperbolas. (Note that the hyperbolas are oriented differently for $-1<k<1$ than for $k<-1$ or $k>1$.) The traces in $y=k$ are $x^2-z^2=1-k^2$, a similar family of hyperbolas. The traces in $z=k$ are $x^2+y^2=1+k^2$, a family of circles. For $k=0$, the trace in the xy –plane, the circle is of radius 1 . As $|k|$ increases, so does the radius of the circle. This behavior, combined with the hyperbolic vertical traces, gives the graph of the hyperboloid of one sheet in Table 1.
(b) The shape of the surface is unchanged, but the hyperboloid is rotated so that its axis is the y –axis. Traces in $y=k$ are circles, while traces in $x=k$ and $z=k$ are hyperbolas.

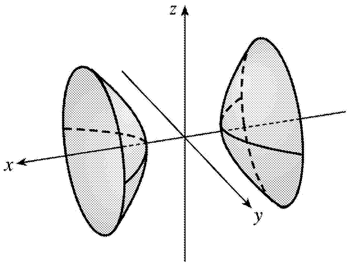


(c) Completing the square in y gives $x^2+(y+1)^2-z^2=1$. The surface is a hyperboloid identical to the one in part (a) but shifted one unit in the negative y –direction.

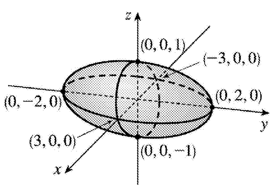


10. **(a)** The traces of $-x^2-y^2+z^2=1$ in $x=k$ are $-y^2+z^2=1+k^2$, a family of hyperbolas, as are the traces in $y=k$, $-x^2+z^2=1+k^2$. The traces in $z=k$ are $x^2+y^2=k^2-1$, a family of circles for $|k|>1$. As $|k|$ increases, the radii of the circles increase; the traces are empty for $|k|<1$. This behavior, combined with the vertical traces, gives the graph of the hyperboloid of two sheets in Table 1.

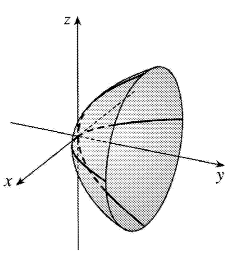
(b) The graph has the same shape as the hyperboloid in part (a) but is rotated so that its axis is the x –axis. Traces in $x=k$, $|k|>1$, are circles, while traces in $y=k$ and $z=k$ are hyperbolas.



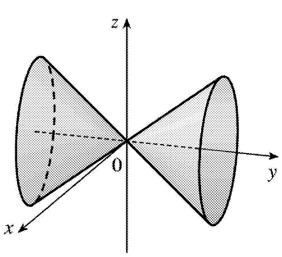
11. Traces: $x=k$, $9y^2+36z^2=36-4k^2$, an ellipse for $|k|<3$;
$y=k$, $4x^2+36z^2=36-9k^2$, an ellipse for $|k|<2$; $z=k$, $4x^2+9y^2=36(1-k^2)$, an ellipse for $|k|<1$. Thus the surface is an ellipsoid with center at the origin and axes along the x –, y – and z –axes.



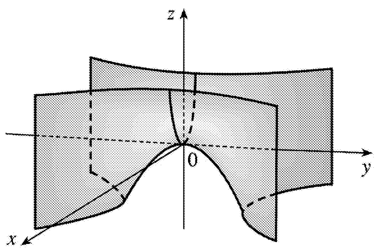
12. Traces: $x=k$, $4y=k^2+z^2$, a parabola; $y=k$, $4k=x^2+z^2$, a circle for $k>0$; $z=k$, $4y=x^2+k^2$ a parabola. Thus the surface is a circular paraboloid with axis the y –axis and vertex at $(0,0,0)$.



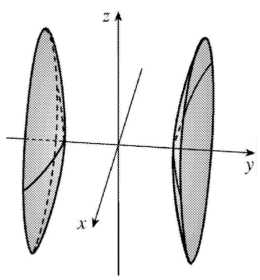
13. Traces: $x=k$, $y^2=k^2+z^2$ or $y^2-z^2=k^2$, a hyperbola for $k\neq 0$ and two intersecting lines for $k=0$; $y=k$, $x^2+z^2=k^2$, a circle for $k\neq 0$; $z=k$, $y^2=x^2+k^2$ or $y^2-x^2=k^2$, a hyperbola for $k\neq 0$ and two intersecting lines for $k=0$. Thus the surface is a cone (right circular) with axis the y –axis and vertex the origin.



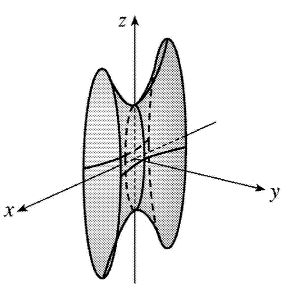
14. Traces: $x=k$, $z-k^2=-y^2$, a parabola; $y=k$, $z+k^2=x^2$, a parabola; $z=k$, $x^2-y^2=k$, a hyperbola. Thus the surface is a hyperbolic paraboloid with saddle point (0 , 0 , 0) (and since $c>0$, the saddle is upside down).



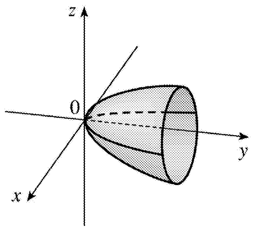
15. Traces: $x=k$, $4y^2-z^2=4+k^2$, a hyperbola; $y=k$, $x^2+z^2=4k^2-4$, a circle for $|k|>1$; $z=k$, $4y^2-x^2=4+k^2$, a hyperbola. Thus the surface is a hyperboloid of two sheets with axis the y –axis.



16. Traces: $x=k$, $25y^2+z^2=100+4k^2$, an ellipse; $y=k$, $25k^2+z^2=100+4x^2$ or $z^2-4x^2=100-25k^2$, a hyperbola for $|k|<2$; $z=k$, $25y^2+k^2=100+4x^2$ or $25y^2-4x^2=100-k^2$, a hyperbola for $|k|<10$. Thus the surface is a hyperboloid of one sheet with axis the x –axis.

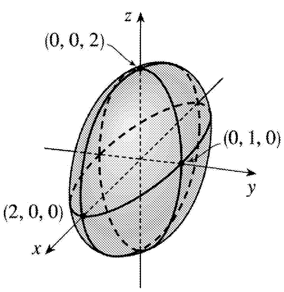


17. Traces: $x=k$, $k^2+4z^2-y=0$ or $y-k^2=4z^2$, a parabola;
$y=k$, $x^2+4z^2=k$, an ellipse for $k>0$; $z=k$, $x^2+4k^2-y=0$ or $y-4k^2=x^2$, a parabola. Thus the surface is an elliptic paraboloid with axis the y –axis and vertex the origin.

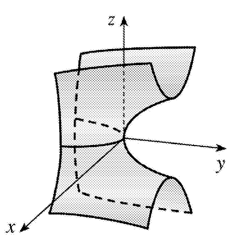


18. Traces: $x=k$, $|k|\leq 2 \Rightarrow y^2+\frac{z^2}{4}=1-\frac{k^2}{4}$, ellipses;

$y=k$, $|k|\leq 1 \Rightarrow x^2+z^2=4\left(1-k^2\right)$, circles; $z=k$, $|k|\leq 2 \Rightarrow \frac{x^2}{4}+y^2=1-\frac{k^2}{4}$, ellipses. $x^2+4y^2+z^2=4 \Leftrightarrow$ $\frac{x^2}{2^2}+\frac{y^2}{1^2}+\frac{z^2}{2^2}=1$, which is the equation of an ellipsoid.

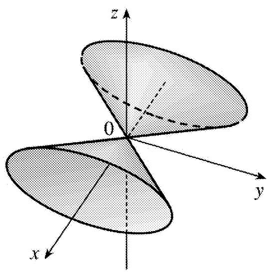


19. $y=z^2-x^2$. The traces in $x=k$ are the parabolas $y=z^2-k^2$;
the traces in $y=k$ are $k=z^2-x^2$, which are hyperbolas (note the hyperbolas are oriented differently for $k>0$ than for $k<0$); and the traces in $z=k$ are the parabolas $y=k^2-x^2$. Thus, $\frac{y}{1}=\frac{z^2}{1^2}-\frac{x^2}{1^2}$ is a hyperbolic paraboloid.



20. Traces: $x=k \Rightarrow y^2+4z^2=16k^2$, ellipses; $y=k \Rightarrow 16x^2-4z^2=k^2$, hyperbolas if $k\neq 0$ and two intersecting lines if $k=0$; $z=k \Rightarrow 16x^2-y^2=4k^2$, hyperbolas if $k\neq 0$ and two intersecting lines if $k=0$.
$16x^2=y^2+4z^2 \Leftrightarrow$

$x^2 = \frac{y^2}{4^2} + \frac{z^2}{2^2}$ is an elliptic cone with axis the x –axis and vertex the origin.



21. This is the equation of an ellipsoid: $x^2+4y^2+9z^2=x^2+\frac{y^2}{(1/2)^2}+\frac{z^2}{(1/3)^2}=1$, with x –intercepts ± 1 , y –intercepts $\pm\frac{1}{2}$ and z –intercepts $\pm\frac{1}{3}$. So the major axis is the x –axis and the only possible graph is VII.

22. This is the equation of an ellipsoid: $9x^2+4y^2+z^2=\frac{x^2}{(1/3)^2}+\frac{y^2}{(1/2)^2}+z^2=1$, with x –intercepts $\pm\frac{1}{3}$, y –intercepts $\pm\frac{1}{2}$ and z –intercepts ± 1 . So the major axis is the z –axis and the only possible graph is IV.

23. This is the equation of a hyperboloid of one sheet, with $a=b=c=1$. Since the coefficient of y^2 is negative, the axis of the hyperboloid is the y – axis, hence the correct graph is II.

24. This is a hyperboloid of two sheets, with $a=b=c=1$. This surface does not intersect the xz –plane at all, so the axis of the hyperboloid is the y –axis and the graph is III.

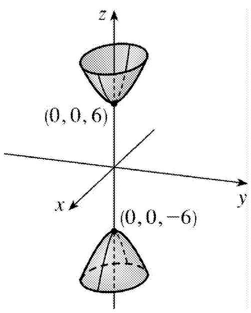
25. There are no real values of x and z that satisfy this equation for $y<0$, so this surface does not extend to the left of the xz –plane. The surface intersects the plane $y=k>0$ in an ellipse. Notice that y occurs to the first power whereas x and z occur to the second power. So the surface is an elliptic paraboloid with axis the y –axis. Its graph is VI.

26. This is the equation of a cone with axis the y –axis, so the graph is I.

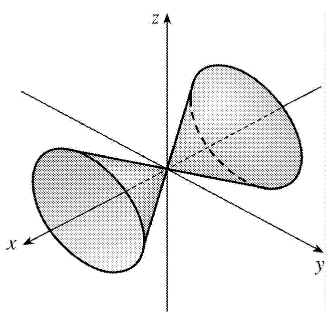
27. This surface is a cylinder because the variable y is missing from the equation. The intersection of the surface and the xz –plane is an ellipse. So the graph is VIII.

28. This is the equation of a hyperbolic paraboloid. The trace in the $xy-$ plane is the parabola $y=x^2$. So the correct graph is V.

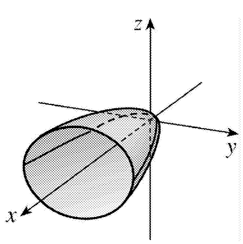
29. $z^2=4x^2+9y^2+36$ or $-4x^2-9y^2+z^2=36$ or $-\frac{x^2}{9}-\frac{y^2}{4}+\frac{z^2}{36}=1$ represents a hyperboloid of two sheets with axis the z –axis.



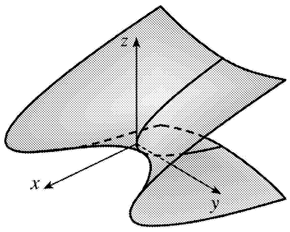
30. $x^2=2y^2+3z^2$ or $x^2=\frac{y^2}{1/2}+\frac{z^2}{1/3}$ or $\frac{x^2}{6}=\frac{y^2}{3}+\frac{z^2}{2}$ represents an elliptic cone with vertex $(0,0,0)$ and axis the x –axis.



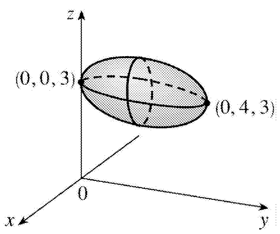
31. $x=2y^2+3z^2$ or $x=\frac{y^2}{1/2}+\frac{z^2}{1/3}$ or $\frac{x}{6}=\frac{y^2}{3}+\frac{z^2}{2}$ represents an elliptic paraboloid with vertex $(0,0,0)$ and axis the x –axis.



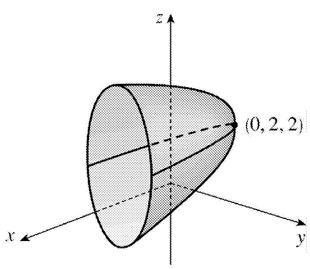
32. $4x-y^2+4z^2=0$ or $4x=y^2-4z^2$ or $x=\frac{y^2}{4}-z^2$ represents a hyperbolic paraboloid with center $(0,0,0)$.



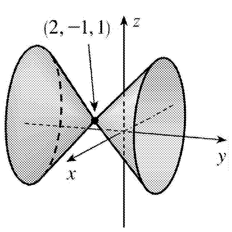
33. Completing squares in y and z gives $4x^2+(y-2)^2+4(z-3)^2=4$ or $x^2+\frac{(y-2)^2}{4}+(z-3)^2=1$, an ellipsoid with center $(0,2,3)$.



34. Completing squares in y and z gives $4(y-2)^2+(z-2)^2-x=0$ or $\frac{x}{4}=(y-2)^2+\frac{(z-2)^2}{4}$, an elliptic paraboloid with vertex $(0,2,2)$ and axis the horizontal line $y=2$, $z=2$.

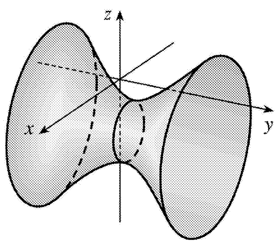


35. Completing squares in all three variables gives $(x-2)^2-(y+1)^2+(z-1)^2=0$ or $(y+1)^2=(x-2)^2+(z-1)^2$, a circular cone with center $(2,-1,1)$ and axis the horizontal line $x=2$, $z=1$.

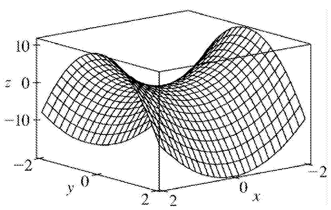


36. Completing squares in all three variables gives $(x-1)^2-(y-1)^2+(z+2)^2=2$ or

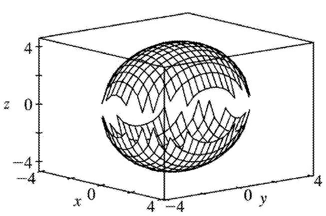
$\frac{(x-1)^2}{2} - \frac{(y-1)^2}{2} + \frac{(z+2)^2}{2} = 1$, a hyperboloid of one sheet with center $(1,1,-2)$ and axis the horizontal line $x=1$, $z=-2$.



37.

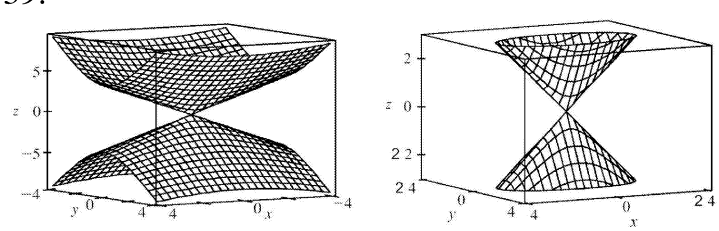


38.



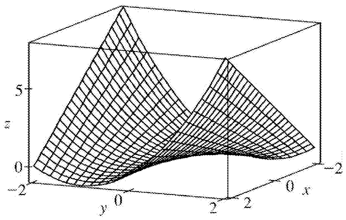
In Section 17.6 [ET 16.6], we will be able to graph ellipsoids without gaps; see Exercise 17.6.53 [ET 16.6.53].

39.

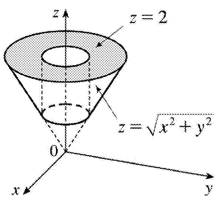


To restrict the z –range as in the second graph, we can use the option *view* = $-2..2$ in Maple's *plot3d* command, or *PlotRange* –> $\{-2, 2\}$ in Mathematica's *Plot3D* command.

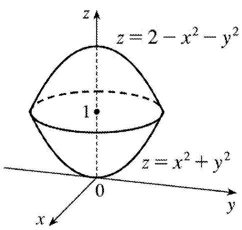
40.



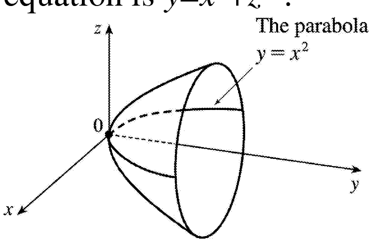
41.



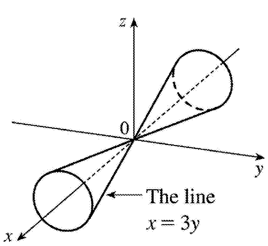
42.



43. The surface is a paraboloid of revolution (circular paraboloid) with vertex at the origin, axis the y –axis and opens to the right. Thus the trace in the yz –plane is also a parabola: $y=z^2$, $x=0$. The equation is $y=x^2+z^2$.



44. The surface is a right circular cone with vertex at $(0,0,0)$ and axis the x –axis. For $x=k\neq 0$, the trace is a circle with center $(k,0,0)$ and radius $r=y=\frac{x}{3}=\frac{k}{3}$. Thus the equation is $\frac{1}{3}x^2=y^2+z^2$.



45. Let $P=(x,y,z)$ be an arbitrary point equidistant from $(-1,0,0)$ and the plane $x=1$. Then the

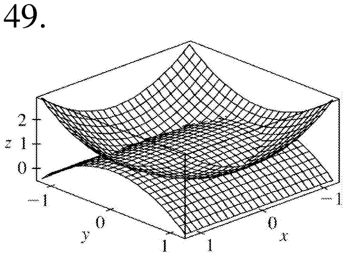
distance from P to $(-1,0,0)$ is $\sqrt{(x+1)^2+y^2+z^2}$ and the distance from P to the plane $x=1$ is $|x-1|/\sqrt{1^2}=|x-1|$ (by Equation 13.5.9 [ET 12.5.9]). So $|x-1|=\sqrt{(x+1)^2+y^2+z^2} \Leftrightarrow$ $(x-1)^2=(x+1)^2+y^2+z^2 \Leftrightarrow x^2-2x+1=x^2+2x+1+y^2+z^2 \Leftrightarrow -4x=y^2+z^2$. Thus the collection of all such points P is a circular paraboloid with vertex at the origin, axis the x –axis, which opens in the negative direction.

46. Let $P=(x,y,z)$ be an arbitrary point whose distance from the x –axis is twice its distance from the yz –plane. The distance from P to the x –axis is $\sqrt{(x-x)^2+y^2+z^2}=\sqrt{y^2+z^2}$ and the distance from P to the yz –plane ($x=0$) is $|x|/1=|x|$. Thus $\sqrt{y^2+z^2}=2|x| \Leftrightarrow y^2+z^2=4x^2 \Leftrightarrow x^2=\left(y^2/2^2\right)+\left(z^2/2^2\right)$. So the surface is a right circular cone with vertex the origin and axis the x –axis.

47. If (a,b,c) satisfies $z=y^2-x^2$, then $c=b^2-a^2$. L_1 : $x=a+t$, $y=b+t$, $z=c+2(b-a)t$, L_2 : $x=a+t$, $y=b-t$, $z=c-2(b+a)t$. Substitute the parametric equations of L_1 into the equation of the hyperbolic paraboloid in order to find the points of intersection: $z=y^2-x^2 \Rightarrow$ $c+2(b-a)t=(b+t)^2-(a+t)^2=b^2-a^2+2(b-a)t \Rightarrow c=b^2-a^2$. As this is true for all values of t , L_1 lies on $z=y^2-x^2$. Performing similar operations with L_2 gives: $z=y^2-x^2 \Rightarrow$ $c-2(b+a)t=(b-t)^2-(a+t)^2=b^2-a^2-2(b+a)t \Rightarrow c=b^2-a^2$. This tells us that all of L_2 also lies on $z=y^2-x^2$.

48. Any point on the curve of intersection must satisfy both $2x^2+4y^2-2z^2+6x=2$ and $2x^2+4y^2-2z^2-5y=0$. Subtracting, we get $6x+5y=2$, which is linear and therefore the equation of a plane. Thus the curve of intersection lies in this plane.

49.



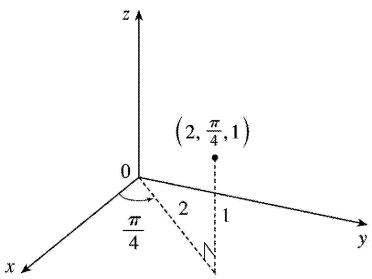
The curve of intersection looks like a bent ellipse. The projection of this curve onto the xy –plane is the set of points $(x,y,0)$ which satisfy $x^2+y^2=1-y^2 \Leftrightarrow x^2+2y^2=1 \Leftrightarrow$

$x^2+\dfrac{y^2}{\left(1/\sqrt{2}\right)^2}=1$. This is an equation of an ellipse.

1. See Figure 1 and the accompanying discussion; see the paragraph accompanying Figure 3.

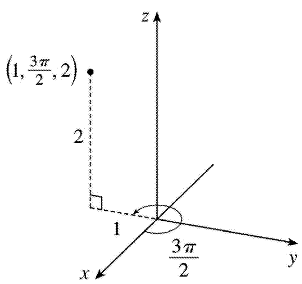
2. See Figure 5 and the accompanying discussion.

3.



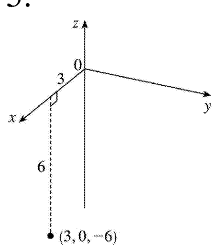
$x=2\cos\frac{\pi}{4}=\sqrt{2}$, $y=2\sin\frac{\pi}{4}=\sqrt{2}$, $z=1$, so the point is $(\sqrt{2},\sqrt{2},1)$ in rectangular coordinates.

4.



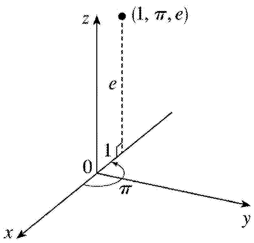
$x=1\cos\frac{3\pi}{2}=0$, $y=1\sin\frac{3\pi}{2}=-1$, $z=2$, so the point is $(0,-1,2)$ in rectangular coordinates.

5.



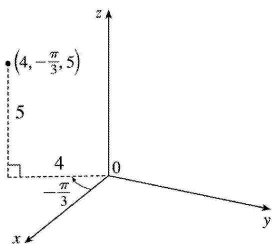
$x=3\cos 0=3$, $y=3\sin 0=0$, and $z=-6$, so the point is $(3,0,-6)$ in rectangular coordinates.

6.



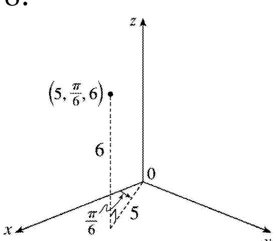
$x=1\cos\pi=-1$, $y=1\sin\pi=0$, and $z=e$, so the point is $(-1,0,e)$ in rectangular coordinates.

7.



$x=4\cos\left(-\frac{\pi}{3}\right)=2$, $y=4\sin\left(-\frac{\pi}{3}\right)=-2\sqrt{3}$, and $z=5$, so the point is $(2,-2\sqrt{3},5)$ in rectangular coordinates.

8.



$x=5\cos\left(\frac{\pi}{6}\right)=\frac{5\sqrt{3}}{2}$, $y=5\sin\left(\frac{\pi}{6}\right)=\frac{5}{2}$, and $z=6$, so the point is $(\frac{5\sqrt{3}}{2},\frac{5}{2},6)$ in rectangular coordinates.

9. $r^2=x^2+y^2=1^2+(-1)^2=2$ so $r=\sqrt{2}$; $\tan\theta=\frac{y}{x}=\frac{-1}{1}=-1$ and the point $(1,-1)$ is in the fourth quadrant of the xy –plane, so $\theta=\frac{7\pi}{4}+2n\pi$; $z=4$. Thus, one set of cylindrical coordinates is $\left(\sqrt{2},\frac{7\pi}{4},4\right)$.

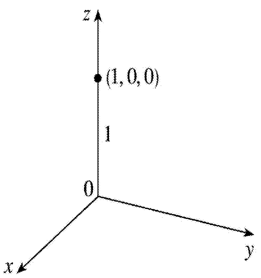
10. $r^2=x^2+y^2=3^2+3^2=18$ so $r=\sqrt{18}=3\sqrt{2}$; $\tan\theta=\frac{y}{x}=\frac{3}{3}=1$ and the point $(3,3)$ is in the first quadrant of the xy –plane, so $\theta=\frac{\pi}{4}+2n\pi$; $z=-2$. Thus, one set of cylindrical coordinates is $\left(3\sqrt{2},\frac{\pi}{4},-2\right)$.

11. $r^2=(-1)^2+(-\sqrt{3})^2=4$ so $r=2$; $\tan\theta=\frac{-\sqrt{3}}{-1}=\sqrt{3}$ and the point $(-1,-\sqrt{3})$ is in the third quadrant of

the xy –plane, so $\theta=\frac{4\pi}{3}+2n\pi$; $z=2$. Thus, one set of cylindrical coordinates is $\left(2,\frac{4\pi}{3},2\right)$.

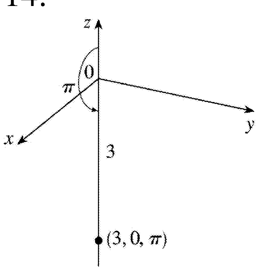
12. $r^2=3^2+4^2=25$ so $r=5$; $\tan\theta=\frac{4}{3}$ and the point $(3,4)$ is in the first quadrant of the xy –plane, so $\theta=\tan^{-1}\left(\frac{4}{3}\right)+2n\pi\approx0.93+2n\pi$; $z=5$. Thus, one set of cylindrical coordinates is $\left(5,\tan^{-1}\left(\frac{4}{3}\right),5\right)\approx(5,0.93,5)$.

13.



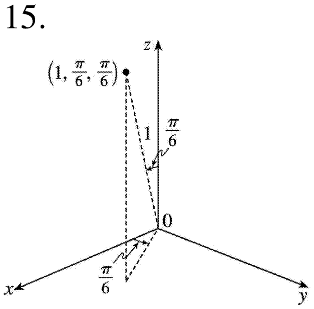
$x=\rho\sin\phi\cos\theta=(1)\sin 0\cos 0=0$, $y=\rho\sin\phi\sin\theta=(1)\sin 0\sin 0=0$, and $z=\rho\cos\phi=(1)\cos 0=1$ so the point is $(0,0,1)$ in rectangular coordinates.

14.



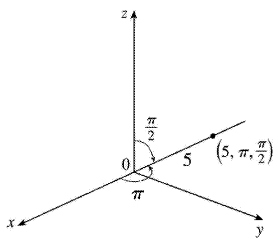
$x=3\sin\pi\cos0=0$, $y=3\sin\pi\sin0=0$, $z=3\cos\pi=-3$ and in rectangular coordinates the point is $(0,0,-3)$.

15.



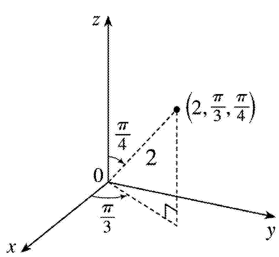
$x = \sin\frac{\pi}{6}\cos\frac{\pi}{6} = \frac{\sqrt{3}}{4}$, $y = \sin\frac{\pi}{6}\sin\frac{\pi}{6} = \frac{1}{4}$, and $z = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, so the point is $\left(\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right)$ in rectangular coordinates.

16.

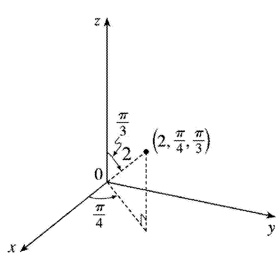


$x = 5\sin\frac{\pi}{2}\cos\pi = -5$, $y = 5\sin\frac{\pi}{2}\sin\pi = 0$, $z = 5\cos\frac{\pi}{2} = 0$ so the point is $(-5,0,0)$ in rectangular coordinates.

17. $x = 2\sin\frac{\pi}{4}\cos\frac{\pi}{3} = \frac{\sqrt{2}}{2}$, $y = 2\sin\frac{\pi}{4}\sin\frac{\pi}{3} = \frac{\sqrt{6}}{2}$, $z = 2\cos\frac{\pi}{4} = \sqrt{2}$ so the point is $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}, \sqrt{2}\right)$ in rectangular coordinates.



18. $x = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4} = \frac{\sqrt{6}}{2}$, $y = 2\sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{6}}{2}$, $z = 2\cos\frac{\pi}{3} = 1$ so the point is $\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}, 1\right)$ in rectangular coordinates.



19. $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{1+3+12} = 4$, $\cos\phi = \frac{z}{\rho} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6}$, and

$\cos\theta = \frac{x}{\rho\sin\phi} = \frac{1}{4\sin(\pi/6)} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ (since $y>0$). Thus spherical coordinates are $\left(4, \frac{\pi}{3}, \frac{\pi}{6}\right)$.

20. $\rho=\sqrt{x^2+y^2+z^2}=\sqrt{0+3+1}=2$, $\cos\phi=\frac{z}{\rho}=\frac{1}{2}\Rightarrow\phi=\frac{\pi}{3}$, and $\cos\theta=\frac{x}{\rho\sin\phi}=\frac{0}{2\sin(\pi/3)}=0\Rightarrow$ $\theta=\frac{\pi}{2}$ (since $y>0$). Thus spherical coordinates are $\left(2,\frac{\pi}{2},\frac{\pi}{3}\right)$.

21. $\rho=\sqrt{0+1+1}=\sqrt{2}$, $\cos\phi=\frac{-1}{\sqrt{2}}\Rightarrow\phi=\frac{3\pi}{4}$, and $\cos\theta=\frac{0}{\sqrt{2}\sin(3\pi/4)}=0\Rightarrow\theta=\frac{3\pi}{2}$ (since $y<0$). Thus spherical coordinates are $\left(\sqrt{2},\frac{3\pi}{2},\frac{3\pi}{4}\right)$.

22. $\rho=\sqrt{1+1+6}=2\sqrt{2}$, $\cos\phi=\frac{\sqrt{6}}{2\sqrt{2}}=\frac{\sqrt{3}}{2}\Rightarrow\phi=\frac{\pi}{6}$, and $\cos\theta=\frac{-1}{2\sqrt{2}\sin(\pi/6)}=-\frac{1}{\sqrt{2}}\Rightarrow\theta=\frac{3\pi}{4}$ (since $y>0$). Thus spherical coordinates are $\left(2\sqrt{2},\frac{3\pi}{4},\frac{\pi}{6}\right)$.

23. $\rho=\sqrt{x^2+y^2+z^2}=\sqrt{r^2+z^2}=\sqrt{1+3}=2$; $\theta=\frac{\pi}{6}$; $\cos\phi=\frac{z}{\rho}=\frac{\sqrt{3}}{2}\Rightarrow\phi=\frac{\pi}{6}$, thus in spherical coordinates the point is $\left(2,\frac{\pi}{6},\frac{\pi}{6}\right)$.

24. $\rho=\sqrt{r^2+z^2}=\sqrt{6+2}=2\sqrt{2}$; $\theta=\frac{\pi}{4}$; $\cos\phi=\frac{z}{\rho}=\frac{\sqrt{2}}{2\sqrt{2}}=\frac{1}{2}\Rightarrow\phi=\frac{\pi}{3}$, thus in spherical coordinates the point is $\left(2\sqrt{2},\frac{\pi}{4},\frac{\pi}{3}\right)$.

25. $\rho=\sqrt{r^2+z^2}=\sqrt{3+1}=2$; $\theta=\frac{\pi}{2}$; $\cos\phi=\frac{z}{\rho}=\frac{-1}{2}\Rightarrow\phi=\frac{2\pi}{3}$, so in spherical coordinates the point is $\left(2,\frac{\pi}{2},\frac{2\pi}{3}\right)$.

26. $\rho=\sqrt{16+9}=5$; $\theta=\frac{\pi}{8}$; $\cos\phi=\frac{3}{5}\Rightarrow\phi=\cos^{-1}\left(\frac{3}{5}\right)$, so in spherical coordinates the point is $\left(5,\frac{\pi}{8},\cos^{-1}\left(\frac{3}{5}\right)\right)\approx\left(5,\frac{\pi}{8},0.927\right)$.

27. $z=\rho\cos\phi=2\cos 0=2$, $\rho^2=x^2+y^2+z^2=r^2+z^2\Rightarrow r=\sqrt{\rho^2-z^2}=\sqrt{2^2-2^2}=0$, (or $r=2\sin 0=0$), $\theta=0$ and the point is $(0,0,2)$.

28. $z=2\sqrt{2}\cos\frac{\pi}{2}=0$, $r=2\sqrt{2}\sin\frac{\pi}{2}=2\sqrt{2}$, $\theta=\frac{3\pi}{2}$ and the point is $\left(2\sqrt{2},\frac{3\pi}{2},0\right)$.

29.

$z=8\cos\frac{\pi}{2}=0$, $r=8\sin\frac{\pi}{2}=8$, $\theta=\frac{\pi}{6}$ and the point is $\left(8,\frac{\pi}{6},0\right)$.

30. $z=4\cos\frac{\pi}{3}=2$, $r=4\sin\frac{\pi}{3}=2\sqrt{3}$, $\theta=\frac{\pi}{4}$ and the point is $\left(2\sqrt{3},\frac{\pi}{4},2\right)$.

31. Since $r=3$, $x^2+y^2=9$ and the surface is a circular cylinder with radius 3 and axis the z –axis.

32. Since $\rho=3$, $x^2+y^2+z^2=9$ and the surface is a sphere with center the origin and radius 3 .

33. Since $\phi=0$, $x=0$ and $y=0$ while $z=\rho\geq 0$. Thus the ''surface'' is the positive z –axis including the origin.

34. Since $\phi=\frac{\pi}{2}$, $z=0$ but there are no restrictions on x and y ($x=\rho\cos\theta$, $y=\rho\sin\theta$). Thus the surface is the xy –plane.

35. Since $\phi=\frac{\pi}{3}$, the surface is the top half of the right circular cone with vertex at the origin and axis the positive z –axis.

36. Whether spherical or cylindrical coordinates, since $\theta=\frac{\pi}{3}$ the surface is a half–plane including the z –axis and intersecting the xy –plane in the half–line $y=\sqrt{3}\,x$, $x>0$.

37. $z=r^2=x^2+y^2$, so the surface is a circular paraboloid with vertex at the origin and axis the positive z –axis.

38. Since $r=4\sin\theta$ and $y=r\sin\theta$, $y=4\sin^2\theta$. Also $r^2=x^2+y^2$ so $x^2+y^2=16\sin^2\theta$. Thus $x^2+y^2-4y=16\sin^2\theta-16\sin^2\theta=0$ or $x^2+(y-2)^2=4$, a circular cylinder of radius 2 and with axis parallel to the z –axis.

39. $2=\rho\cos\phi=z$ is a plane through the point $(0,0,2)$ and parallel to the xy –plane.

40. Since $\rho\sin\phi=2$ and $x=\rho\sin\phi\cos\theta$, $x=2\cos\theta$. Also $y=\rho\sin\phi\sin\theta$ so $y=2\sin\theta$. Then $x^2+y^2=4\cos^2\theta+4\sin^2\theta=4$, a circular cylinder of radius 2 about the z –axis.

41. $r=2\cos\theta\Rightarrow r^2=x^2+y^2=2r\cos\theta=2x\Leftrightarrow(x-1)^2+y^2=1$, which is the equation of a circular cylinder with radius 1 , whose axis is the vertical line $x=1$, $y=0$, $z=z$.

42. $\rho=2\cos\phi \Rightarrow \rho^2=2\rho\cos\phi=2z \Leftrightarrow x^2+y^2+z^2=2z \Leftrightarrow x^2+y^2+(z-1)^2=1$. Therefore, the surface is a sphere of radius 1 centered at $(0,0,1)$.

43. Since $r^2+z^2=25$ and $r^2=x^2+y^2$, we have $x^2+y^2+z^2=25$, a sphere with radius 5 and center at the origin.

44. Since $r^2-2z^2=4$ and $r^2=x^2+y^2$, we have $x^2+y^2-2z^2=4$ or $\frac{1}{4}x^2+\frac{1}{4}y^2-\frac{1}{2}z^2=1$, a hyperboloid of one sheet with axis the z -axis.

45. Since $x^2=\rho^2\sin^2\phi\cos^2\theta$ and $z^2=\rho^2\cos^2\phi$, the equation of the surface in rec tan gular coordinates is $x^2+z^2=4$. Thus the surface is a circular cylinder of radius 2 about the y -axis.

46. Since $\rho^2\left(\sin^2\phi-4\cos^2\phi\right)=1$, $\rho^2\left(\sin^2\phi-4\cos^2\phi\right)+\rho^2\cos^2\phi-\rho^2\cos^2\phi=1$ or $\rho^2\left(\sin^2\phi+\cos^2\phi-5\cos^2\phi\right)=1$ or $\rho^2\left(1-5\cos^2\phi\right)=1$. But $\rho^2=x^2+y^2+z^2$ and $z^2=\rho^2\cos^2\phi$, so we can rewrite the equation of the surface as $x^2+y^2+z^2-5z^2=1$ or $x^2+y^2-4z^2=1$. Thus the surface is a hyperboloid of one sheet with axis the z -axis.

47. Since $r^2-r=0$, $r=0$ or $r=1$. But $x^2+y^2=r^2$. Thus the surface consists of the right circular cylinder of radius 1 and axis the z -axis along with the surface given by $x^2+y^2=0$, that is, the z -axis.

48. Since $\rho^2-6\rho+8=0$, either $\rho=2$ or $\rho=4$. Thus the surface consists of two concentric spheres (centered at the origin), one with radius 2 and the other with radius 4 .

49. **(a)** $x^2+y^2=r^2$, so the equation becomes $z=r^2$.
(b) $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, and $z=\rho\cos\phi$, so the equation becomes $\rho\cos\phi=(\rho\sin\phi\cos\theta)^2+(\rho\sin\phi\sin\theta)^2$ or $\rho\cos\phi=\rho^2\sin^2\phi$ or $\rho\sin^2\phi=\cos\phi$.

50. **(a)** $x^2+y^2=r^2$, so the equation becomes $r^2+z^2=2$.
(b) $x^2+y^2+z^2=\rho^2$, so the equation becomes $\rho^2=2$ or $\rho=\sqrt{2}$.

51. **(a)** $x=r\cos\theta$, so the equation becomes $r\cos\theta=3$ or $r=3\sec\theta$ (since $\cos\theta\neq0$ here).
(b) $x=\rho\sin\phi\cos\theta$, so the equation becomes $\rho\sin\phi\cos\theta=3$.

52. **(a)** $x^2+y^2=r^2$, so the equation becomes $r^2+z^2+2z=0$ or $r^2+(z+1)^2=1$.

(b) $x^2+y^2+z^2=\rho^2$ and $z=\rho\cos\phi$, so the equation becomes $\rho^2+2\rho\cos\phi=0$ or $\rho=-2\cos\phi$.

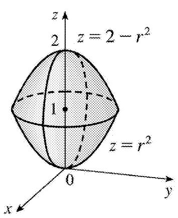
53. **(a)** $r^2(\cos^2\theta-\sin^2\theta)-2z^2=4$ or $2z^2=r^2\cos 2\theta-4$.
(b) $\rho^2\left(\sin^2\phi\cos^2\theta-\sin^2\phi\sin^2\theta-2\cos^2\phi\right)=4$ or $\rho^2\left(\sin^2\phi\cos 2\theta-2\cos^2\phi\right)=4$.

54. **(a)** $r^2\sin^2\theta+z^2=1$
(b) $\rho^2\sin^2\phi\sin^2\theta+\rho^2\cos^2\phi=1$ or $\rho^2\left(\sin^2\phi\sin^2\theta+\cos^2\phi\right)=1$.

55. **(a)** $r^2=2r\sin\theta$ or $r=2\sin\theta$.
(b) $\rho^2\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right)=2\rho\sin\phi\sin\theta$ or $\rho\sin^2\phi=2\sin\phi\sin\theta$ or $\rho\sin\phi=2\sin\theta$.

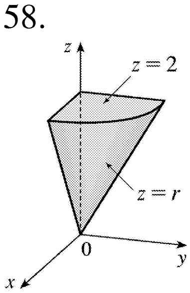
56. **(a)** $z=r^2(\cos^2\theta-\sin^2\theta)$ or $z=r^2\cos 2\theta$.
(b) $\rho\cos\phi=\rho^2\sin^2\phi(\cos^2\theta-\sin^2\theta)$ or $\cos\phi=\rho\sin^2\phi\cos 2\theta$.

57.



$z=r^2=x^2+y^2$ is a circular paraboloid with vertex (0,0,0) , opening upward. $z=2-r^2 \Rightarrow z-2=-\left(x^2+y^2\right)$ is a circular paraboloid with vertex (0,0,2) opening downward. Thus $r^2\leq z\leq 2-r^2$ is the solid region enclosed by these two surfaces.

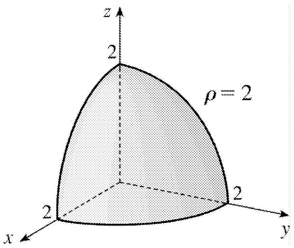
58.



$z=r=\sqrt{x^2+y^2}$ is a cone that opens upward. Thus $r\leq z\leq 2$ is the region above this cone and beneath the horizontal plane $z=2$.

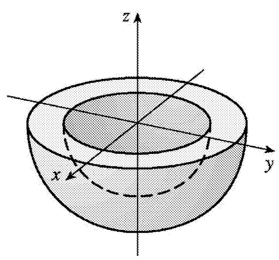
$0 \le \theta \le \dfrac{\pi}{2}$ restricts the solid to that part of this region in the first octant.

59.



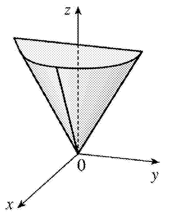
$\rho = 2$ represents a sphere of radius 2, centered at the origin, so $\rho \le 2$ is this sphere and its interior. $0 \le \phi \le \dfrac{\pi}{2}$ restricts the solid to that portion of the region that lies on or above the xy –plane, and $0 \le \theta \le \dfrac{\pi}{2}$ further restricts the solid to the first octant. Thus the solid is the portion in the first octant of the solid ball centered at the origin with radius 2.

60.

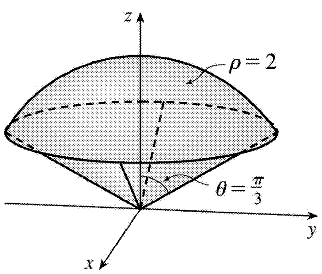


$2 \le \rho \le 3$ represents the solid region between and including the spheres of radii 2 and 3, centered at the origin. $\dfrac{\pi}{2} \le \phi \le \pi$ restricts the solid to that portion on or below the xy –plane.

61. $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$ restricts the solid to the 4 octants in which x is positive. $\rho = \sec\phi \Rightarrow \rho\cos\phi = z = 1$, which is the equation of a horizontal plane. $0 \le \phi \le \dfrac{\pi}{6}$ describes a cone, opening upward. So the solid lies above the cone $\phi = \dfrac{\pi}{6}$ and below the plane $z = 1$.



62. $\rho=2 \Leftrightarrow x^2+y^2+z^2=4$, which is a sphere of radius 2 , centered at the origin. Hence $\rho \leq 2$ is this sphere and its interior. $0 \leq \phi \leq \frac{\pi}{3}$ restricts the solid to that section of this ball that lies above the cone $\phi = \frac{\pi}{3}$.

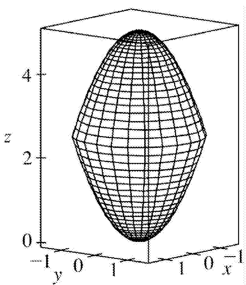


63. We can position the cylindrical shell vertically so that its axis coincides with the z –axis and its base lies in the xy –plane. If we use centimeters as the unit of measurement, then cylindrical coordinates conveniently describe the shell as $6 \leq r \leq 7$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq 20$.

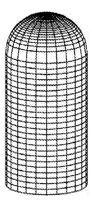
64. **(a)** The hollow ball is a spherical shell with outer radius 15 cm and inner radius 14.5 cm. If we center the ball at the origin of the coordinate system and use centimeters as the unit of measurement, then spherical coordinates conveniently describe the hollow ball as $14.5 \leq \rho \leq 15$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$.
(b) If we position the ball as in part (a), one possibility is to take the half of the ball that is above the xy –plane which is described by $14.5 \leq \rho \leq 15$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi/2$.

65. $z \geq \sqrt{x^2+y^2}$ because the solid lies above the cone. Squaring both sides of this inequality gives $z^2 \geq x^2+y^2 \Rightarrow 2z^2 \geq x^2+y^2+z^2=\rho^2 \Rightarrow z^2=\rho^2\cos^2\phi \geq \frac{1}{2}\rho^2 \Rightarrow \cos^2\phi \geq \frac{1}{2}$. The cone opens upward so that the inequality is $\cos\phi \geq \frac{1}{\sqrt{2}}$, or equivalently $0 \leq \phi \leq \frac{\pi}{4}$. In spherical coordinates the sphere $z=x^2+y^2+z^2$ is $\rho\cos\phi=\rho^2 \Rightarrow \rho=\cos\phi$. $0 \leq \rho \leq \cos\phi$ because the solid lies below the sphere. The solid can therefore be described as the region in spherical coordinates satisfying $0 \leq \rho \leq \cos\phi$, $0 \leq \phi \leq \frac{\pi}{4}$.

66. In cylindrical coordinates, the equations are $z=r^2$ and $z=5-r^2$. The curve of intersection is $r^2=5-r^2$ or $r=\sqrt{5/2}$. So we graph the surfaces in cylindrical coordinates, with $0\le r\le\sqrt{5/2}$. In Maple, we can use either the *coords=cylindrical* option in a regular *plot* command, or the *plots[cylinderplot]* command. In Mathematica, we can use *ParametricPlot3d* .



67. In cylindrical coordinates, the equation of the cylinder is $r=3$, $0\le z\le 10$. The hemisphere is the upper part of the sphere radius 3 , center $(0,0,10)$, equation $r^2+(z-10)^2=3^2$, $z\ge 10$. In Maple, we can use either the *coords=cylindrical* option in a regular *plot* command, or the *plots[cylinderplot]* command. In Mathematica, we can use *ParametricPlot3d* .



68. We begin by finding the positions of Los Angeles and Montreal in spherical coordinates, using the method described in the exercise:

Montreal	Los Angeles
$\rho=3960$ mi $\theta=360°-73.60°=286.40°$ $\phi=90°-45.50°=44.50°$	$\rho=3960$ mi $\theta=360°-118.25°=241.75°$ $\phi=90°-34.06°=55.94°$

Now we change the above to Cartesian coordinates using $x=\rho\cos\theta\sin\phi$, $y=\rho\sin\theta\sin\phi$ and $z=\rho\cos\phi$ to get two position vectors of length 3960 mi (since both cities must lie on the surface of the Earth). In particular:

Montreal: $\langle 783.67,-2662.67,2824.47\rangle$ Los Angeles: $\langle -1552.80,-2889.91,2217.84\rangle$

To find the angle α between these two vectors we use the dot product:

$\langle 783.67, -2662.67, 2824.47\rangle \cdot \langle -1552.80, -2889.91, 2217.84\rangle = (3960)^2 \cos\alpha \Rightarrow \cos\alpha \approx 0.8126 \Rightarrow$ $\alpha \approx 0.6223$ rad. The great circle distance between the cities is $s = \rho\theta \approx 3960(0.6223) \approx 2464$ mi.

1. The component functions t^2 , $\sqrt{t-1}$, and $\sqrt{5-t}$ are all defined when $t-1\geq 0\Rightarrow t\geq 1$ and $5-t\geq 0\Rightarrow$ $t\leq 5$, so the domain of $\boldsymbol{r}(t)$ is $[1,5]$.

2. The component functions $\dfrac{t-2}{t+2}$, $\sin t$, and $\ln(9-t^2)$ are all defined when $t\neq -2$ and $9-t^2>0\Rightarrow$ $-3<t<3$, so the domain of $\boldsymbol{r}(t)$ is $(-3,-2)\cup(-2,3)$.

3. $\lim\limits_{t\to 0^+}\cos t=\cos 0=1$, $\lim\limits_{t\to 0^+}\sin t=\sin 0=0$, $\lim\limits_{t\to 0^+} t\ln t=\lim\limits_{t\to 0^+}\dfrac{\ln t}{1/t}=\lim\limits_{t\to 0^+}\dfrac{1/t}{-1/t^2}=\lim\limits_{t\to 0^+}-t=0$. Thus $\lim\limits_{t\to 0^+}\langle\cos t,\sin t,t\ln t\rangle=\left\langle\lim\limits_{t\to 0^+}\cos t,\lim\limits_{t\to 0^+}\sin t,\lim\limits_{t\to 0^+}t\ln t\right\rangle=\langle 1,0,0\rangle$.

4. $\lim\limits_{t\to 0}\dfrac{e^t-1}{t}=\lim\limits_{t\to 0}\dfrac{e^t}{1}=1$ [using l'Hospital's Rule],

$\lim\limits_{t\to 0}\dfrac{\sqrt{1+t}-1}{t}=\lim\limits_{t\to 0}\dfrac{\sqrt{1+t}-1}{t}\cdot\dfrac{\sqrt{1+t}+1}{\sqrt{1+t}+1}=\lim\limits_{t\to 0}\dfrac{1}{\sqrt{1+t}+1}=\dfrac{1}{2}$, $\lim\limits_{t\to 0}\dfrac{3}{1+t}=3$.

Thus the given limit equals $\left\langle 1,\dfrac{1}{2},3\right\rangle$.

5. $\lim\limits_{t\to 1}\sqrt{t+3}=2$, $\lim\limits_{t\to 1}\dfrac{t-1}{t^2-1}=\lim\limits_{t\to 1}\dfrac{1}{t+1}=\dfrac{1}{2}$, $\lim\limits_{t\to 1}\left(\dfrac{\tan t}{t}\right)=\tan 1$.

Thus the given limit equals $\left\langle 2,\dfrac{1}{2},\tan 1\right\rangle$.

6. $\lim\limits_{t\to\infty}\arctan t=\dfrac{\pi}{2}$, $\lim\limits_{t\to\infty}e^{-2t}=0$, $\lim\limits_{t\to\infty}\dfrac{\ln t}{t}=\lim\limits_{t\to\infty}\dfrac{1/t}{1}=0$ [by l'Hospital's Rule].

Thus $\lim\limits_{t\to\infty}\left\langle \arctan t, e^{-2t},\dfrac{\ln t}{t}\right\rangle=\left\langle\dfrac{\pi}{2},0,0\right\rangle$.

7. The corresponding parametric equations for this curve are $x=t^4+1$, $y=t$. We can make a table of values, or we can eliminate the parameter: $t=y\Rightarrow x=y^4+1$, with $y\in R$. By comparing different values of t , we find the direction in which t increases as indicated in the graph.

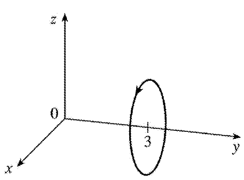
8. The corresponding parametric equations for this curve are $x=t^3, y=t^2$. We can make a table of values, or we can eliminate the parameter: $x=t^3\Rightarrow t=\sqrt[3]{x}\Rightarrow y=t^2=\left(\sqrt[3]{x}\right)^2=x^{2/3}$, with $t\in R\Rightarrow x\in R$. By

comparing different values of t , we find the direction in which t increases as indicated in the graph.

9. The corresponding parametric equations are $x=t$, $y=\cos 2t$, $z=\sin 2t$. Note that $y^2+z^2=\cos^2 2t+\sin^2 2t=1$, so the curve lies on the circular cylinder $y^2+z^2=1$. Since $x=t$, the curve is a helix.

10. The corresponding parametric equations are $x=1+t$, $y=3t$, $z=-t$, which are parametric equations of a line through the point $(1,0,0)$ and with direction vector $\langle 1,3,-1\rangle$.

11. The parametric equations give $x^2+z^2=\sin^2 t+\cos^2 t=1$, $y=3$, which is a circle of radius 1 , center $(0,3,0)$ in the plane $y=3$.



12. The parametric equations are $x=t$, $y=t$, $z=\cos t$. Thus $x=y$, so the curve must lie in the plane $x=y$. Combine this with $z=\cos t$ to determine that the curve traces out the cosine curve in the vertical plane $x=y$.

13. The parametric equations are $x=t^2$, $y=t^4$, $z=t^6$. These are positive for $t\neq 0$ and 0 when $t=0$. So the curve lies entirely in the first quadrant. The projection of the graph onto the xy –plane is $y=x^2$, $y>0$, a half parabola. On the xz –plane $z=x^3$, $z>0$, a half cubic, and the yz –plane, $y^3=z^2$.

14. The parametric equations give $x^2+y^2+z^2=2\sin^2 t+2\cos^2 t=2$, so the curve lies on the sphere with radius $\sqrt{2}$ and center $(0,0,0)$. Furthermore $x=y=\sin t$, so the curve is the intersection of this sphere with the plane $x=y$, that is, the curve is the circle of radius $\sqrt{2}$, center $(0,0,0)$ in the plane $x=y$.

15. Taking $\boldsymbol{r}_0=\langle 0,0,0\rangle$ and $\boldsymbol{r}_1=\langle 1,2,3\rangle$, we have from Equation 13.5.4
$\boldsymbol{r}(t)=(1-t)\boldsymbol{r}_0+t\boldsymbol{r}_1=(1-t)\langle 0,0,0\rangle+t\langle 1,2,3\rangle$, $0\leq t\leq 1$ or $\boldsymbol{r}(t)=\langle \mathrm{t},2\mathrm{t},3t\rangle$, $0\leq t\leq 1$.
Parametric equations are $x=t$, $y=2t$, $z=3t$, $0\leq t\leq 1$.

16. Taking $\boldsymbol{r}_0=\langle 1,0,1\rangle$ and $\boldsymbol{r}_1=\langle 2,3,1\rangle$, we have from Equation 13.5.4 $\boldsymbol{r}(t)=(1-t)\boldsymbol{r}_0+t\boldsymbol{r}_1=(1-t)\langle 1,0,1\rangle+t\langle 2,3,1\rangle$, $0\le t\le 1$ or $\boldsymbol{r}(t)=\langle 1+t,3t,1\rangle$, $0\le t\le 1$. Parametric equations are $x=1+t$, $y=3t$, $z=1$, $0\le t\le 1$.

17. Taking $\boldsymbol{r}_0=\langle 1,-1,2\rangle$ and $\boldsymbol{r}_1=\langle 4,1,7\rangle$, we have $\boldsymbol{r}(t)=(1-t)\boldsymbol{r}_0+t\boldsymbol{r}_1=(1-t)\langle 1,-1,2\rangle+t\langle 4,1,7\rangle$, $0\le t\le 1$ or $\boldsymbol{r}(t)=\langle 1+3t,-1+2t,2+5t\rangle$, $0\le t\le 1$. Parametric equations are $x=1+3t$, $y=-1+2t$, $z=2+5t$, $0\le t\le 1$.

18. Taking $\boldsymbol{r}_0=\langle -2,4,0\rangle$ and $\boldsymbol{r}_1=\langle 6,-1,2\rangle$, we have $\boldsymbol{r}(t)=(1-t)\boldsymbol{r}_0+t\boldsymbol{r}_1=(1-t)\langle -2,4,0\rangle+t\langle 6,-1,2\rangle$, $0\le t\le 1$ or $\boldsymbol{r}(t)=\langle -2+8t,4-5t,2t\rangle$, $0\le t\le 1$. Parametric equations are $x=-2+8t$, $y=4-5t$, $z=2t$, $0\le t\le 1$.

19. $x=\cos 4t$, $y=t$, $z=\sin 4t$. At any point (x,y,z) on the curve, $x^2+z^2=\cos^2 4t+\sin^2 4t=1$. So the curve lies on a circular cylinder with axis the y –axis. Since $y=t$, this is a helix. So the graph is VI.

20. $x=t$, $y=t^2$, $z=e^{-t}$. At any point on the curve, $y=x^2$. So the curve lies on the parabolic cylinder $y=x^2$. Note that y and z are positive for all t , and the point $(0,0,1)$ is on the curve (when $t=0$). As $t\to\infty$, $(x,y,z)\to(\infty,\infty,0)$, while as $t\to-\infty$, $(x,y,z)\to(-\infty,\infty,\infty)$, so the graph must be II.

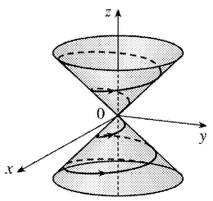
21. $x=t$, $y=1/(1+t^2)$, $z=t^2$. Note that y and z are positive for all t . The curve passes through $(0,1,0)$ when $t=0$. As $t\to\infty$, $(x,y,z)\to(\infty,0,\infty)$, and as $t\to-\infty$, $(x,y,z)\to(-\infty,0,\infty)$. So the graph is IV.

22. $x=e^{-t}\cos 10t$, $y=e^{-t}\sin 10t$, $z=e^{-t}$. $x^2+y^2=e^{-2t}\cos^2 10t+e^{-2t}\sin^2 10t=e^{-2t}(\cos^2 10t+\sin^2 10t)=e^{-2t}=z^2$, so the curve lies on the cone $x^2+y^2=z^2$. Also, z is always positive; the graph must be I.

23. $x=\cos t$, $y=\sin t$, $z=\sin 5t$. $x^2+y^2=\cos^2 t+\sin^2 t=1$, so the curve lies on a circular cylinder with axis the z –axis. Each of x , y and z is periodic, and at $t=0$ and $t=2\pi$ the curve passes through the same point, so the curve repeats itself and the graph is V.

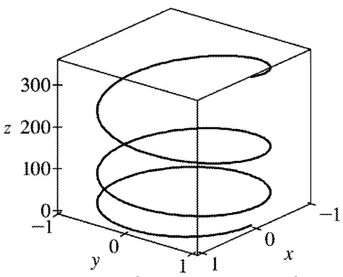
24. $x=\cos t$, $y=\sin t$, $z=\ln t$. $x^2+y^2=\cos^2 t+\sin^2 t=1$, so the curve lies on a circular cylinder with axis the z –axis. As $t\to 0$, $z\to-\infty$, so the graph is III.

25. If $x=t\cos t$, $y=t\sin t$, and $z=t$, then $x^2+y^2=t^2\cos^2 t+t^2\sin^2 t=t^2=z^2$, so the curve lies on the cone $z^2=x^2+y^2$. Since $z=t$, the curve is a spiral on this cone.

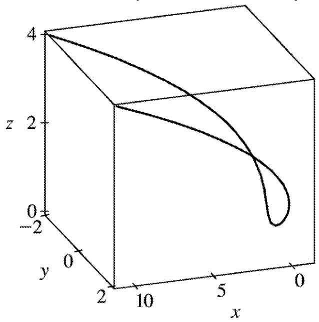


26. Here $x^2=\sin^2 t=z$ and $x^2+y^2=\sin^2 t+\cos^2 t=1$, so the curve is the intersection of the parabolic cylinder $z=x^2$ with the circular cylinder $x^2+y^2=1$.

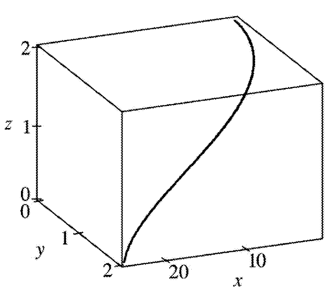
27. $\boldsymbol{r}(t)=\left\langle \sin t,\cos t,t^2 \right\rangle$



28. $\boldsymbol{r}(t)=\left\langle t^4-t^2+1,t,t^2 \right\rangle$

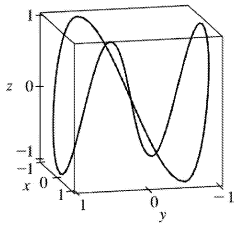


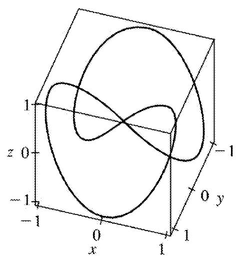
29. $\boldsymbol{r}(t)=\left\langle t^2,\sqrt{t-1},\sqrt{5-t} \right\rangle$



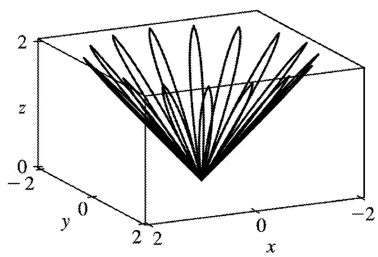
30. We have the computer plot the parametric equations $x=\sin t$, $y=\sin 2t$, $z=\sin 3t$, $0\leq t\leq 2\pi$. The

shape of the curve is not clear from just one viewpoint, so we include a second plot drawn from a different angle.





31.



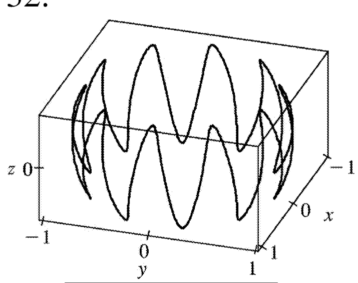
$x=(1+\cos 16t)\cos t$, $y=(1+\cos 16t)\sin t$, $z=1+\cos 16t$. At any point on the graph,

$$x^2+y^2=(1+\cos 16t)^2\cos^2 t+(1+\cos 16t)^2\sin^2 t$$

$$=(1+\cos 16t)^2=z^2$$, so the graph lies on the cone

$x^2+y^2=z^2$. From the graph at left, we see that this curve looks like the projection of a leaved two–dimensional curve onto a cone.

32.



$x=\sqrt{1-0.25\cos^2 10t}\cos t$, $y=\sqrt{1-0.25\cos^2 10t}\sin t$, $z=0.5\cos 10t$. At any point on the graph,

$$\begin{aligned} x^2+y^2+z^2 &= (1-0.25\cos^2 10t)\cos^2 t \\ &\quad +(1-0.25\cos^2 10t)\sin^2 t+0.25\cos^2 t \\ &= 1-0.25\cos^2 10t+0.25\cos^2 10t=1 , \end{aligned}$$

so the graph lies on the sphere $x^2+y^2+z^2=1$, and since $z=0.5\cos 10t$ the graph resembles a trigonometric curve with ten peaks projected onto the sphere. The graph is generated by $t\in[0,2\pi]$.

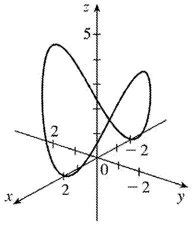
33. If $t=-1$, then $x=1,y=4,z=0$, so the curve passes through the point $(1,4,0)$. If $t=3$, then $x=9,y=-8,z=28$, so the curve passes through the point $(9,-8,28)$. For the point $(4,7,-6)$ to be on the curve, we require $y=1-3t=7\Rightarrow t=-2$. But then $z=1+(-2)^3=-7\neq -6$, so $(4,7,-6)$ is not on the curve.

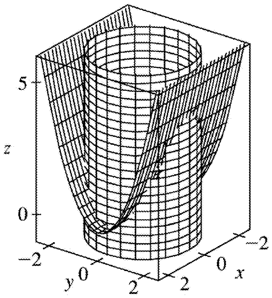
34. The projection of the curve C of intersection onto the xy -plane is the circle $x^2+y^2=4,z=0$. Then we can write $x=2\cos t,y=2\sin t,0\leq t\leq 2\pi$. Since C also lies on the surface $z=xy$, we have $z=xy=(2\cos t)(2\sin t)=4\cos t\sin t$, or $2\sin(2t)$. Then parametric equations for C are $x=2\cos t,y=2\sin t,z=2\sin(2t),0\leq t\leq 2\pi$, and the corresponding vector function is $\boldsymbol{r}(t)=2\cos t\,\boldsymbol{i}+2\sin t\,\boldsymbol{j}+2\sin(2t)\boldsymbol{k}$, $0\leq t\leq 2\pi$.

35. Both equations are solved for z , so we can substitute to eliminate z : $\sqrt{x^2+y^2}=1+y\Rightarrow x^2+y^2=1+2y+y^2\Rightarrow x^2=1+2y\Rightarrow y=\frac{1}{2}(x^2-1)$. We can form parametric equations for the curve C of intersection by choosing a parameter $x=t$, then $y=\frac{1}{2}(t^2-1)$ and $z=1+y=1+\frac{1}{2}(t^2-1)=\frac{1}{2}(t^2+1)$. Thus a vector function representing C is $\boldsymbol{r}(t)=t\,\boldsymbol{i}+\frac{1}{2}(t^2-1)\boldsymbol{j}+\frac{1}{2}(t^2+1)\boldsymbol{k}$.

36. The projection of the curve C of intersection onto the xy -plane is the parabola $y=x^2,z=0$. Then we can choose the parameter $x=t\Rightarrow y=t^2$. Since C also lies on the surface $z=4x^2+y^2$, we have $z=4x^2+y^2=4t^2+(t^2)^2$. Then parametric equations for C are $x=t$, $y=t^2$, $z=4t^2+t^4$, and the corresponding vector function is $\boldsymbol{r}(t)=t\,\boldsymbol{i}+t^2\boldsymbol{j}+(4t^2+t^4)\boldsymbol{k}$.

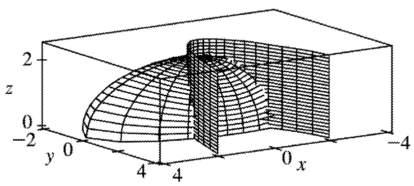
37.

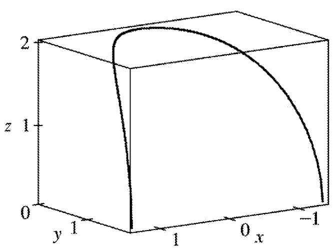




The projection of the curve C of intersection onto the xy –plane is the circle $x^2+y^2=4, z=0$. Then we can write $x=2\cos t$, $y=2\sin t$, $0\leq t\leq 2\pi$. Since C also lies on the surface $z=x^2$, we have $z=x^2=(2\cos t)^2=4\cos^2 t$. Then parametric equations for C are $x=2\cos t$, $y=2\sin t$, $z=4\cos^2 t, 0\leq t\leq 2\pi$.

38.





$x=t \Rightarrow y=t^2 \Rightarrow 4z^2=16-x^2-4y^2=16-t^2-4t^4 \Rightarrow z=\sqrt{4-\left(\frac{1}{2}t\right)^2-t^4}$. Note that z is positive because the intersection is with the top half of the ellipsoid. Hence the curve is given by $x=t$, $y=t^2$, $z=\sqrt{4-\frac{1}{4}t^2-t^4}$.

39. For the particles to collide, we require $\boldsymbol{r}_1(t)=\boldsymbol{r}_2(t) \Leftrightarrow \left\langle t^2, 7t-12, t^2\right\rangle=\left\langle 4t-3, t^2, 5t-6\right\rangle$. Equating components gives $t^2=4t-3$, $7t-12=t^2$, and $t^2=5t-6$. From the first equation, $t^2-4t+3=0 \Leftrightarrow (t-3)(t-1)=0$ so $t=1$ or $t=3$. $t=1$ does not satisfy the other two equations, but $t=3$ does. The particles collide when $t=3$, at the point $(9,9,9)$.

40. The particles collide provided $\boldsymbol{r}_1(t)=\boldsymbol{r}_2(t) \Leftrightarrow \left\langle t, t^2, t^3\right\rangle=\left\langle 1+2t, 1+6t, 1+14t\right\rangle$. Equating components

gives $t=1+2t$, $t^2=1+6t$, and $t^3=1+14t$. The first equation gives $t=-1$, but this does not satisfy the other equations, so the particles do not collide. For the paths to intersect, we need to find a value for t and a value for s where $\boldsymbol{r}_1(t)=\boldsymbol{r}_2(s)\Leftrightarrow\left\langle t,t^2,t^3\right\rangle=\left\langle 1+2s,1+6s,1+14s\right\rangle$. Equating components, $t=1+2s$, $t^2=1+6s$, and $t^3=1+14s$. Substituting the first equation into the second gives $(1+2s)^2=1+6s\Rightarrow$ $4s^2-2s=0\Rightarrow 2s(2s-1)=0\Rightarrow s=0$ or $s=\frac{1}{2}$. From the first equation, $s=0\Rightarrow t=1$ and $s=\frac{1}{2}\Rightarrow t=2$. Checking, we see that both pairs of values satisfy the third equation. Thus the paths intersect twice, at the point $(1,1,1)$ when $s=0$ and $t=1$, and at $(2,4,8)$ when $s=\frac{1}{2}$ and $t=2$.

41. Let $\boldsymbol{u}(t)=\left\langle u_1(t),u_2(t),u_3(t)\right\rangle$ and $\boldsymbol{v}(t)=\left\langle v_1(t),v_2(t),v_3(t)\right\rangle$. In each part of this problem the basic procedure is to use Equation 1 and then analyze the individual component functions using the limit properties we have already developed for real–valued functions.

(a) $\lim_{t\to a}\boldsymbol{u}(t)+\lim_{t\to a}\boldsymbol{v}(t)=\left\langle \lim_{t\to a}u_1(t),\lim_{t\to a}u_2(t),\lim_{t\to a}u_3(t)\right\rangle+\left\langle \lim_{t\to a}v_1(t),\lim_{t\to a}v_2(t),\lim_{t\to a}v_3(t)\right\rangle$ and the limits of these component functions must each exist since the vector functions both possess limits as $t\to a$. Then
adding the two vectors and using the addition property of limits for real–valued functions, we have that

$$\begin{aligned}\lim_{t\to a}\boldsymbol{u}(t)+\lim_{t\to a}\boldsymbol{v}(t) &= \left\langle \lim_{t\to a}u_1(t)+\lim_{t\to a}v_1(t),\lim_{t\to a}u_2(t)+\lim_{t\to a}v_2(t),\lim_{t\to a}u_3(t)+\lim_{t\to a}v_3(t)\right\rangle\\ &= \left\langle \lim_{t\to a}\left[u_1(t)+v_1(t)\right],\lim_{t\to a}\left[u_2(t)+v_2(t)\right],\lim_{t\to a}\left[u_3(t)+v_3(t)\right]\right\rangle\\ &= \lim_{t\to a}\left\langle u_1(t)+v_1(t),u_2(t)+v_2(t),u_3(t)+v_3(t)\right\rangle \text{ [using(1)backward}\\ &= \lim_{t\to a}\left[\boldsymbol{u}(t)+\boldsymbol{v}(t)\right]\end{aligned}$$

(b)

$$\begin{aligned}\lim_{t\to a}c\boldsymbol{u}(t) &=\lim_{t\to a}\left\langle cu_1(t),cu_2(t),cu_3(t)\right\rangle=\left\langle \lim_{t\to a}cu_1(t),\lim_{t\to a}cu_2(t),\lim_{t\to a}cu_3(t)\right\rangle\\ &=\left\langle c\lim_{t\to a}u_1(t),c\lim_{t\to a}u_2(t),c\lim_{t\to a}u_3(t)\right\rangle=c\left\langle \lim_{t\to a}u_1(t),\lim_{t\to a}u_2(t),\lim_{t\to a}u_3(t)\right\rangle\\ &=c\lim_{t\to a}\left\langle u_1(t),u_2(t),u_3(t)\right\rangle=c\lim_{t\to a}\boldsymbol{u}(t)\end{aligned}$$

(c)

$$\lim_{t\to a}\boldsymbol{u}(t)\cdot\lim_{t\to a}\boldsymbol{v}(t)=\left\langle \lim_{t\to a}u_1(t),\lim_{t\to a}u_2(t),\lim_{t\to a}u_3(t)\right\rangle\cdot\left\langle \lim_{t\to a}v_1(t),\lim_{t\to a}v_2(t),\lim_{t\to a}v_3(t)\right\rangle$$

$$=\left[\lim_{t\to a}u_1(t)\right]\left[\lim_{t\to a}v_1(t)\right]+\left[\lim_{t\to a}u_2(t)\right]\left[\lim_{t\to a}v_2(t)\right]+\left[\lim_{t\to a}u_3(t)\right]\left[\lim_{t\to a}v_3(t)\right]$$

$$=\lim_{t\to a}u_1(t)v_1(t)+\lim_{t\to a}u_2(t)v_2(t)+\lim_{t\to a}u_3(t)v_3(t)$$

$$=\lim_{t\to a}\left[u_1(t)v_1(t)+u_2(t)v_2(t)+u_3(t)v_3(t)\right]=\lim_{t\to a}\left[\boldsymbol{u}(t)\cdot\boldsymbol{v}(t)\right]$$

(d)

$$\lim_{t\to a}\boldsymbol{u}(t)\times\lim_{t\to a}\boldsymbol{v}(t)=\left\langle \lim_{t\to a}u_1(t),\lim_{t\to a}u_2(t),\lim_{t\to a}u_3(t)\right\rangle\times\left\langle \lim_{t\to a}v_1(t),\lim_{t\to a}v_2(t),\lim_{t\to a}v_3(t)\right\rangle$$

$$=\left\langle\left[\lim_{t\to a}u_2(t)\right]\left[\lim_{t\to a}v_3(t)\right]-\left[\lim_{t\to a}u_3(t)\right]\left[\lim_{t\to a}v_2(t)\right],\right.$$

$$\left[\lim_{t\to a}u_3(t)\right]\left[\lim_{t\to a}v_1(t)\right]-\left[\lim_{t\to a}u_1(t)\right]\left[\lim_{t\to a}v_3(t)\right],$$

$$\left.\left[\lim_{t\to a}u_1(t)\right]\left[\lim_{t\to a}v_2(t)\right]-\left[\lim_{t\to a}u_2(t)\right]\left[\lim_{t\to a}v_1(t)\right]\right\rangle$$

$$=\left\langle\lim_{t\to a}\left[u_2(t)v_3(t)-u_3(t)v_2(t)\right],\lim_{t\to a}\left[u_3(t)v_1(t)-u_1(t)v_3(t)\right],\right.$$

$$\left.\lim_{t\to a}\left[u_1(t)v_2(t)-u_2(t)v_1(t)\right]\right\rangle$$

$$=\lim_{t\to a}\left\langle u_2(t)v_3(t)-u_3(t)v_2(t),\mathrm{u}_3(t)v_1(t)-u_1(t)v_3(t),\right.$$

$$\left.u_1(t)v_2(t)-u_2(t)v_1(t)\right\rangle$$

$$=\lim_{t\to a}\left[\boldsymbol{u}(t)\times\boldsymbol{v}(t)\right]$$

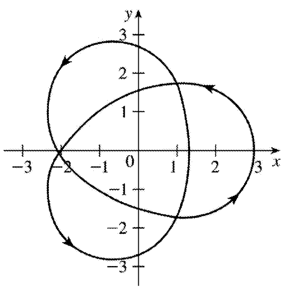
42. The projection of the curve onto the xy –plane is given by the parametric equations $x=(2+\cos 1.5t)\cos t$, $y=(2+\cos 1.5t)\sin t$. If we convert to polar coordinates, we have

$$r^2 = x^2+y^2=[(2+\cos 1.5t)\cos t]^2+[(2+\cos 1.5t)\sin t]^2$$

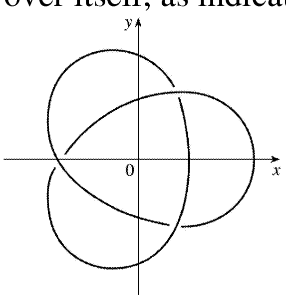
$$= (2+\cos 1.5t)^2(\cos^2 t+\sin^2 t)$$

$$= (2+\cos 1.5t)^2$$

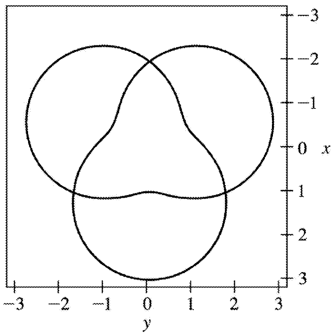
$\Rightarrow r=2+\cos 1.5t$. Also, $\tan\theta=\dfrac{y}{x}=\dfrac{(2+\cos 1.5t)\sin t}{(2+\cos 1.5t)\cos t}=\tan t\Rightarrow\theta=t$.
Thus the polar equation of the curve is $r=2+\cos 1.5\theta$. At $\theta=0$, we have $r=3$, and r decreases to 1 as θ increases to $\dfrac{2\pi}{3}$. For $\dfrac{2\pi}{3}\le\theta\le\dfrac{4\pi}{3}$, r increases to 3 ; r decreases to 1 again at $\theta=2\pi$, increases to 3 at $\theta=\dfrac{8\pi}{3}$, decreases to 1 at $\theta=\dfrac{10\pi}{3}$, and completes the closed curve by increasing to 3 at $\theta=4\pi$. We sketch an approximate graph as shown in the figure.



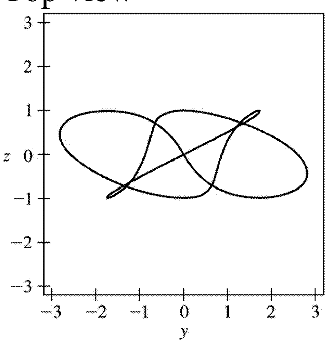
We can determine how the curve passes over itself by investigating the maximum and minimum values of z for $t=\theta\in[0,4\pi]$. Since $z=\sin 1.5t$, z is maximized where $\sin 1.5t=1\Rightarrow 1.5t=\dfrac{\pi}{2}$, $\dfrac{5\pi}{2}$, or $\dfrac{9\pi}{2}\Rightarrow$
$t=\dfrac{\pi}{3}$, $\dfrac{5\pi}{3}$, or 3π . z is minimized where $\sin 1.5t=-1\Rightarrow 1.5t=\dfrac{3\pi}{2}$, $\dfrac{7\pi}{2}$, or $\dfrac{11\pi}{2}\Rightarrow t=\pi$, $\dfrac{7\pi}{3}$, or $\dfrac{11\pi}{3}$. Note that these are precisely the values for which $\cos 1.5t=0\Rightarrow r=2$, and on the graph of the projection, these six points appear to be at the three self–intersections we see. Comparing the maximum and minimum values of z at these intersections, we can determine where the curve passes over itself, as indicated in the figure.



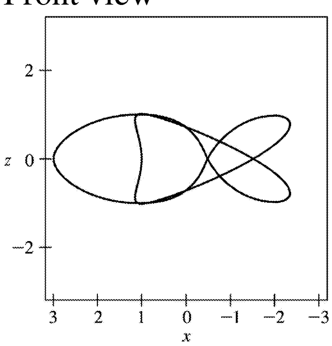
We show a computer–drawn graph of the curve from above, as well as views from the front and from the right side.



Top view

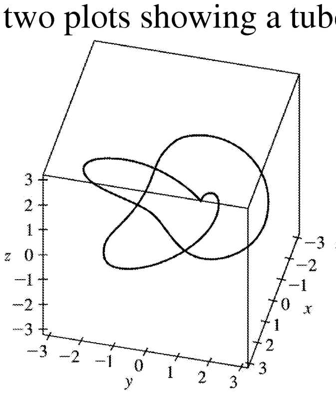


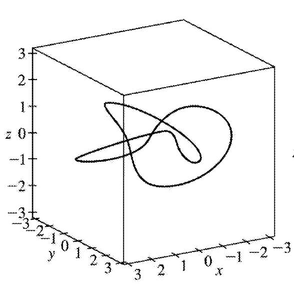
Front view

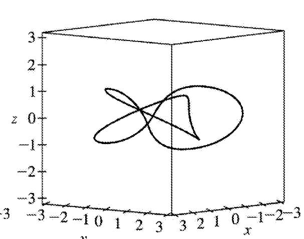


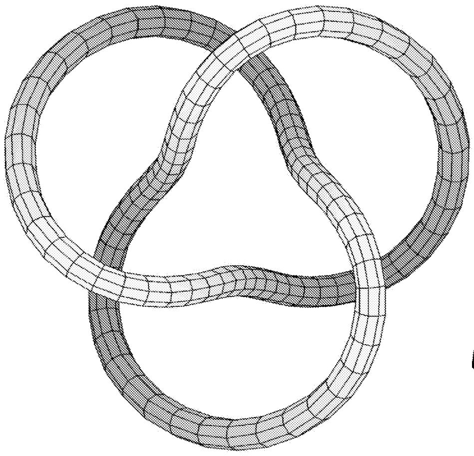
Side view

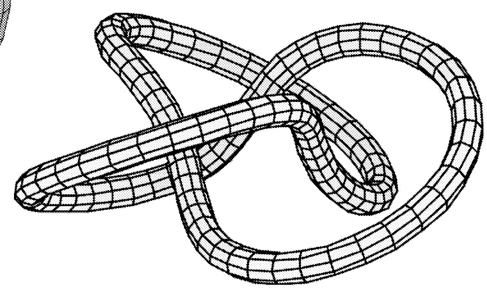
The top view graph shows a more accurate representation of the projection of the trefoil knot on the xy –plane (the axes are rotated 90°). Notice the indentations the graph exhibits at the points corresponding to $r=1$. Finally, we graph several additional viewpoints of the trefoil knot, along with two plots showing a tube of radius 0.2 around the curve.











43. Let $\boldsymbol{r}(t)=\langle f(t),g(t),h(t)\rangle$ and $\boldsymbol{b}=\langle b_1,b_2,b_3\rangle$. If $\lim_{t\to a}\boldsymbol{r}(t)=\boldsymbol{b}$, then $\lim_{t\to a}\boldsymbol{r}(t)$ exists, so by (1), $\boldsymbol{b}=\lim_{t\to a}\boldsymbol{r}(t)=\left\langle \lim_{t\to a}f(t),\lim_{t\to a}g(t),\lim_{t\to a}h(t)\right\rangle$. By the definition of equal vectors we have $\lim_{t\to a}f(t)=b_1$, $\lim_{t\to a}g(t)=b_2$ and $\lim_{t\to a}h(t)=b_3$. But these are limits of real–valued functions, so by the definition of limits, for every $\varepsilon>0$ there exists $\delta_1>0$, $\delta_2>0$, $\delta_3>0$ so $\left|f(t)-b_1\right|<\varepsilon/3$ whenever $0<|t-a|<\delta_1$, $\left|g(t)-b_2\right|<\varepsilon/3$ whenever $0<|t-a|<\delta_2$, and $\left|h(t)-b_3\right|<\varepsilon/3$ whenever $0<|t-a|<\delta_3$. Letting $\delta=$ minimum of $\{\delta_1,\delta_2,\delta_3\}$, we have $\left|f(t)-b_1\right|+\left|g(t)-b_2\right|+\left|h(t)-b_3\right|<\varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon$ whenever $0<|t-a|<\delta$. But

$|\boldsymbol{r}(t)-\boldsymbol{b}|=\left|\left\langle f(t)-b_1,g(t)-b_2,h(t)-b_3\right\rangle\right|$

$=\sqrt{(f(t)-b_1)^2+(g(t)-b_2)^2+(h(t)-b_3)^2}\le\sqrt{[f(t)-b_1]^2}+\sqrt{[g(t)-b_2]^2}+\sqrt{[h(t)-b_3]^2}$

$=\left|f(t)-b_1\right|+\left|g(t)-b_2\right|+\left|h(t)-b_3\right|$. Thus for every $\varepsilon>0$ there exists $\delta>0$ such that $|\boldsymbol{r}(t)-b|\le\left|f(t)-b_1\right|+\left|g(t)-b_2\right|+\left|h(t)-b_3\right|<\varepsilon$ whenever $0<|t-a|<\delta$. Conversely, if for every $\varepsilon>0$, there exists $\delta>0$ such that $|\boldsymbol{r}(t)-\boldsymbol{b}|<\varepsilon$ whenever $0<|t-a|<\delta$, then

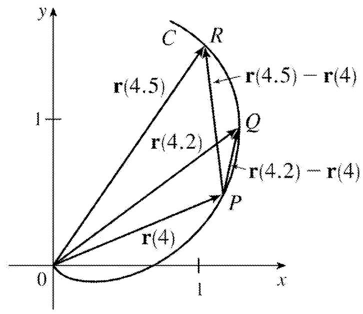
$\left|\left\langle f(t)-b_1,g(t)-b_2,h(t)-b_3\right\rangle\right|<\varepsilon\Leftrightarrow\sqrt{[f(t)-b_1]^2+[g(t)-b_2]^2+[h(t)-b_3]^2}<\varepsilon\Leftrightarrow$

$[f(t)-b_1]^2+[g(t)-b_2]^2+[h(t)-b_3]^2<\varepsilon^2$ whenever $0<|t-a|<\delta$. But each term on the left side of this inequality is positive so $[f(t)-b_1]^2<\varepsilon^2$, $[g(t)-b_2]^2<\varepsilon^2$ and $[h(t)-b_3]^2<\varepsilon^2$ whenever $0<|t-a|<\delta$, or taking the square root of both sides in each of the above we have $\left|f(t)-b_1\right|<\varepsilon$, $\left|g(t)-b_2\right|<\varepsilon$ and $\left|h(t)-b_3\right|<\varepsilon$ whenever $0<|t-a|<\delta$. And by definition of limits of real–valued functions we have

$\lim_{t \to a} f(t) = b_1$, $\lim_{t \to a} g(t) = b_2$ and $\lim_{t \to a} h(t) = b_3$. But by (1), $\lim_{t \to a} \boldsymbol{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$, so $\lim_{t \to a} \boldsymbol{r}(t) = \left\langle b_1, \mathrm{b}_2, \mathrm{b}_3 \right\rangle = \boldsymbol{b}$.

1.

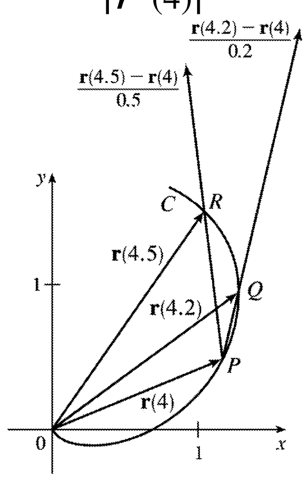
(a)



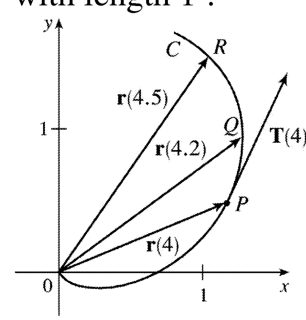
(b) $\dfrac{\boldsymbol{r}(4.5)-\boldsymbol{r}(4)}{0.5}=2[\boldsymbol{r}(4.5)-\boldsymbol{r}(4)]$, so we draw a vector in the same direction but with twice the length of the vector $\boldsymbol{r}(4.5)-\boldsymbol{r}(4)$. $\dfrac{\boldsymbol{r}(4.2)-\boldsymbol{r}(4)}{0.2}=5[\boldsymbol{r}(4.2)-\boldsymbol{r}(4)]$, so we draw a vector in the same direction but with 5 times the length of the vector $\boldsymbol{r}(4.2)-\boldsymbol{r}(4)$

(c) By Definition 1, $\boldsymbol{r}'(4)=\lim_{h\to 0}\dfrac{\boldsymbol{r}(4+h)-\boldsymbol{r}(4)}{h}$.

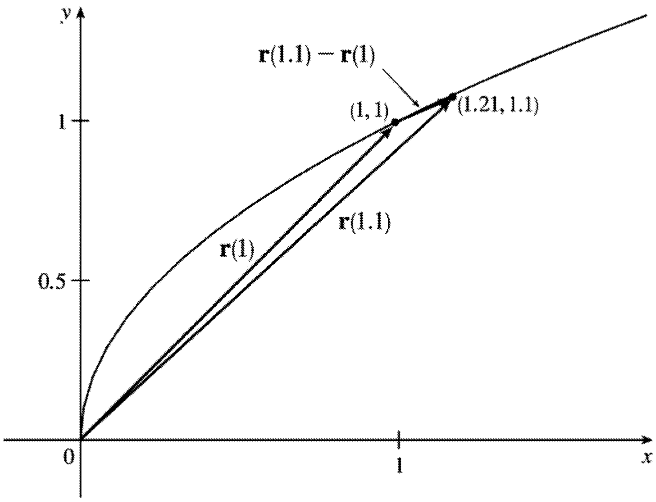
$\boldsymbol{T}(4)=\dfrac{\boldsymbol{r}'(4)}{\left|\boldsymbol{r}'(4)\right|}$.



(d) $\boldsymbol{T}(4)$ is a unit vector in the same direction as $\boldsymbol{r}'(4)$, that is, parallel to the tangent line to the curve at $\boldsymbol{r}(4)$ with length 1 .

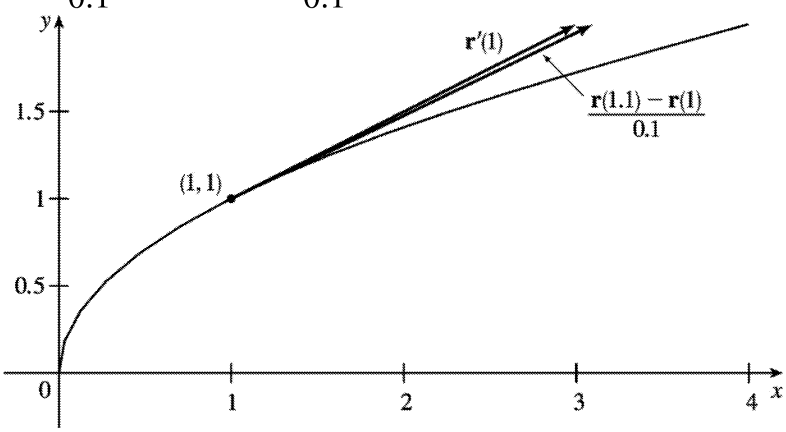


2. **(a)** The curve can be represented by the parametric equations $x=t^2$, $y=t$, $0\leq t\leq 2$. Eliminating the parameter, we have $x=y^2$, $0\leq y\leq 2$, a portion of which we graph here, along with the vectors $\boldsymbol{r}(1)$, $\boldsymbol{r}(1.1)$, and $\boldsymbol{r}(1.1)-\boldsymbol{r}(1)$.



(b) Since $\boldsymbol{r}(t)=\langle t^2,t\rangle$, we differentiate components, giving $\boldsymbol{r}'(t)=\langle 2t,1\rangle$, so $\boldsymbol{r}'(1)=\langle 2,1\rangle$.

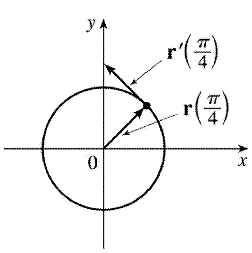
$$\frac{\boldsymbol{r}(1.1)-\boldsymbol{r}(1)}{0.1}=\frac{\langle 1.21,1.1\rangle-\langle 1,1\rangle}{0.1}=10\langle 0.21,0.1\rangle=\langle 2.1,1\rangle \ .$$



As we can see from the graph, these vectors are very close in length and direction. $\boldsymbol{r}'(1)$ is defined to be $\lim_{h\to 0}\frac{\boldsymbol{r}(1+h)-\boldsymbol{r}(1)}{h}$, and we recognize $\frac{\boldsymbol{r}(1.1)-\boldsymbol{r}(1)}{0.1}$ as the expression after the limit sign with $h=0.1$. Since h is close to 0 , we would expect $\frac{\boldsymbol{r}(1.1)-\boldsymbol{r}(1)}{0.1}$ to be a vector close to $\boldsymbol{r}'(1)$.

3.

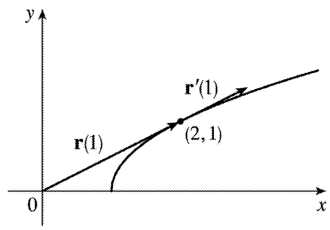
(a), (c)



(b) $\boldsymbol{r}'(t)=\langle -\sin t, \cos t\rangle$

4.

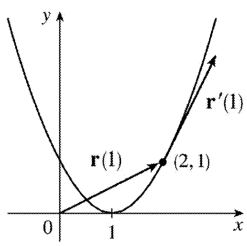
(a), (c)



(b) $\boldsymbol{r}'(t)=\left\langle 1, \frac{1}{2\sqrt{t}} \right\rangle$

5. Since $(x-1)^2=t^2=y$, the curve is a parabola.

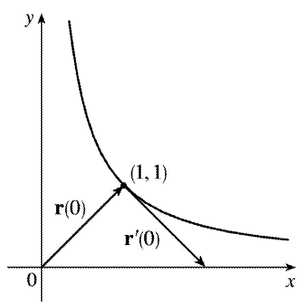
(a), (c)



(b) $\boldsymbol{r}'(t)=\boldsymbol{i}+2t\,\boldsymbol{j}$

6.

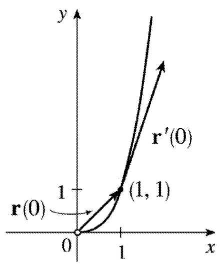
(a), (c)



(b) $\boldsymbol{r}'(t)=e^{t}\boldsymbol{i}-e^{-t}\boldsymbol{j}$

7.

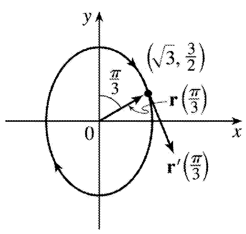
(a), (c)



(b) $\boldsymbol{r}'(t)=e^{t}\boldsymbol{i}+3e^{3t}\boldsymbol{j}$

8. $x=2\sin t$, $y=3\cos t$, so $(x/2)^{2}+(y/3)^{2}=\sin^{2}t+\cos^{2}t=1$ and the curve is an ellipse .

(a), (c)



(b) $\boldsymbol{r}'(t)=2\cos t\,\boldsymbol{i}-3\sin t\,\boldsymbol{j}$

9. $\boldsymbol{r}'(t)=\left\langle \frac{d}{dt}\left[t^{2}\right], \frac{d}{dt}\left[1-t\right], \frac{d}{dt}\left[\sqrt{t}\,\right] \right\rangle=\left\langle 2\mathrm{t}, -1, \frac{1}{2\sqrt{t}} \right\rangle$

10. $\boldsymbol{r}(t)=\langle\cos 3\mathrm{t},\mathrm{t},\sin 3t\rangle \Rightarrow \boldsymbol{r}'(t)=\langle -3\sin 3\mathrm{t},1,3\cos 3t\rangle$

11. $\boldsymbol{r}(t)=\boldsymbol{i}-\boldsymbol{j}+e^{4t}\boldsymbol{k}\Rightarrow \boldsymbol{r}'(t)=0\boldsymbol{i}+0\boldsymbol{j}+4e^{4t}\boldsymbol{k}=4e^{4t}\boldsymbol{k}$

12. $\boldsymbol{r}(t)=\sin^{-1}t\,\boldsymbol{i}+\sqrt{1-t^{2}}\,\boldsymbol{j}+\boldsymbol{k}\Rightarrow \boldsymbol{r}'(t)=\frac{1}{\sqrt{1-t^{2}}}\,\boldsymbol{i}-\frac{t}{\sqrt{1-t^{2}}}\,\boldsymbol{j}$

13. $\boldsymbol{r}(t)=e^{t^{2}}\boldsymbol{i}-\boldsymbol{j}+\ln(1+3t)\boldsymbol{k}\Rightarrow \boldsymbol{r}'(t)=2te^{t^{2}}\boldsymbol{i}+\frac{3}{1+3t}\,\boldsymbol{k}$

14.

$$\begin{aligned}\boldsymbol{r}'(t) &=[at(-3\sin 3t)+a\cos 3t]\boldsymbol{i}+b\cdot 3\sin^{2}t\cos t\,\boldsymbol{j}+c\cdot 3\cos^{2}t(-\sin t)\boldsymbol{k}\\ &=(a\cos 3t-3at\sin 3t)\boldsymbol{i}+3b\sin^{2}t\cos t\,\boldsymbol{j}-3c\cos^{2}t\sin t\,\boldsymbol{k}\end{aligned}$$

15. $\boldsymbol{r}'(t)=\boldsymbol{0}+\boldsymbol{b}+2t\,\boldsymbol{c}=\boldsymbol{b}+2t\,\boldsymbol{c}$ by Formulas 1 and 3 of Theorem 3.

16. To find $\mathbf{r}'(t)$, we first expand $\mathbf{r}(t)=t\,\mathbf{a}\times(\mathbf{b}+t\,\mathbf{c})=t(\mathbf{a}\times\mathbf{b})+t^2(\mathbf{a}\times\mathbf{c})$, so $\mathbf{r}'(t)=\mathbf{a}\times\mathbf{b}+2t(\mathbf{a}\times\mathbf{c})$.

17. $\mathbf{r}'(t)=\left\langle 30t^4,12t^2,2\right\rangle \Rightarrow \mathbf{r}'(1)=\langle 30,12,2\rangle$. So $\left|\mathbf{r}'(1)\right|=\sqrt{30^2+12^2+2^2}=\sqrt{1048}=2\sqrt{262}$ and

$$\mathbf{T}(1)=\frac{\mathbf{r}'(1)}{\left|\mathbf{r}'(1)\right|}=\frac{1}{2\sqrt{262}}\langle 30,12,2\rangle=\left\langle \frac{15}{\sqrt{262}},\frac{6}{\sqrt{262}},\frac{1}{\sqrt{262}}\right\rangle .$$

18. $\mathbf{r}'(t)=\frac{2}{\sqrt{t}}\,\mathbf{i}+2t\,\mathbf{j}+\mathbf{k}\Rightarrow \mathbf{r}'(1)=2\mathbf{i}+2\mathbf{j}+\mathbf{k}$. Thus

$$\mathbf{T}(1)=\frac{\mathbf{r}'(1)}{\left|\mathbf{r}'(1)\right|}=\frac{1}{\sqrt{2^2+2^2+1^2}}(2\mathbf{i}+2\mathbf{j}+\mathbf{k})=\frac{1}{3}(2\mathbf{i}+2\mathbf{j}+\mathbf{k})=\frac{2}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{1}{3}\mathbf{k} .$$

19. $\mathbf{r}'(t)=-\sin t\,\mathbf{i}+3\,\mathbf{j}+4\cos 2t\,\mathbf{k}\Rightarrow \mathbf{r}'(0)=3\,\mathbf{j}+4\mathbf{k}$. Thus

$$\mathbf{T}(0)=\frac{\mathbf{r}'(0)}{\left|\mathbf{r}'(0)\right|}=\frac{1}{\sqrt{0^2+3^2+4^2}}(3\mathbf{j}+4\mathbf{k})=\frac{1}{5}(3\mathbf{j}+4\mathbf{k})=\frac{3}{5}\mathbf{j}+\frac{4}{5}\mathbf{k} .$$

20. $\mathbf{r}'(t)=2\cos t\,\mathbf{i}-2\sin t\,\mathbf{j}+\sec^2 t\,\mathbf{k}\Rightarrow \mathbf{r}'\left(\frac{\pi}{4}\right)=\sqrt{2}\,\mathbf{i}-\sqrt{2}\,\mathbf{j}+2\mathbf{k}$ and $\left|\mathbf{r}'\left(\frac{\pi}{4}\right)\right|=\sqrt{2+2+4}=2\sqrt{2}$.

Thus $$\mathbf{T}\left(\frac{\pi}{4}\right)=\frac{\mathbf{r}'\left(\frac{\pi}{4}\right)}{\left|\mathbf{r}'\left(\frac{\pi}{4}\right)\right|}=\frac{1}{2\sqrt{2}}\left(\sqrt{2}\,\mathbf{i}-\sqrt{2}\,\mathbf{j}+2\mathbf{k}\right)=\frac{1}{2}\mathbf{i}-\frac{1}{2}\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k} .$$

21. $\mathbf{r}(t)=\left\langle \mathrm{t},\mathrm{t}^2,\mathrm{t}^3\right\rangle \Rightarrow \mathbf{r}'(t)=\left\langle 1,2\mathrm{t},3t^2\right\rangle$. Then $\mathbf{r}'(1)=\langle 1,2,3\rangle$ and $\left|\mathbf{r}'(1)\right|=\sqrt{1^2+2^2+3^2}=\sqrt{14}$, so

$$\mathbf{T}(1)=\frac{\mathbf{r}'(1)}{\left|\mathbf{r}'(1)\right|}=\frac{1}{\sqrt{14}}\langle 1,2,3\rangle=\left\langle \frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}}\right\rangle .\ \mathbf{r}''(t)=\langle 0,2,6t\rangle ,\text{ so}$$

$$\mathbf{r}'(t)\times\mathbf{r}''(t)=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2\mathrm{t} & 3\mathrm{t}^2 \\ 0 & 2 & 6\mathrm{t}\end{vmatrix}=\begin{vmatrix} 2\mathrm{t} & 3\mathrm{t}^2 \\ 2 & 6\mathrm{t}\end{vmatrix}\mathbf{i}-\begin{vmatrix} 1 & 3\mathrm{t}^2 \\ 0 & 6\mathrm{t}\end{vmatrix}\mathbf{j}+\begin{vmatrix} 1 & 2\mathrm{t} \\ 0 & 2\end{vmatrix}\mathbf{k}$$

$$=(12t^2-6t^2)\mathbf{i}-(6t-0)\mathbf{j}+(2-0)\mathbf{k}=\left\langle 6t^2,-6\mathrm{t},2\right\rangle .$$

22. $\mathbf{r}(t)=\left\langle e^{2t},\mathrm{e}^{-2t},\mathrm{te}^{2t}\right\rangle \Rightarrow \mathbf{r}'(t)=\left\langle 2e^{2t},-2e^{-2t},(2t+1)e^{2t}\right\rangle \Rightarrow \mathbf{r}'(0)=\left\langle 2e^0,-2e^0,(0+1)e^0\right\rangle=\langle 2,-2,1\rangle$ and

$\left|\boldsymbol{r}'(0)\right|=\sqrt{2^2+(-2)^2+1^2}=3$. Then $\boldsymbol{T}(0)=\dfrac{\boldsymbol{r}'(0)}{\left|\boldsymbol{r}'(0)\right|}=\dfrac{1}{3}\left\langle 2,-2,1\right\rangle=\left\langle \dfrac{2}{3},-\dfrac{2}{3},\dfrac{1}{3}\right\rangle$.

$\boldsymbol{r}''(t)=\left\langle 4e^{2t},4e^{-2t},(4t+4)e^{2t}\right\rangle \Rightarrow \boldsymbol{r}''(0)=\left\langle 4e^{0},4e^{0},(0+4)e^{0}\right\rangle=\left\langle 4,4,4\right\rangle$.

$$\begin{aligned}\boldsymbol{r}'(t)\cdot\boldsymbol{r}''(t) &=\left\langle 2e^{2t},-2e^{-2t},(2t+1)e^{2t}\right\rangle\cdot\left\langle 4e^{2t},4e^{-2t},(4t+4)e^{2t}\right\rangle\\ &=(2e^{2t})(4e^{2t})+(-2e^{-2t})(4e^{-2t})+((2t+1)e^{2t})((4t+4)e^{2t})\\ &=8e^{4t}-8e^{-4t}+(8t^2+12t+4)e^{4t}=(8t^2+12t+12)e^{4t}-8e^{-4t}\end{aligned}$$

23. The vector equation for the curve is $\boldsymbol{r}(t)=\left\langle t^5,t^4,t^3\right\rangle$, so $\boldsymbol{r}'(t)=\left\langle 5t^4,4t^3,3t^2\right\rangle$. The point $(1,1,1)$ corresponds to $t=1$, so the tangent vector there is $\boldsymbol{r}'(1)=\left\langle 5,4,3\right\rangle$. Thus, the tangent line goes through the point $(1,1,1)$ and is parallel to the vector $\left\langle 5,4,3\right\rangle$. Parametric equations are $x=1+5t$, $y=1+4t$, $z=1+3t$.

24. The vector equation for the curve is $\boldsymbol{r}(t)=\left\langle t^2-1,t^2+1,t+1\right\rangle$, so $\boldsymbol{r}'(t)=\left\langle 2t,2t,1\right\rangle$. The point $(-1,1,1)$ corresponds to $t=0$, so the tangent vector there is $\boldsymbol{r}'(0)=\left\langle 0,0,1\right\rangle$. Thus, the tangent line is parallel to the vector $\left\langle 0,0,1\right\rangle$ and parametric equations are $x=-1+0\cdot t=-1$, $y=1+0\cdot t=1$, $z=1+1\cdot t=1+t$.

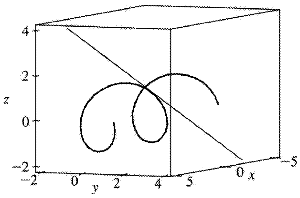
25. The vector equation for the curve is $\boldsymbol{r}(t)=\left\langle e^{-t}\cos t,e^{-t}\sin t,e^{-t}\right\rangle$, so

$$\begin{aligned}\boldsymbol{r}'(t) &=\left\langle e^{-t}(-\sin t)+(\cos t)(-e^{-t}),e^{-t}\cos t+(\sin t)(-e^{-t}),(-e^{-t})\right\rangle\\ &=\left\langle -e^{-t}(\cos t+\sin t),e^{-t}(\cos t-\sin t),-e^{-t}\right\rangle .\end{aligned}$$

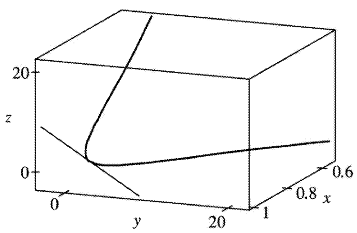
The point $(1,0,1)$ corresponds to $t=0$, so the tangent vector there is $\boldsymbol{r}'(0)=\left\langle -e^{0}(\cos 0+\sin 0),e^{0}(\cos 0-\sin 0),-e^{0}\right\rangle=\left\langle -1,1,-1\right\rangle$. Thus, the tangent line is parallel to the vector $\left\langle -1,1,-1\right\rangle$ and parametric equations are $x=1+(-1)t=1-t$, $y=0+1\cdot t=t$, $z=1+(-1)t=1-t$.

26. $\boldsymbol{r}(t)=\left\langle \ln t,2\sqrt{t},t^2\right\rangle$, $\boldsymbol{r}'(t)=\left\langle 1/t,1/\sqrt{t},2t\right\rangle$. At $(0,2,1)$, $t=1$ and $\boldsymbol{r}'(1)=\left\langle 1,1,2\right\rangle$. Thus, parametric equations of the tangent line are $x=t$, $y=2+t$, $z=1+2t$.

27. $\boldsymbol{r}(t)=\left\langle t,\sqrt{2}\cos t,\sqrt{2}\sin t\right\rangle \Rightarrow \boldsymbol{r}'(t)=\left\langle 1,-\sqrt{2}\sin t,\sqrt{2}\cos t\right\rangle$. At $\left(\dfrac{\pi}{4},1,1\right)$, $t=\dfrac{\pi}{4}$ and $\boldsymbol{r}'\left(\dfrac{\pi}{4}\right)=\left\langle 1,-1,1\right\rangle$. Thus, parametric equations of the tangent line are $x=\dfrac{\pi}{4}+t$, $y=1-t$, $z=1+t$.



28. $\boldsymbol{r}(t)=\langle \cos t, 3e^{2t}, 3e^{-2t}\rangle$, $\boldsymbol{r}'(t)=\langle -\sin t, 6e^{2t}, -6e^{-2t}\rangle$. At $(1,3,3)$, $t=0$ and $\boldsymbol{r}'(0)=\langle 0,6,-6\rangle$. Thus, parametric equations of the tangent line are $x=1$, $y=3+6t$, $z=3-6t$.

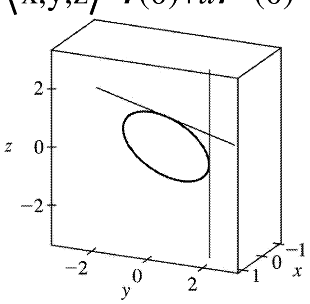


29. **(a)** $\boldsymbol{r}(t)=\langle t^3, t^4, t^5\rangle \Rightarrow \boldsymbol{r}'(t)=\langle 3t^2, 4t^3, 5t^4\rangle$, and since $\boldsymbol{r}'(0)=\langle 0,0,0\rangle=\mathbf{0}$, the curve is not smooth.

(b) $\boldsymbol{r}(t)=\langle t^3+t, t^4, t^5\rangle \Rightarrow \boldsymbol{r}'(t)=\langle 3t^2+1, 4t^3, 5t^4\rangle$. $\boldsymbol{r}'(t)$ is continuous since its component functions are continuous. Also, $\boldsymbol{r}'(t)\neq\mathbf{0}$, as the y – and z –components are 0 only for $t=0$, but $\boldsymbol{r}'(0)=\langle 1,0,0\rangle\neq\mathbf{0}$. Thus, the curve is smooth.

(c) $\boldsymbol{r}(t)=\langle \cos^3 t, \sin^3 t\rangle \Rightarrow \boldsymbol{r}'(t)=\langle -3\cos^2 t\sin t, 3\sin^2 t\cos t\rangle$. Since $\boldsymbol{r}'(0)=\langle -3\cos^2 0\sin 0, 3\sin^2 0\cos 0\rangle=\langle 0,0\rangle=\mathbf{0}$, the curve is not smooth.

30. **(a)** p250pt The tangent line at $t=0$ is the line through the point with position vector $\boldsymbol{r}(0)=\langle \sin 0, 2\sin 0, \cos 0\rangle=\langle 0,0,1\rangle$,
and in the direction of the tangent vector, $\boldsymbol{r}'(0)=\langle \pi\cos 0, 2\pi\cos 0, -\pi\sin 0\rangle=\langle \pi, 2\pi, 0\rangle$.
So an equation of the line is
$\langle x,y,z\rangle=\boldsymbol{r}(0)+u\boldsymbol{r}'(0)=\langle 0+\pi u, 0+2\pi u, 1\rangle=\langle \pi u, 2\pi u, 1\rangle$. (b)



$\boldsymbol{r}\left(\frac{1}{2}\right)=\left\langle \sin\frac{\pi}{2},2\sin\frac{\pi}{2},\cos\frac{\pi}{2}\right\rangle=\langle 1,2,0\rangle$,

$\boldsymbol{r}'\left(\frac{1}{2}\right)=\left\langle \pi\cos\frac{\pi}{2},2\pi\cos\frac{\pi}{2},-\pi\sin\frac{\pi}{2}\right\rangle=\langle 0,0,-\pi\rangle$.

So the equation of the second line is $\langle x,y,z\rangle=\langle 1,2,0\rangle+v\langle 0,0,-\pi\rangle=\langle 1,2,-\pi v\rangle$. The lines intersect where $\langle \pi u,2\pi u,1\rangle=\langle 1,2,-\pi v\rangle$, so the point of intersection is $(1,2,1)$.

31. The angle of intersection of the two curves is the angle between the two tangent vectors to the curves at the point of intersection. Since $\boldsymbol{r}_1'(t)=\langle 1,2t,3t^2\rangle$ and $t=0$ at $(0,0,0)$, $\boldsymbol{r}_1'(0)=\langle 1,0,0\rangle$ is a tangent vector to $\boldsymbol{r}_1$ at $(0,0,0)$. Similarly, $\boldsymbol{r}_2'(t)=\langle \cos t,2\cos 2t,1\rangle$ and since $\boldsymbol{r}_2(0)=\langle 0,0,0\rangle$, $\boldsymbol{r}_2'(0)=\langle 1,2,1\rangle$ is a tangent vector to $\boldsymbol{r}_2$ at $(0,0,0)$. If θ is the angle between these two tangent vectors, then $\cos\theta=\frac{1}{\sqrt{1}\sqrt{6}}\langle 1,0,0\rangle\cdot\langle 1,2,1\rangle=\frac{1}{\sqrt{6}}$ and $\theta=\cos^{-1}\left(\frac{1}{\sqrt{6}}\right)\approx 66°$.

32. To find the point of intersection, we must find the values of t and s which satisfy the following three equations simultaneously: $t=3-s$, $1-t=s-2$, $3+t^2=s^2$. Solving the last two equations gives $t=1$, $s=2$ (check these in the first equation). Thus the point of intersection is $(1,0,4)$. To find the angle θ of intersection, we proceed as in Exercise 31 . The tangent vectors to the respective curves at $(1,0,4)$ are $\boldsymbol{r}_1'(1)=\langle 1,-1,2\rangle$ and $\boldsymbol{r}_2'(2)=\langle -1,1,4\rangle$. So $\cos\theta=\frac{1}{\sqrt{6}\sqrt{18}}(-1-1+8)=\frac{6}{6\sqrt{3}}=\frac{1}{\sqrt{3}}$ and $\theta=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)\approx 55°$.

Note: In Exercise 31 , the curves intersect when the value of both parameters is zero. However, as seen in this exercise, it is not necessary for the parameters to be of equal value at the point of intersection.

33.

$$\int_0^1(16t^3\boldsymbol{i}-9t^2\boldsymbol{j}+25t^4\boldsymbol{k})dt=\left(\int_0^1 16t^3dt\right)\boldsymbol{i}-\left(\int_0^1 9t^2dt\right)\boldsymbol{j}+\left(\int_0^1 25t^4dt\right)\boldsymbol{k}$$
$$=\left[4t^4\right]_0^1\boldsymbol{i}-\left[3t^3\right]_0^1\boldsymbol{j}+\left[5t^5\right]_0^1\boldsymbol{k}=4\boldsymbol{i}-3\boldsymbol{j}+5\boldsymbol{k}$$

34. $\int_0^1\left(\frac{4}{1+t^2}\boldsymbol{j}+\frac{2t}{1+t^2}\boldsymbol{k}\right)dt=\left[4\tan^{-1}t\,\boldsymbol{j}+\ln(1+t^2)\boldsymbol{k}\right]_0^1$

$=\left[4\tan^{-1}1\,\boldsymbol{j}+\ln 2\,\boldsymbol{k}\right]-\left[4\tan^{-1}0\,\boldsymbol{j}+\ln 1\,\boldsymbol{k}\right]=4\left(\frac{\pi}{4}\right)\boldsymbol{j}+\ln 2\,\boldsymbol{k}-0\,\boldsymbol{j}-0\,\boldsymbol{k}=\pi\,\boldsymbol{j}+\ln 2\,\boldsymbol{k}$

35. $\int_0^{\pi/2}(3\sin^2 t\cos t\,\boldsymbol{i}+3\sin t\cos^2 t\,\boldsymbol{j}+2\sin t\cos t\,\boldsymbol{k})\,dt$

$=\left(\int_0^{\pi/2}3\sin^2 t\cos t\,dt\right)\boldsymbol{i}+\left(\int_0^{\pi/2}3\sin t\cos^2 t\,dt\right)\boldsymbol{j}+\left(\int_0^{\pi/2}2\sin t\cos t\,dt\right)\boldsymbol{k}$
$=\left[\sin^3 t\right]_0^{\pi/2}\boldsymbol{i}+\left[-\cos^3 t\right]_0^{\pi/2}\boldsymbol{j}+\left[\sin^2 t\right]_0^{\pi/2}\boldsymbol{k}$
$=(1-0)\boldsymbol{i}+(0+1)\boldsymbol{j}+(1-0)\boldsymbol{k}=\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$

36. $\int_1^4\left(\sqrt{t}\,\boldsymbol{i}+te^{-t}\,\boldsymbol{j}+t^{-2}\boldsymbol{k}\right)dt=\left[\frac{2}{3}t^{3/2}\boldsymbol{i}-t^{-1}\boldsymbol{k}\right]_1^4+\left(\left[-te^{-t}\right]_1^4+\int_1^4 e^{-t}\,dt\right)\boldsymbol{j}$
$=\left(\frac{16}{3}-\frac{2}{3}\right)\boldsymbol{i}-\left(\frac{1}{4}-1\right)\boldsymbol{k}+(-4e^{-4}+e^{-1}-e^{-4}+e^{-1})\boldsymbol{j}=\frac{14}{3}\boldsymbol{i}+e^{-1}(2-5e^{-3})\boldsymbol{j}+\frac{3}{4}\boldsymbol{k}$

37. $\int\left(e^t\boldsymbol{i}+2t\,\boldsymbol{j}+\ln t\,\boldsymbol{k}\right)dt=\left(\int e^t dt\right)\boldsymbol{i}+\left(\int 2t\,dt\right)\boldsymbol{j}+\left(\int \ln t\,dt\right)\boldsymbol{k}$
$=e^t\boldsymbol{i}+t^2\boldsymbol{j}+(t\ln t-t)\boldsymbol{k}+\boldsymbol{C}$, where $\boldsymbol{C}$ is a vector constant of integration.

38.
$\int(\cos\pi t\,\boldsymbol{i}+\sin\pi t\,\boldsymbol{j}+t\,\boldsymbol{k})\,dt=\left(\int\cos\pi t\,dt\right)\boldsymbol{i}+\left(\int\sin\pi t\,dt\right)\boldsymbol{j}+\left(\int t\,dt\right)\boldsymbol{k}$
$=\frac{1}{\pi}\sin\pi t\,\boldsymbol{i}-\frac{1}{\pi}\cos\pi t\,\boldsymbol{j}+\frac{1}{2}t^2\boldsymbol{k}+\boldsymbol{C}$

39. $\boldsymbol{r}'(t)=t^2\boldsymbol{i}+4t^3\boldsymbol{j}-t^2\boldsymbol{k}\Rightarrow\boldsymbol{r}(t)=\frac{1}{3}t^3\boldsymbol{i}+t^4\boldsymbol{j}-\frac{1}{3}t^3\boldsymbol{k}+\boldsymbol{C}$, where $\boldsymbol{C}$ is a constant vector.

But $\boldsymbol{j}=\boldsymbol{r}(0)=(0)\boldsymbol{i}+(0)\boldsymbol{j}-(0)\boldsymbol{k}+\boldsymbol{C}$. Thus $\boldsymbol{C}=\boldsymbol{j}$ and $\boldsymbol{r}(t)=\frac{1}{3}t^3\boldsymbol{i}+t^4\boldsymbol{j}-\frac{1}{3}t^3\boldsymbol{k}+\boldsymbol{j}=\frac{1}{3}t^3\boldsymbol{i}+(t^4+1)\boldsymbol{j}-\frac{1}{3}t^3\boldsymbol{k}$.

40. $\boldsymbol{r}'(t)=\sin t\,\boldsymbol{i}-\cos t\,\boldsymbol{j}+2t\,\boldsymbol{k}\Rightarrow\boldsymbol{r}(t)=(-\cos t)\boldsymbol{i}-(\sin t)\boldsymbol{j}+t^2\boldsymbol{k}+\boldsymbol{C}$. But $\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}=\boldsymbol{r}(0)=-\boldsymbol{i}+(0)\boldsymbol{j}+(0)\boldsymbol{k}+\boldsymbol{C}$.
Thus $\boldsymbol{C}=2\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}$ and $\boldsymbol{r}(t)=(2-\cos t)\boldsymbol{i}+(1-\sin t)\boldsymbol{j}+(2+t^2)\boldsymbol{k}$.

41.
$\frac{d}{dt}\left[\boldsymbol{u}(t)+\boldsymbol{v}(t)\right]=\frac{d}{dt}\left\langle u_1(t)+v_1(t),u_2(t)+v_2(t),u_3(t)+v_3(t)\right\rangle$
$=\left\langle\frac{d}{dt}\left[u_1(t)+v_1(t)\right],\frac{d}{dt}\left[u_2(t)+v_2(t)\right],\frac{d}{dt}\left[u_3(t)+v_3(t)\right]\right\rangle$

$$=\left\langle u_1'(t)+v_1'(t), u_2'(t)+v_2'(t), u_3'(t)+v_3'(t)\right\rangle$$

$$=\left\langle u_1'(t), u_2'(t), u_3'(t)\right\rangle+\left\langle v_1'(t), v_2'(t), v_3'(t)\right\rangle=\boldsymbol{u}'(t)+\boldsymbol{v}'(t)\ .$$

42.

$$\frac{d}{dt}\left[f(t)\,\boldsymbol{u}(t)\right]=\frac{d}{dt}\left\langle f(t)u_1(t), f(t)u_2(t), f(t)u_3(t)\right\rangle$$

$$=\left\langle \frac{d}{dt}\left[f(t)u_1(t)\right], \frac{d}{dt}\left[f(t)u_2(t)\right], \frac{d}{dt}\left[f(t)u_3(t)\right]\right\rangle$$

$$=\left\langle f'(t)u_1(t)+f(t)u_1'(t), f'(t)u_2(t)+f(t)u_2'(t), f'(t)u_3(t)+f(t)u_3'(t)\right\rangle$$

$$=f'(t)\left\langle u_1(t), u_2(t), u_3(t)\right\rangle+f(t)\left\langle u_1'(t), u_2'(t), u_3'(t)\right\rangle$$

$$=f'(t)\,\boldsymbol{u}(t)+f(t)\,\boldsymbol{u}'(t)$$

43. $\frac{d}{dt}\left[\boldsymbol{u}(t)\times\boldsymbol{v}(t)\right]$

$$=\frac{d}{dt}\left\langle u_2(t)v_3(t)-u_3(t)v_2(t), u_3(t)v_1(t)-u_1(t)v_3(t), u_1(t)v_2(t)-u_2(t)v_1(t)\right\rangle$$

$$=\left\langle u_2'v_3(t)+u_2(t)v_3'(t)-u_3'(t)v_2(t)-u_3(t)v_2'(t),\right.$$

$$u_3'(t)v_1(t)+u_3(t)v_1'(t)-u_1'(t)v_3(t)-u_1(t)v_3'(t),$$

$$\left. u_1'(t)v_2(t)+u_1(t)v_2'(t)-u_2'(t)v_1(t)-u_2(t)v_1'(t)\right\rangle$$

$$=\left\langle u_2'(t)v_3(t)-u_3'(t)v_2(t), u_3'(t)v_1(t)-u_1'(t)v_3(t), u_1'(t)v_2(t)-u_2'(t)v_1(t)\right\rangle$$

$$+\left\langle u_2(t)v_3'(t)-u_3(t)v_2'(t), u_3(t)v_1'(t)-u_1(t)v_3'(t), u_1(t)v_2'(t)-u_2(t)v_1'(t)\right\rangle$$

$$=\boldsymbol{u}'(t)\times\boldsymbol{v}(t)+\boldsymbol{u}(t)\times\boldsymbol{v}'(t)$$

Alternate solution: Let $\boldsymbol{r}(t)=\boldsymbol{u}(t)\times\boldsymbol{v}(t)$. Then

$$\boldsymbol{r}(t+h)-\boldsymbol{r}(t)=\left[\boldsymbol{u}(t+h)\times\boldsymbol{v}(t+h)\right]-\left[\boldsymbol{u}(t)\times\boldsymbol{v}(t)\right]$$

$$=\left[\boldsymbol{u}(t+h)\times\boldsymbol{v}(t+h)\right]-\left[\boldsymbol{u}(t)\times\boldsymbol{v}(t)\right]+\left[\boldsymbol{u}(t+h)\times\boldsymbol{v}(t)\right]-\left[\boldsymbol{u}(t+h)\times\boldsymbol{v}(t)\right]$$

$$=\boldsymbol{u}(t+h)\times[\boldsymbol{v}(t+h)-\boldsymbol{v}(t)]+[\boldsymbol{u}(t+h)-\boldsymbol{u}(t)]\times\boldsymbol{v}(t)$$

(Be careful of the order of the cross product.)
Dividing through by h and taking the limit as $h\to 0$ we have

$$\boldsymbol{r}'(t)=\lim_{h\to 0}\frac{\boldsymbol{u}(t+h)\times[\boldsymbol{v}(t+h)-\boldsymbol{v}(t)]}{h}+\lim_{h\to 0}\frac{[\boldsymbol{u}(t+h)-\boldsymbol{u}(t)]\times\boldsymbol{v}(t)}{h}$$

$$=\boldsymbol{u}(t)\times\boldsymbol{v}'(t)+\boldsymbol{u}'(t)\times\boldsymbol{v}(t)$$

by Exercise 14.1.41(a) and Definition 1.

44.

$$\frac{d}{dt}[\boldsymbol{u}(f(t))]=\frac{d}{dt}\left\langle u_1(f(t)),\mathrm{u}_2(f(t)),\mathrm{u}_3(f(t))\right\rangle=\left\langle\frac{d}{dt}\left[u_1(f(t))\right],\frac{d}{dt}\left[u_2(f(t))\right],\frac{d}{dt}\left[u_3(f(t))\right]\right\rangle$$

$$=\left\langle f'(t)u_1'(f(t)),\mathrm{f}'(t)u_2'(f(t)),\mathrm{f}'(t)u_3'(f(t))\right\rangle=f'(t)\boldsymbol{u}'(t)$$

45.

$$\frac{d}{dt}[\boldsymbol{u}(t)\cdot\boldsymbol{v}(t)]=\boldsymbol{u}'(t)\cdot\boldsymbol{v}(t)+\boldsymbol{u}(t)\cdot\boldsymbol{v}'(t)$$

$$=(-4t\,\boldsymbol{j}+9t^2\boldsymbol{k})\cdot(t\,\boldsymbol{i}+\cos t\,\boldsymbol{j}+\sin t\,\boldsymbol{k})+(\boldsymbol{i}-2t^2\boldsymbol{j}+3t^3\boldsymbol{k})\cdot(\boldsymbol{i}-\sin t\,\boldsymbol{j}+\cos t\,\boldsymbol{k})$$

$$=-4t\cos t+9t^2\sin t+1+2t^2\sin t+3t^3\cos t$$

$$=1-4t\cos t+11t^2\sin t+3t^3\cos t$$

46.

$$\frac{d}{dt}[\boldsymbol{u}(t)\times\boldsymbol{v}(t)]=\boldsymbol{u}'(t)\times\boldsymbol{v}(t)+\boldsymbol{u}(t)\times\boldsymbol{v}'(t)$$

$$=(-4t\,\boldsymbol{j}+9t^2\boldsymbol{k})\times(t\,\boldsymbol{i}+\cos t\,\boldsymbol{j}+\sin t\,\boldsymbol{k})+(\boldsymbol{i}-2t^2\boldsymbol{j}+3t^3\boldsymbol{k})\times(\boldsymbol{i}-\sin t\,\boldsymbol{j}+\cos t\,\boldsymbol{k})$$

$$=(-4t\sin t-9t^2\cos t)\boldsymbol{i}+(9t^3-0)\boldsymbol{j}+(0+4t^2)\boldsymbol{k}$$

$$+(-2t^2\cos t+3t^3\sin t)\boldsymbol{i}+(3t^3-\cos t)\boldsymbol{j}+(-\sin t+2t^2)\boldsymbol{k}$$

$$=[(\sin t)(3t^3-4t)-11t^2\cos t]\boldsymbol{i}+(12t^3-\cos t)\boldsymbol{j}+(6t^2-\sin t)\boldsymbol{k}$$

47. $\frac{d}{dt}\left[\boldsymbol{r}(t)\times\boldsymbol{r}'(t)\right]=\boldsymbol{r}'(t)\times\boldsymbol{r}'(t)+\boldsymbol{r}(t)\times\boldsymbol{r}''(t)$ by Formula 5 of Theorem 3. But $\boldsymbol{r}'(t)\times\boldsymbol{r}'(t)=\boldsymbol{0}$ (see Example 13.4.2). Thus,

$\frac{d}{dt}\left[\boldsymbol{r}(t)\times\boldsymbol{r}'(t)\right]=\boldsymbol{r}(t)\times\boldsymbol{r}''(t)$.

48.

$$\begin{aligned}\frac{d}{dt}\left(\boldsymbol{u}(t)\cdot[\boldsymbol{v}(t)\times\boldsymbol{w}(t)]\right) &=\boldsymbol{u}'(t)\cdot[\boldsymbol{v}(t)\times\boldsymbol{w}(t)]+\boldsymbol{u}(t)\cdot\frac{d}{dt}[\boldsymbol{v}(t)\times\boldsymbol{w}(t)]\\ &=\boldsymbol{u}'(t)\cdot[\boldsymbol{v}(t)\times\boldsymbol{w}(t)]+\boldsymbol{u}(t)\cdot\left[\boldsymbol{v}'(t)\times\boldsymbol{w}(t)+\boldsymbol{v}(t)\times\boldsymbol{w}'(t)\right]\\ &=\boldsymbol{u}'(t)\cdot[\boldsymbol{v}(t)\times\boldsymbol{w}(t)]+\boldsymbol{u}(t)\cdot\left[\boldsymbol{v}'(t)\times\boldsymbol{w}(t)\right]+\boldsymbol{u}(t)\cdot\left[\boldsymbol{v}(t)\times\boldsymbol{w}'(t)\right]\\ &=\boldsymbol{u}'(t)\cdot[\boldsymbol{v}(t)\times\boldsymbol{w}(t)]-\boldsymbol{v}'(t)\cdot[\boldsymbol{u}(t)\times\boldsymbol{w}(t)]+\boldsymbol{w}'(t)\cdot[\boldsymbol{u}(t)\times\boldsymbol{v}(t)]\end{aligned}$$

49. $\frac{d}{dt}|\boldsymbol{r}(t)|=\frac{d}{dt}[\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)]^{1/2}=\frac{1}{2}[\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)]^{-1/2}\left[2\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)\right]=\frac{1}{|\boldsymbol{r}(t)|}\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)$

50. Since $\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)=0$, we have $0=2\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)=\frac{d}{dt}[\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)]=\frac{d}{dt}|\boldsymbol{r}(t)|^2$. Thus $|\boldsymbol{r}(t)|^2$, and so $|\boldsymbol{r}(t)|$, is a constant, and hence the curve lies on a sphere with center the origin.

51. Since $\boldsymbol{u}(t)=\boldsymbol{r}(t)\cdot\left[\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right]$, $\boldsymbol{u}'(\text{t})=\boldsymbol{r}'(\text{t})\cdot\left[\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right]+\boldsymbol{r}(\text{t})\cdot\frac{d}{dt}\left[\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right]$
$=0+\boldsymbol{r}(\text{t})\cdot\left[\boldsymbol{r}''(t)\times\boldsymbol{r}''(t)+\boldsymbol{r}'(t)\times\boldsymbol{r}'''(t)\right]=\boldsymbol{r}(\text{t})\cdot\left[\boldsymbol{r}'(t)\times\boldsymbol{r}'''(t)\right]$

1. $\mathbf{r}'(t)=\langle 2\cos t,5,-2\sin t\rangle \Rightarrow \left|\mathbf{r}'(t)\right|=\sqrt{(2\cos t)^2+5^2+(-2\sin t)^2}=\sqrt{29}$. Then using Formula 3, we have

$L=\int_{-10}^{10}\left|\mathbf{r}'(t)\right|dt=\int_{-10}^{10}\sqrt{29}\,dt=\left.\sqrt{29}\,t\right]_{-10}^{10}=20\sqrt{29}$.

2. $\mathbf{r}'(t)=\langle 2t,\cos t+t\sin t-\cos t,-\sin t+t\cos t+\sin t\rangle=\langle 2t,t\sin t,t\cos t\rangle \Rightarrow$
$\left|\mathbf{r}'(t)\right|=\sqrt{(2t)^2+(t\sin t)^2+(t\cos t)^2}=\sqrt{4t^2+t^2(\sin^2 t+\cos^2 t)}=\sqrt{5}\,|t|=\sqrt{5}\,t$ for $0\le t\le \pi$. Then using Formula 3, we have $L=\int_0^{\pi}\left|\mathbf{r}'(t)\right|dt=\int_0^{\pi}\sqrt{5}\,t\,dt=\left.\sqrt{5}\,\frac{t^2}{2}\right]_0^{\pi}=\frac{\sqrt{5}}{2}\pi^2$.

3. $\mathbf{r}'(t)=\sqrt{2}\,\mathbf{i}+e^t\mathbf{j}-e^{-t}\mathbf{k} \Rightarrow$
$\left|\mathbf{r}'(t)\right|=\sqrt{(\sqrt{2})^2+(e^t)^2+(-e^{-t})^2}=\sqrt{2+e^{2t}+e^{-2t}}=\sqrt{(e^t+e^{-t})^2}=e^t+e^{-t}$ (since $e^t+e^{-t}>0$).
Then $L=\int_0^1\left|\mathbf{r}'(t)\right|dt=\int_0^1(e^t+e^{-t})dt=\left[e^t-e^{-t}\right]_0^1=e-e^{-1}$.

4. $\mathbf{r}'(t)=\langle 2t,2,1/t\rangle$, $\left|\mathbf{r}'(t)\right|=\sqrt{4t^2+4+(1/t)^2}=\frac{1+2t^2}{|t|}=\frac{1+2t^2}{t}$ for $1\le t\le e$.
$L=\int_1^e\frac{1+2t^2}{t}\,dt=\int_1^e\left(\frac{1}{t}+2t\right)dt=\left[\ln t+t^2\right]_1^e=e^2$

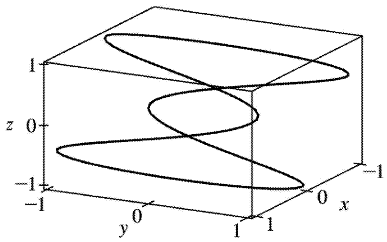
5. $\mathbf{r}'(t)=2t\,\mathbf{j}+3t^2\mathbf{k}\Rightarrow \left|\mathbf{r}'(t)\right|=\sqrt{4t^2+9t^4}=t\sqrt{4+9t^2}$ (since $t\ge 0$).
Then $L=\int_0^1\left|\mathbf{r}'(t)\right|dt=\int_0^1 t\sqrt{4+9t^2}\,dt=\left.\frac{1}{18}\cdot\frac{2}{3}(4+9t^2)^{3/2}\right]_0^1=\frac{1}{27}(13^{3/2}-4^{3/2})=\frac{1}{27}(13^{3/2}-8)$.

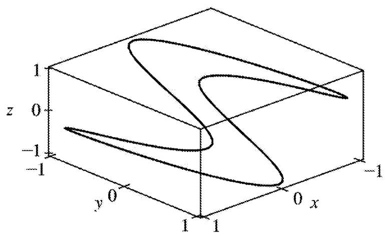
6. $\mathbf{r}'(t)=12\mathbf{i}+12\sqrt{t}\,\mathbf{j}+6t\,\mathbf{k}\Rightarrow \left|\mathbf{r}'(t)\right|=\sqrt{144+144t+36t^2}=\sqrt{36(t+2)^2}=6|t+2|=6(t+2)$ for $0\le t\le 1$.
Then $L=\int_0^1\left|\mathbf{r}'(t)\right|dt=\int_0^1 6(t+2)\,dt=\left[3t^2+12t\right]_0^1=15$.

7. The point $(2,4,8)$ corresponds to $t=2$, so by Equation 2, $L=\int_0^2\sqrt{(1)^2+(2t)^2+(3t^2)^2}\,dt$. If $f(t)=\sqrt{1+4t^2+9t^4}$, then Simpson's Rule gives

$L \approx \frac{2-0}{10 \cdot 3}\left[f(0)+4f(0.2)+2f(0.4)+\cdots+4f(1.8)+f(2)\right] \approx 9.5706$.

8. Here are two views of the curve with parametric equations $x=\cos t$, $y=\sin 3t$, $z=\sin t$:





The complete curve is given by the parameter interval $[0,2\pi]$, so

$L=\int_0^{2\pi}\sqrt{(-\sin t)^2+(3\cos 3t)^2+(\cos t)^2}\,dt=\int_0^{2\pi}\sqrt{1+9\cos^2 3t}\,dt \approx 13.9744$.

9. $\boldsymbol{r}'(t)=2\boldsymbol{i}-3\boldsymbol{j}+4\boldsymbol{k}$ and $\frac{ds}{dt}=\left|\boldsymbol{r}'(t)\right|=\sqrt{4+9+16}=\sqrt{29}$. Then $s=s(t)=\int_0^t\left|\boldsymbol{r}'(u)\right|du=\int_0^t\sqrt{29}\,du=\sqrt{29}\,t$.

Therefore, $t=\frac{1}{\sqrt{29}}s$, and substituting for t in the original equation, we have

$\boldsymbol{r}(t(s))=\frac{2}{\sqrt{29}}s\boldsymbol{i}+\left(1-\frac{3}{\sqrt{29}}s\right)\boldsymbol{j}+\left(5+\frac{4}{\sqrt{29}}s\right)\boldsymbol{k}$.

10. $\boldsymbol{r}'(t)=2e^{2t}(\cos 2t-\sin 2t)\boldsymbol{i}+2e^{2t}(\cos 2t+\sin 2t)\boldsymbol{k}$,

$\frac{ds}{dt}=\left|\boldsymbol{r}'(t)\right|=2e^{2t}\sqrt{(\cos 2t-\sin 2t)^2+(\cos 2t+\sin 2t)^2}=2e^{2t}\sqrt{2\cos^2 2t+2\sin^2 2t}=2\sqrt{2}\,e^{2t}$.

$s=s(t)=\int_0^t\left|\boldsymbol{r}'(u)\right|du=\int_0^t 2\sqrt{2}\,e^{2u}du=\left[\sqrt{2}\,e^{2u}\right]_0^t=\sqrt{2}\,(e^{2t}-1) \Rightarrow$

$\frac{s}{\sqrt{2}}+1=e^{2t} \Rightarrow t=\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)$. Substituting, we have

$\boldsymbol{r}(t(s))=e^{2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)}\cos 2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)\boldsymbol{i}+2\boldsymbol{j}+e^{2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)}\sin 2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)\boldsymbol{k}$

$$=\left(\frac{s}{\sqrt{2}}+1\right)\cos\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)\boldsymbol{i}+2\,\boldsymbol{j}+\left(\frac{s}{\sqrt{2}}+1\right)\sin\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)\boldsymbol{k}\ .$$

11. $\left|\boldsymbol{r}'(t)\right|=\sqrt{(3\cos t)^2+16+(-3\sin t)^2}=\sqrt{9+16}=5$ and $s(t)=\int_0^t\left|\boldsymbol{r}'(u)\right|du=\int_0^t 5\,du=5t\Rightarrow t(s)=\frac{1}{5}s$.

Therefore,

$\boldsymbol{r}(t(s))=3\sin\left(\frac{1}{5}s\right)\boldsymbol{i}+\frac{4}{5}s\,\boldsymbol{j}+3\cos\left(\frac{1}{5}s\right)\boldsymbol{k}$.

12. $\boldsymbol{r}'(t)=\frac{-4t}{(t^2+1)^2}\,\boldsymbol{i}+\frac{-2t^2+2}{(t^2+1)^2}\,\boldsymbol{j}$,

$$\frac{ds}{dt}=\left|\boldsymbol{r}'(t)\right|=\sqrt{\left[\frac{-4t}{(t^2+1)^2}\right]^2+\left[\frac{-2t^2+2}{(t^2+1)^2}\right]^2}=\sqrt{\frac{4t^4+8t^2+4}{(t^2+1)^4}}=\sqrt{\frac{4(t^2+1)^2}{(t^2+1)^4}}$$

$$=\sqrt{\frac{4}{(t^2+1)^2}}=\frac{2}{t^2+1}$$

Since the initial point $(1,0)$ corresponds to $t=0$, the arc length function

$s(t)=\int_0^t\left|\boldsymbol{r}'(u)\right|du=\int_0^t\frac{2}{u^2+1}\,du=2\arctan t$. Then $\arctan t=\frac{1}{2}s\Rightarrow t=\tan\frac{1}{2}s$. Substituting, we have

$$\boldsymbol{r}(t(s))=\left[\frac{2}{\tan^2\left(\frac{1}{2}s\right)+1}-1\right]\boldsymbol{i}+\frac{2\tan\left(\frac{1}{2}s\right)}{\tan^2\left(\frac{1}{2}s\right)+1}\,\boldsymbol{j}=\frac{1-\tan^2\left(\frac{1}{2}s\right)}{1+\tan^2\left(\frac{1}{2}s\right)}\,\boldsymbol{i}+\frac{2\tan\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)}\,\boldsymbol{j}$$

$$=\frac{1-\tan^2\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)}\,\boldsymbol{i}+2\tan\left(\frac{1}{2}s\right)\cos^2\left(\frac{1}{2}s\right)\boldsymbol{j}$$

$$=\left[\cos^2\left(\frac{1}{2}s\right)-\sin^2\left(\frac{1}{2}s\right)\right]\boldsymbol{i}+2\sin\left(\frac{1}{2}s\right)\cos\left(\frac{1}{2}s\right)\boldsymbol{j}=\cos s\,\boldsymbol{i}+\sin s\,\boldsymbol{j}$$

With this parametrization, we recognize the function as representing the unit circle. Note here that the curve approaches, but does not include, the point $(-1,0)$, since $\cos s=-1$ for $s=\pi+2k\pi$ (k an integer) but then $t=\tan\left(\frac{1}{2}s\right)$ is undefined.

13. **(a)** $\boldsymbol{r}'(t)=\langle 2\cos t,5,-2\sin t\rangle \Rightarrow \left|\boldsymbol{r}'(t)\right|=\sqrt{4\cos^2 t+25+4\sin^2 t}=\sqrt{29}$. Then

$\boldsymbol{T}(t)=\dfrac{\boldsymbol{r}'(t)}{\left|\boldsymbol{r}'(t)\right|}=\dfrac{1}{\sqrt{29}}\langle 2\cos t,5,-2\sin t\rangle$ or $\left\langle \dfrac{2}{\sqrt{29}}\cos t,\dfrac{5}{\sqrt{29}},-\dfrac{2}{\sqrt{29}}\sin t\right\rangle$.

$\boldsymbol{T}'(t)=\dfrac{1}{\sqrt{29}}\langle -2\sin t,0,-2\cos t\rangle \Rightarrow \left|\boldsymbol{T}'(t)\right|=\dfrac{1}{\sqrt{29}}\sqrt{4\sin^2 t+0+4\cos^2 t}=\dfrac{2}{\sqrt{29}}$. Thus

$\boldsymbol{N}(t)=\dfrac{\boldsymbol{T}'(t)}{\left|\boldsymbol{T}'(t)\right|}=\dfrac{1/\sqrt{29}}{2/\sqrt{29}}\langle -2\sin t,0,-2\cos t\rangle=\langle -\sin t,0,-\cos t\rangle$.

(b) $\kappa(t)=\dfrac{\left|\boldsymbol{T}'(t)\right|}{\left|\boldsymbol{r}'(t)\right|}=\dfrac{2/\sqrt{29}}{\sqrt{29}}=\dfrac{2}{29}$.

14. **(a)** $\boldsymbol{r}'(t)=\langle 2t,t\sin t,t\cos t\rangle \Rightarrow \left|\boldsymbol{r}'(t)\right|=\sqrt{4t^2+t^2\sin^2 t+t^2\cos^2 t}=\sqrt{5t^2}=\sqrt{5}\,t$ (since $t>0$).

Then $\boldsymbol{T}(t)=\dfrac{\boldsymbol{r}'(t)}{\left|\boldsymbol{r}'(t)\right|}=\dfrac{1}{\sqrt{5}\,t}\langle 2t,t\sin t,t\cos t\rangle=\dfrac{1}{\sqrt{5}}\langle 2,\sin t,\cos t\rangle$. $\boldsymbol{T}'(t)=\dfrac{1}{\sqrt{5}}\langle 0,\cos t,-\sin t\rangle \Rightarrow$

$\left|\boldsymbol{T}'(t)\right|=\dfrac{1}{\sqrt{5}}\sqrt{0+\cos^2 t+\sin^2 t}=\dfrac{1}{\sqrt{5}}$.

Thus $\boldsymbol{N}(t)=\dfrac{\boldsymbol{T}'(t)}{\left|\boldsymbol{T}'(t)\right|}=\dfrac{1/\sqrt{5}}{1/\sqrt{5}}\langle 0,\cos t,-\sin t\rangle=\langle 0,\cos t,-\sin t\rangle$.

(b) $\kappa(t)=\dfrac{\left|\boldsymbol{T}'(t)\right|}{\left|\boldsymbol{r}'(t)\right|}=\dfrac{1/\sqrt{5}}{\sqrt{5}\,t}=\dfrac{1}{5t}$.

15. **(a)** $\boldsymbol{r}'(t)=\left\langle \sqrt{2},e^t,-e^{-t}\right\rangle \Rightarrow \left|\boldsymbol{r}'(t)\right|=\sqrt{2+e^{2t}+e^{-2t}}=\sqrt{(e^t+e^{-t})^2}=e^t+e^{-t}$. Then

$\boldsymbol{T}(t)=\dfrac{\boldsymbol{r}'(t)}{\left|\boldsymbol{r}'(t)\right|}=\dfrac{1}{e^t+e^{-t}}\left\langle \sqrt{2},e^t,-e^{-t}\right\rangle=\dfrac{1}{e^{2t}+1}\left\langle \sqrt{2}\,e^t,e^{2t},-1\right\rangle$ $\left(\textit{after multiplying by } \dfrac{e^t}{e^t}\right)$ and

$$\begin{aligned}
\boldsymbol{T}'(t) &= \frac{1}{e^{2t}+1}\left\langle \sqrt{2}\,e^t,2e^{2t},0\right\rangle-\frac{2e^{2t}}{\left(e^{2t}+1\right)^2}\left\langle \sqrt{2}\,e^t,e^{2t},-1\right\rangle \\
&= \frac{1}{(e^{2t}+1)^2}\left[(e^{2t}+1)\left\langle \sqrt{2}\,e^t,2e^{2t},0\right\rangle-2e^{2t}\left\langle \sqrt{2}\,e^t,e^{2t},-1\right\rangle\right] \\
&= \frac{1}{(e^{2t}+1)^2}\left\langle \sqrt{2}\,e^t\left(1-e^{2t}\right),2e^{2t},2e^{2t}\right\rangle
\end{aligned}$$

Then $|\boldsymbol{T}'(t)| = \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1-2e^{2t}+e^{4t})+4e^{4t}+4e^{4t}} = \frac{1}{\left(e^{2t}+1\right)^2}\sqrt{2e^{2t}(1+2e^{2t}+e^{4t})}$

$= \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}\left(1+e^{2t}\right)^2} = \frac{\sqrt{2}\,e^t(1+e^{2t})}{(e^{2t}+1)^2} = \frac{\sqrt{2}\,e^t}{e^{2t}+1}$

Therefore $\boldsymbol{N}(t) = \frac{\boldsymbol{T}'(t)}{|\boldsymbol{T}'(t)|} = \frac{e^{2t}+1}{\sqrt{2}\,e^t}\,\frac{1}{(e^{2t}+1)^2}\left\langle \sqrt{2}\,e^t(1-e^{2t}),2e^{2t},2e^{2t}\right\rangle$

$= \frac{1}{\sqrt{2}\,e^t(e^{2t}+1)}\left\langle \sqrt{2}\,e^t(1-e^{2t}),2e^{2t},2e^{2t}\right\rangle$

$= \frac{1}{e^{2t}+1}\left\langle 1-e^{2t},\sqrt{2}\,e^t,\sqrt{2}\,e^t\right\rangle$

(b) $\kappa(t) = \frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}'(t)|} = \frac{\sqrt{2}\,e^t}{e^{2t}+1}\cdot\frac{1}{e^t+e^{-t}} = \frac{\sqrt{2}\,e^t}{e^{3t}+2e^t+e^{-t}} = \frac{\sqrt{2}\,e^{2t}}{e^{4t}+2e^{2t}+1} = \frac{\sqrt{2}\,e^{2t}}{(e^{2t}+1)^2}$.

16. **(a)** $\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} = \frac{1}{\sqrt{4t^2+4+(1/t)^2}}\left\langle 2t,2,1/t\right\rangle = \frac{|t|}{2t^2+1}\left\langle 2t,2,1/t\right\rangle$. But since the $\boldsymbol{k}$ –component is $\ln t$, t is positive, $|t|=t$ and

$\boldsymbol{T}(t) = \frac{1}{2t^2+1}\left\langle 2t^2,2t,1\right\rangle$. Then

$\boldsymbol{T}'(t) = \frac{1}{2t^2+1}\left\langle 4t,2,0\right\rangle - \left(2t^2+1\right)^{-2}(4t)\left\langle 2t^2,2t,1\right\rangle = \frac{1}{(2t^2+1)^2}\left\langle 4t,2-4t^2,-4t\right\rangle$, so

$\boldsymbol{N}(t) = \frac{\boldsymbol{T}'(t)}{|\boldsymbol{T}'(t)|} = \frac{\left\langle 4t,2-4t^2,-4t\right\rangle}{\sqrt{(4t)^2+(2-4t^2)^2+(-4t)^2}} = \frac{1}{2t^2+1}\left\langle 2t,1-2t^2,-2t\right\rangle$.

(b) $\kappa(t) = \frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}'(t)|} = \frac{2}{2t^2+1}\left(\frac{t}{2t^2+1}\right) = \frac{2t}{(2t^2+1)^2}$

17. $\boldsymbol{r}'(t)=2t\boldsymbol{i}+\boldsymbol{k}$, $\boldsymbol{r}''(t)=2\boldsymbol{i}$, $|\boldsymbol{r}'(t)| = \sqrt{(2t)^2+0^2+1^2} = \sqrt{4t^2+1}$, $\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)=2\boldsymbol{j}$, $|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)|=2$.

Then

$$\kappa(t)=\frac{\left|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right|}{\left|\boldsymbol{r}'(t)\right|^3}=\frac{2}{\left(\sqrt{4t^2+1}\right)^3}=\frac{2}{(4t^2+1)^{3/2}}\;.$$

18. $\boldsymbol{r}'(t)=\boldsymbol{i}+\boldsymbol{j}+2t\,\boldsymbol{k}$, $\boldsymbol{r}''(t)=2\boldsymbol{k}$, $\left|\boldsymbol{r}'(t)\right|=\sqrt{1^2+1^2+(2t)^2}=\sqrt{4t^2+2}$, $\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)=2\boldsymbol{i}-2\boldsymbol{j}$, $\left|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right|=\sqrt{2^2+2^2+0^2}=\sqrt{8}=2\sqrt{2}$. Then

$$\kappa(t)=\frac{\left|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right|}{\left|\boldsymbol{r}'(t)\right|^3}=\frac{2\sqrt{2}}{\left(\sqrt{4t^2+2}\right)^3}=\frac{2\sqrt{2}}{\left(\sqrt{2}\sqrt{2t^2+1}\right)^3}=\frac{1}{(2t^2+1)^{3/2}}\;.$$

19. $\boldsymbol{r}'(t)=3\boldsymbol{i}+4\cos t\,\boldsymbol{j}-4\sin t\,\boldsymbol{k}$, $\boldsymbol{r}''(t)=-4\sin t\,\boldsymbol{j}-4\cos t\,\boldsymbol{k}$, $\left|\boldsymbol{r}'(t)\right|=\sqrt{9+16\cos^2 t+16\sin^2 t}=\sqrt{9+16}=5$, $\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)=-16\boldsymbol{i}+12\cos t\,\boldsymbol{j}-12\sin t\,\boldsymbol{k}$, $\left|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right|=\sqrt{256+144\cos^2 t+144\sin^2 t}=\sqrt{400}=20$. Then $\kappa(t)=\dfrac{\left|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right|}{\left|\boldsymbol{r}'(t)\right|^3}=\dfrac{20}{5^3}=\dfrac{4}{25}$.

20. $\boldsymbol{r}'(t)=\left\langle e^t\cos t-e^t\sin t, e^t\cos t+e^t\sin t, 1\right\rangle$. The point $(1,0,0)$ corresponds to $t=0$, and $\boldsymbol{r}'(0)=\langle 1,1,1\rangle\Rightarrow\left|\boldsymbol{r}'(0)\right|=\sqrt{1^2+1^2+1^2}=\sqrt{3}$.

$$\begin{aligned}\boldsymbol{r}''(t)&=\left\langle e^t\cos t-e^t\sin t-e^t\cos t-e^t\sin t, e^t\cos t-e^t\sin t+e^t\cos t+e^t\sin t, 0\right\rangle\\&=\left\langle -2e^t\sin t, 2e^t\cos t, 0\right\rangle\Rightarrow\boldsymbol{r}''(0)=\langle 0,2,0\rangle\;.\end{aligned}$$

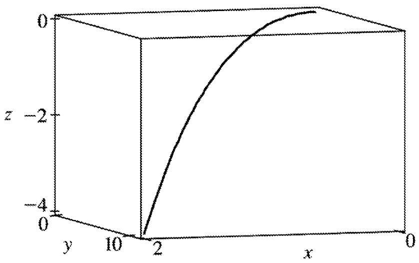
$\boldsymbol{r}'(0)\times\boldsymbol{r}''(0)=\langle -2,0,2\rangle$. $\left|\boldsymbol{r}'(0)\times\boldsymbol{r}''(0)\right|=\sqrt{(-2)^2+0^2+2^2}=\sqrt{8}=2\sqrt{2}$.

Then $\kappa(0)=\dfrac{\left|\boldsymbol{r}'(0)\times\boldsymbol{r}''(0)\right|}{\left|\boldsymbol{r}'(0)\right|^3}=\dfrac{2\sqrt{2}}{\left(\sqrt{3}\right)^3}=\dfrac{2\sqrt{2}}{3\sqrt{3}}$ or $\dfrac{2\sqrt{6}}{9}$.

21. $\boldsymbol{r}'(t)=\left\langle 1,2t,3t^2\right\rangle$. The point $(1,1,1)$ corresponds to $t=1$, and $\boldsymbol{r}'(1)=\langle 1,2,3\rangle\Rightarrow\left|\boldsymbol{r}'(1)\right|=\sqrt{1+4+9}=\sqrt{14}$. $\boldsymbol{r}''(t)=\langle 0,2,6t\rangle\Rightarrow\boldsymbol{r}''(1)=\langle 0,2,6\rangle$. $\boldsymbol{r}'(1)\times\boldsymbol{r}''(1)=\langle 6,-6,2\rangle$, so $\left|\boldsymbol{r}'(1)\times\boldsymbol{r}''(1)\right|=\sqrt{36+36+4}=\sqrt{76}$. Then

$$\kappa(1)=\frac{\left|\boldsymbol{r}'(1)\times\boldsymbol{r}''(1)\right|}{\left|\boldsymbol{r}'(1)\right|^3}=\frac{\sqrt{76}}{\sqrt{14}^3}=\frac{1}{7}\sqrt{\frac{19}{14}}\;.$$

22.



$\boldsymbol{r}(t)=\left\langle t,4t^{3/2},-t^{2}\right\rangle \Rightarrow \boldsymbol{r}'(t)=\left\langle 1,6t^{1/2},-2t\right\rangle$,

$\boldsymbol{r}''(t)=\left\langle 0,3t^{-1/2},-2\right\rangle$, $\left|\boldsymbol{r}'(t)\right|^{3}=\left(1+36t+4t^{2}\right)^{3/2}$,

$\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)=\left\langle -12t^{1/2}+6t^{1/2},2,3t^{-1/2}\right\rangle \Rightarrow$

$\left|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right|=\sqrt{36t+4+9t^{-1}}=\left[\dfrac{36t^{2}+4t+9}{t}\right]^{1/2}$

$$\kappa(t)=\frac{\left|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right|}{\left|\boldsymbol{r}'(t)\right|^{3}}=\left(\frac{36t^{2}+4t+9}{t}\right)^{1/2}\frac{1}{(1+36t+4t^{2})^{3/2}}=\frac{\sqrt{36t^{2}+4t+9}}{t^{1/2}(1+36t+4t^{2})^{3/2}} .$$

The point $(1,4,-1)$ corresponds to $t=1$, so the curvature at this point is $\kappa(1)=\dfrac{\sqrt{36+4+9}}{(1+36+4)^{3/2}}=\dfrac{7}{41\sqrt{41}}$.

23. $f(x)=x^{3}$, $f'(x)=3x^{2}$, $f''(x)=6x$, $\kappa(x)=\dfrac{\left|f''(x)\right|}{\left[1+\left(f'(x)\right)^{2}\right]^{3/2}}=\dfrac{6|x|}{\left(1+9x^{4}\right)^{3/2}}$

24. $f(x)=\cos x$, $f'(x)=-\sin x$, $f''(x)=-\cos x$,

$\kappa(x)=\dfrac{\left|f''(x)\right|}{\left[1+(f'(x))^{2}\right]^{3/2}}=\dfrac{\left|-\cos x\right|}{\left[1+(-\sin x)^{2}\right]^{3/2}}=\dfrac{\left|\cos x\right|}{(1+\sin^{2}x)^{3/2}}$

25. $f(x)=4x^{5/2}$, $f'(x)=10x^{3/2}$, $f''(x)=15x^{1/2}$,

$\kappa(x)=\dfrac{\left|f''(x)\right|}{\left[1+(f'(x))^{2}\right]^{3/2}}=\dfrac{\left|15x^{1/2}\right|}{\left[1+(10x^{3/2})^{2}\right]^{3/2}}=\dfrac{15\sqrt{x}}{(1+100x^{3})^{3/2}}$

26. $y'=\dfrac{1}{x}$, $y''=-\dfrac{1}{x^{2}}$,

$$\kappa(x)=\frac{\left|y''(x)\right|}{\left[1+\left(y'(x)\right)^2\right]^{3/2}}=\left|\frac{-1}{x^2}\right|\frac{1}{(1+1/x^2)^{3/2}}=\frac{1}{x^2}\frac{(x^2)^{3/2}}{(x^2+1)^{3/2}}=\frac{|x|}{(x^2+1)^{3/2}}=\frac{x}{(x^2+1)^{3/2}}$$

(since $x>0$). To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

$$\kappa'(x)=\frac{(x^2+1)^{3/2}-x\left(\frac{3}{2}\right)(x^2+1)^{1/2}(2x)}{\left[(x^2+1)^{3/2}\right]^2}=\frac{(x^2+1)^{1/2}[(x^2+1)-3x^2]}{(x^2+1)^3}=\frac{1-2x^2}{(x^2+1)^{5/2}}\ ;$$

$\kappa'(x)=0\Rightarrow 1-2x^2=0$, so the only critical number in the domain is $x=\frac{1}{\sqrt{2}}$. Since $\kappa'(x)>0$ for $0<x<\frac{1}{\sqrt{2}}$ and $\kappa'(x)<0$ for $x>\frac{1}{\sqrt{2}}$, $\kappa(x)$ attains its maximum at $x=\frac{1}{\sqrt{2}}$. Thus, the maximum curvature occurs at $\left(\frac{1}{\sqrt{2}},\ln\frac{1}{\sqrt{2}}\right)$. Since $\lim_{x\to\infty}\frac{x}{(x^2+1)^{3/2}}=0$, $\kappa(x)$ approaches 0 as $x\to\infty$.

27. Since $y'=y''=e^x$, the curvature is $\kappa(x)=\frac{\left|y''(x)\right|}{\left[1+(y'(x))^2\right]^{3/2}}=\frac{e^x}{(1+e^{2x})^{3/2}}=e^x(1+e^{2x})^{-3/2}$.

To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

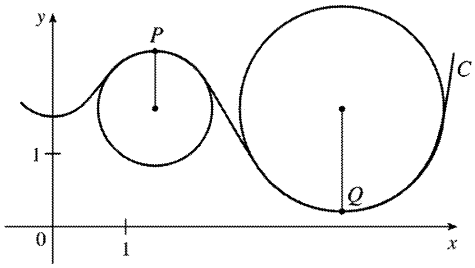
$$\kappa'(x)=e^x(1+e^{2x})^{-3/2}+e^x\left(-\frac{3}{2}\right)(1+e^{2x})^{-5/2}(2e^{2x})=e^x\frac{1+e^{2x}-3e^{2x}}{(1+e^{2x})^{5/2}}=e^x\frac{1-2e^{2x}}{(1+e^{2x})^{5/2}}\ .$$

$\kappa'(x)=0$ when $1-2e^{2x}=0$, so $e^{2x}=\frac{1}{2}$ or $x=-\frac{1}{2}\ln 2$. And since $1-2e^{2x}>0$ for $x<-\frac{1}{2}\ln 2$ and $1-2e^{2x}<0$ for $x>-\frac{1}{2}\ln 2$, the maximum curvature is attained at the point $\left(-\frac{1}{2}\ln 2,e^{(-\ln 2)/2}\right)=\left(-\frac{1}{2}\ln 2,\frac{1}{\sqrt{2}}\right)$. Since $\lim_{x\to\infty}e^x(1+e^{2x})^{-3/2}=0,\kappa(x)$ approaches 0 as $x\to\infty$.

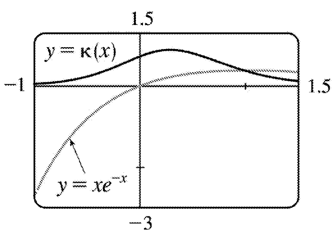
28. We can take the parabola as having its vertex at the origin and opening upward, so the equation is $f(x)=ax^2,a>0$. Then by Equation 11, $\kappa(x)=\frac{\left|f''(x)\right|}{[1+(f'(x))^2]^{3/2}}=\frac{|2a|}{[1+(2ax)^2]^{3/2}}=\frac{2a}{(1+4a^2x^2)^{3/2}}$, thus $\kappa(0)=2a$. We want $\kappa(0)=4$, so $a=2$ and the equation is $y=2x^2$.

29. **(a)** C appears to be changing direction more quickly at P than Q , so we would expect the curvature to be greater
at P .
(b) First we sketch approximate osculating circles at P and Q . Using the axes scale as a guide, we measure the radius of the osculating circle at P to be approximately 0.8 units, thus

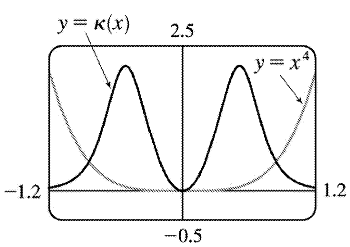
$\rho = \frac{1}{\kappa} \Rightarrow \kappa = \frac{1}{\rho} \approx \frac{1}{0.8} \approx 1.3$. Similarly, we estimate the radius of the osculating circle at Q to be 1.4 units, so $\kappa = \frac{1}{\rho} \approx \frac{1}{1.4} \approx 0.7$.



30. $y = xe^{-x} \Rightarrow y' = e^{-x}(1-x)$, $y'' = e^{-x}(x-2)$, and $\kappa(x) = \frac{|y''|}{[1+(y')^2]^{3/2}} = \frac{e^{-x}|x-2|}{[1+e^{-2x}(1-x)^2]^{3/2}}$. The graph of the curvature here is what we would expect. The graph of xe^{-x} is bending most sharply slightly to the right of the origin. As $x \to \infty$, the graph of xe^{-x} is asymptotic to the x –axis, and so the curvature approaches zero.



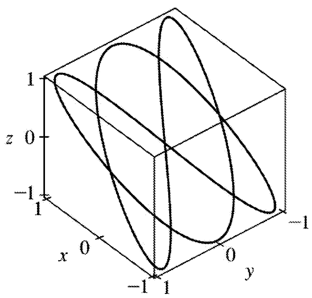
31. $y = x^4 \Rightarrow y' = 4x^3$, $y'' = 12x^2$, and $\kappa(x) = \frac{|y''|}{[1+(y')^2]^{3/2}} = \frac{12x^2}{(1+16x^6)^{3/2}}$. The appearance of the two humps in this graph is perhaps a little surprising, but it is explained by the fact that $y = x^4$ is very flat around the origin, and so here the curvature is zero.



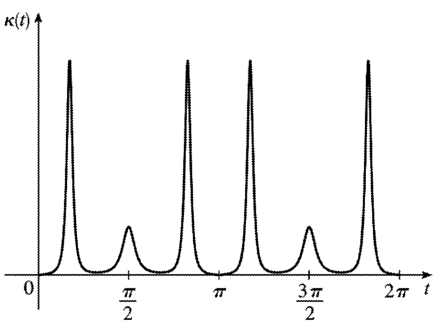
32. Notice that the curve a is highest for the same x –values at which curve b is turning more sharply, and a is 0 or near 0 where b is nearly straight. So, a must be the graph of $y = \kappa(x)$, and b is the graph of $y = f(x)$.

33. Notice that the curve b has two inflection points at which the graph appears almost straight. We would expect the curvature to be 0 or nearly 0 at these values, but the curve a isn't near 0 there. Thus, a must be the graph of $y=f(x)$ rather than the graph of curvature, and b is the graph of $y=\kappa(x)$.

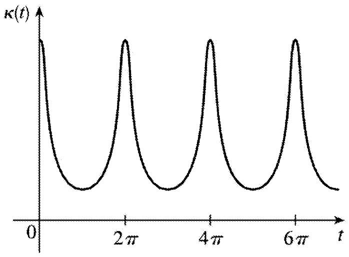
34. **(a)** The complete curve is given by $0\leq t\leq 2\pi$. Curvature appears to have a local (or absolute) maximum at 6 points. (Look at points where the curve appears to turn more sharply.)



(b) Using a CAS, we find (after simplifying) $\kappa(t)=\dfrac{3\sqrt{2}\sqrt{(5\sin t+\sin 5t)^2}}{(9\cos 6t+2\cos 4t+11)^{3/2}}$. (To compute cross products in Maple, use the Linalg package and the crossprod(a,b) command; in Mathematica, use Cross.) The graph shows 6 local (or absolute) maximum points for $0\leq t\leq 2\pi$, as observed in part (a).



35. Using a CAS, we find (after simplifying) $\kappa(t)=\dfrac{6\sqrt{4\cos^2 t-12\cos t+13}}{(17-12\cos t)^{3/2}}$. (To compute cross products in Maple, use the Linalg package and the crossprod (a,b) command; in Mathematica, use Cross.) Curvature is largest at integer multiples of 2π .



36. Here $\mathbf{r}(t)=\langle f(t),g(t)\rangle$, $\mathbf{r}'(t)=\langle f'(t),g'(t)\rangle$,

$\mathbf{r}''(t)=\langle f''(t),g''(t)\rangle$, $|\mathbf{r}'(t)|^3=\left[\sqrt{(f'(t))^2+(g'(t))^2}\right]^3=[(f'(t))^2+(g'(t))^2]^{3/2}=(\dot{x}^2+\dot{y}^2)^{3/2}$, and

$|\mathbf{r}'(t)\times\mathbf{r}''(t)|=|\langle 0,0,f'(t)g''(t)-f''(t)g'(t)\rangle|=\left[(\dot{x}\ddot{y}-\ddot{x}\dot{y})^2\right]^{1/2}=|\dot{x}\ddot{y}-\dot{y}\ddot{x}|$. Thus

$$\kappa(t)=\frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} .$$

37. $x=e^t\cos t\Rightarrow\dot{x}=e^t(\cos t-\sin t)\Rightarrow\ddot{x}=e^t(-\sin t-\cos t)+e^t(\cos t-\sin t)=-2e^t\sin t$,

$y=e^t\sin t\Rightarrow\dot{y}=e^t(\cos t+\sin t)\Rightarrow\ddot{y}=e^t(-\sin t+\cos t)+e^t(\cos t+\sin t)=2e^t\cos t$. Then

$$\begin{aligned}\kappa(t) &= \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}=\frac{|e^t(\cos t-\sin t)(2e^t\cos t)-e^t(\cos t+\sin t)(-2e^t\sin t)|}{\left([e^t(\cos t-\sin t)]^2+[e^t(\cos t+\sin t)]^2\right)^{3/2}}\\ &= \frac{|2e^{2t}(\cos^2 t-\sin t\cos t+\sin t\cos t+\sin^2 t)|}{\left[e^{2t}(\cos^2 t-2\cos t\sin t+\sin^2 t+\cos^2 t+2\cos t\sin t+\sin^2 t)\right]^{3/2}}\\ &= \frac{|2e^{2t}(1)|}{\left[e^{2t}(1+1)\right]^{3/2}}=\frac{2e^{2t}}{e^{3t}(2)^{3/2}}=\frac{1}{\sqrt{2}\,e^t}\end{aligned}$$

38. $x=1+t^3\Rightarrow\dot{x}=3t^2\Rightarrow\ddot{x}=6t$, $y=t+t^2\Rightarrow\dot{y}=1+2t\Rightarrow\ddot{y}=2$. Then

$$\kappa(t) = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}=\frac{|(3t^2)(2)-(1+2t)(6t)|}{\left[(3t^2)^2+(1+2t)^2\right]^{3/2}}=\frac{|-6t^2-6t|}{(9t^4+4t^2+4t+1)^{3/2}}$$

$$= \frac{6\left|t^2+t\right|}{(9t^4+4t^2+4t+1)^{3/2}}$$

39. $\left(1,\frac{2}{3},1\right)$ corresponds to $t=1$. $\boldsymbol{T}(t)=\frac{\boldsymbol{r}'(t)}{\left|\boldsymbol{r}'(t)\right|}=\frac{\left\langle 2t,2t^2,1\right\rangle}{\sqrt{4t^2+4t^4+1}}=\frac{\left\langle 2t,2t^2,1\right\rangle}{2t^2+1}$, so

$\boldsymbol{T}(1)=\left\langle \frac{2}{3},\frac{2}{3},\frac{1}{3} \right\rangle$.

$\boldsymbol{T}'(t)=-4t(2t^2+1)^{-2}\left\langle 2t,2t^2,1\right\rangle+(2t^2+1)^{-1}\left\langle 2,4t,0\right\rangle$ [by Theorem 14.2.3 [ET 13.2.3] #3]

$=(2t^2+1)^{-2}\left\langle -8t^2+4t^2+2,-8t^3+8t^3+4t,-4t\right\rangle=2(2t^2+1)^{-2}\left\langle 1-2t^2,2t,-2t\right\rangle$

$$\boldsymbol{N}(t)=\frac{\boldsymbol{T}'(t)}{\left|\boldsymbol{T}'(t)\right|}=\frac{2(2t^2+1)^{-2}\left\langle 1-2t^2,2t,-2t\right\rangle}{2(2t^2+1)^{-2}\sqrt{(1-2t^2)^2+(2t)^2+(-2t)^2}}=\frac{\left\langle 1-2t^2,2t,-2t\right\rangle}{\sqrt{1-4t^2+4t^4+8t^2}}$$

$$=\frac{\left\langle 1-2t^2,2t,-2t\right\rangle}{1+2t^2}$$

$\boldsymbol{N}(1)=\left\langle -\frac{1}{3},\frac{2}{3},-\frac{2}{3}\right\rangle$ and $\boldsymbol{B}(1)=\boldsymbol{T}(1)\times\boldsymbol{N}(1)=\left\langle -\frac{4}{9}-\frac{2}{9},-\left(-\frac{4}{9}+\frac{1}{9}\right),\frac{4}{9}+\frac{2}{9}\right\rangle=\left\langle -\frac{2}{3},\frac{1}{3},\frac{2}{3}\right\rangle$.

40. $(1,0,1)$ corresponds to $t=0$. $\boldsymbol{r}(t)=e^t\left\langle 1,\sin t,\cos t\right\rangle$, so

$\boldsymbol{r}'(t)=e^t\left\langle 1,\sin t,\cos t\right\rangle+e^t\left\langle 0,\cos t,-\sin t\right\rangle=e^t\left\langle 1,\sin t+\cos t,\cos t-\sin t\right\rangle$ and

$$\boldsymbol{T}(t)=\frac{\boldsymbol{r}'(t)}{\left|\boldsymbol{r}'(t)\right|}=\frac{e^t\left\langle 1,\sin t+\cos t,\cos t-\sin t\right\rangle}{e^t\sqrt{1+\sin^2 t+2\sin t\cos t+\cos^2 t+\cos^2 t-2\sin t\cos t+\sin^2 t}}$$

$$=\frac{\left\langle 1,\sin t+\cos t,\cos t-\sin t\right\rangle}{\sqrt{3}} ,$$

$\boldsymbol{T}(0)=\left\langle \frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right\rangle$. $\boldsymbol{T}'(t)=\frac{1}{\sqrt{3}}\left\langle 0,\cos t-\sin t,-\sin t-\cos t\right\rangle$, so

$$\boldsymbol{N}(t)=\frac{\boldsymbol{T}'(t)}{|\boldsymbol{T}'(t)|}=\frac{\frac{1}{\sqrt{3}}\langle 0,\cos t-\sin t,-\sin t-\cos t\rangle}{\frac{1}{\sqrt{3}}\sqrt{0^2+\cos^2 t-2\cos t\sin t+\sin^2 t+\sin^2 t+2\sin t\cos t+\cos^2 t}}$$

$$=\frac{1}{\sqrt{2}}\langle 0,\cos t-\sin t,-\sin t-\cos t\rangle .$$

$\boldsymbol{N}(0)=\left\langle 0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right\rangle$ and $\boldsymbol{B}(0)=\boldsymbol{T}(0)\times\boldsymbol{N}(0)=\left\langle -\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right\rangle$.

41. $(0,\pi,-2)$ corresponds to $t=\pi$. $\boldsymbol{r}(t)=\langle 2\sin 3t,t,2\cos 3t\rangle\Rightarrow$

$$\boldsymbol{T}(t)=\frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}=\frac{\langle 6\cos 3t,1,-6\sin 3t\rangle}{\sqrt{36\cos^2 3t+1+36\sin^2 3t}}=\frac{1}{\sqrt{37}}\langle 6\cos 3t,1,-6\sin 3t\rangle .$$

$\boldsymbol{T}(\pi)=\frac{1}{\sqrt{37}}\langle -6,1,0\rangle$ is a normal vector for the normal plane, and so $\langle -6,1,0\rangle$ is also normal. Thus an equation for the plane is $-6(x-0)+1(y-\pi)+0(z+2)=0$ or $y-6x=\pi$.

$$\boldsymbol{T}'(t)=\frac{1}{\sqrt{37}}\langle -18\sin 3t,0,-18\cos 3t\rangle\Rightarrow|\boldsymbol{T}'(t)|=\frac{\sqrt{18^2\sin^2 3t+18^2\cos^2 3t}}{\sqrt{37}}=\frac{18}{\sqrt{37}}\Rightarrow$$

$\boldsymbol{N}(t)=\frac{\boldsymbol{T}'(t)}{|\boldsymbol{T}'(t)|}=\langle -\sin 3t,0,-\cos 3t\rangle$. So $\boldsymbol{N}(\pi)=\langle 0,0,1\rangle$ and

$\boldsymbol{B}(\pi)=\frac{1}{\sqrt{37}}\langle -6,1,0\rangle\times\langle 0,0,1\rangle=\frac{1}{\sqrt{37}}\langle 1,6,0\rangle$. Since $\boldsymbol{B}(\pi)$ is a normal to the osculating plane, so is $\langle 1,6,0\rangle$ and an equation for the plane is $1(x-0)+6(y-\pi)+0(z+2)=0$ or $x+6y=6\pi$.

42. $t=1$ at $(1,1,1)$. $\boldsymbol{r}'(t)=\langle 1,2t,3t^2\rangle$. $\boldsymbol{r}'(1)=\langle 1,2,3\rangle$ is normal to the normal plane, so an equation for this plane is $1(x-1)+2(y-1)+3(z-1)=0$, or $x+2y+3z=6$.

$\boldsymbol{T}(t)=\frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}=\frac{1}{\sqrt{1+4t^2+9t^4}}\langle 1,2t,3t^2\rangle$. Using the product rule on each term of $\boldsymbol{T}(t)$ gives

$$\boldsymbol{T}'(t)=\frac{1}{\left(1+4t^2+9t^4\right)^{3/2}}\left\langle -\frac{1}{2}(8t+36t^3),2(1+4t^2+9t^4)-\frac{1}{2}(8t+36t^3)2t,\right.$$

$$\left. 6t(1+4t^2+9t^4)-\frac{1}{2}(8t+36t^3)3t^2\right\rangle$$

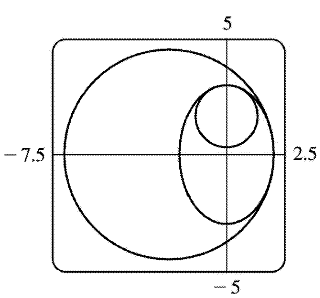
$$=\frac{1}{(1+4t^2+9t^4)^{3/2}}\left\langle -4t-18t^3, 2-18t^4, 6t+12t^3\right\rangle=\frac{-2}{(14)^{3/2}}\left\langle 11,8,-9\right\rangle \text{ when } t=1 .$$

$\boldsymbol{N}(1)\parallel \boldsymbol{T}'(1)\parallel \langle 11,8,-9\rangle$ and $\boldsymbol{T}(1)\parallel \boldsymbol{r}'(1)=\langle 1,2,3\rangle \Rightarrow$ a normal vector to the osculating plane is $\langle 11,8,-9\rangle\times\langle 1,2,3\rangle=\langle 42,-42,14\rangle$ or equivalently $\langle 3,-3,1\rangle$. An equation for the plane is $3(x-1)-3(y-1)+(z-1)=0$ or $3x-3y+z=1$.

43. The ellipse is given by the parametric equations $x=2\cos t$, $y=3\sin t$, so using the result from Exercise 36,

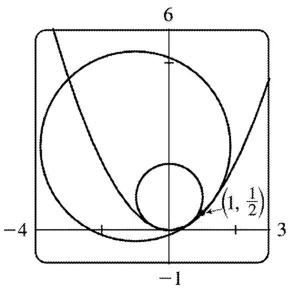
$$\kappa(t)=\frac{\left|\dot{x}\ddot{y}-\ddot{x}\dot{y}\right|}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}=\frac{\left|(-2\sin t)(-3\sin t)-(3\cos t)(-2\cos t)\right|}{(4\sin^2 t+9\cos^2 t)^{3/2}}=\frac{6}{(4\sin^2 t+9\cos^2 t)^{3/2}}$$

At $(2,0)$, $t=0$. Now $\kappa(0)=\frac{6}{27}=\frac{2}{9}$, so the radius of the osculating circle is $1/\kappa(0)=\frac{9}{2}$ and its center is $\left(-\frac{5}{2},0\right)$. Its equation is therefore $\left(x+\frac{5}{2}\right)^2+y^2=\frac{81}{4}$. At $(0,3)$, $t=\frac{\pi}{2}$, and $\kappa\left(\frac{\pi}{2}\right)=\frac{6}{8}=\frac{3}{4}$. So the radius of the osculating circle is $\frac{4}{3}$ and its center is $\left(0,\frac{5}{3}\right)$. Hence its equation is $x^2+\left(y-\frac{5}{3}\right)^2=\frac{16}{9}$.



44. $y=\frac{1}{2}x^2\Rightarrow y'=x$ and $y''=1$, so Formula 11 gives $\kappa(x)=\frac{1}{(1+x^2)^{3/2}}$. So the curvature at $(0,0)$ is $\kappa(0)=1$ and the osculating circle has radius 1 and center $(0,1)$, and hence equation $x^2+(y-1)^2=1$. The curvature at $\left(1,\frac{1}{2}\right)$ is $\kappa(1)=\frac{1}{(1+1^2)^{3/2}}=\frac{1}{2\sqrt{2}}$. The tangent line to the parabola at $\left(1,\frac{1}{2}\right)$ has slope 1 ,
so the normal line has slope -1 . Thus the center of the osculating circle lies in the direction of the

unit vector $\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. The circle has radius $2\sqrt{2}$, so its center has position vector $\left\langle 1, \frac{1}{2} \right\rangle + 2\sqrt{2} \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle -1, \frac{5}{2} \right\rangle$. So the equation of the circle is $(x+1)^2 + \left(y - \frac{5}{2} \right)^2 = 8$.



45. The tangent vector is normal to the normal plane, and the vector $\langle 6,6,-8 \rangle$ is normal to the given plane. But $\boldsymbol{T}(t) \parallel \boldsymbol{r}'(t)$ and $\langle 6,6,-8 \rangle \parallel \langle 3,3,-4 \rangle$, so we need to find t such that $\boldsymbol{r}'(t) \parallel \langle 3,3,-4 \rangle$. $\boldsymbol{r}(t) = \left\langle t^3, 3t, t^4 \right\rangle \Rightarrow \boldsymbol{r}'(t) = \left\langle 3t^2, 3, 4t^3 \right\rangle \parallel \langle 3,3,-4 \rangle$ when $t=-1$. So the planes are parallel at the point $\boldsymbol{r}(-1) = (-1,-3,1)$.

46. To find the osculating plane, we first calculate the tangent and normal vectors.
In Maple, we set x:=t^3; y:=3*t; and z:=t^4; and then calculate the components of the tangent vector $\boldsymbol{T}(t)$ using the diff command. We find that $\boldsymbol{T}(t) = \frac{\left\langle 3t^2, 3, 4t^3 \right\rangle}{\sqrt{16t^6 + 9t^4 + 9}}$. Differentiating the components of $\boldsymbol{T}(t)$, we find that $\boldsymbol{N}(t) = \frac{\boldsymbol{T}'(t)}{\left| \boldsymbol{T}'(t) \right|} = \frac{\left\langle -6t(8t^6-9), 3(48t^5+18t^3), 36t^2(t^4+3) \right\rangle}{\sqrt{144t(8t^6-9)^2 + 9(96t^5+36t^3)^2 + 5{,}184t^{12} + 31{,}104t^8 + 46{,}656t^4}}$.
In Maple, we can calculate $\boldsymbol{B}(t) = \boldsymbol{T}(t) \times \boldsymbol{N}(t)$ using the linalg package. First we define $\boldsymbol{T}$ and $\boldsymbol{N}$ using T:=array(); and N:=array(); where f, g, h, F, G, and H are the components of $\boldsymbol{T}$ and $\boldsymbol{N}$. Then we use the command B:=crossprod(T,N);. After normalization and simplification, we find that $\boldsymbol{B}(t) = b\left\langle 6\mathrm{t}, -2t^3, -3 \right\rangle$, where

$$b = \frac{t\sqrt{16t^6 + 9t^4 + 9}}{\sqrt{16t^2(8t^6-9)^2 + (96t^5+36t^3)^2 + 576t^{12} + 3456t^8 + 5184t^4}}$$

In Mathematica, we use the command Dt to differentiate the components of $\boldsymbol{r}(t)$ and subsequently $\boldsymbol{T}(t)$, and then load the vector analysis package with the command << Calculus‘VectorAnalysis‘. After setting T={f,g,h} and N={F,G,H} , we use CrossProduct [T, N] to find $\boldsymbol{B}$ (before normalization).

Now $\boldsymbol{B}(t)$ is parallel to $\left\langle 6t,-2t^3,-3\right\rangle$, so if $\boldsymbol{B}(t)$ is parallel to $\langle 1,1,1\rangle$ for some t , then $6t=1\Rightarrow t=\frac{1}{6}$, but $-2\left(\frac{1}{6}\right)^3\neq 1$. So there is no such osculating plane.

47. $\kappa=\left|\frac{d\boldsymbol{T}}{ds}\right|=\left|\frac{d\boldsymbol{T}/dt}{ds/dt}\right|=\frac{|d\boldsymbol{T}/dt|}{ds/dt}$ and $\boldsymbol{N}=\frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|}$, so $\kappa\boldsymbol{N}=\frac{\left|\frac{d\boldsymbol{T}}{dt}\right|\frac{d\boldsymbol{T}}{dt}}{\left|\frac{d\boldsymbol{T}}{dt}\right|\frac{ds}{dt}}=\frac{d\boldsymbol{T}/dt}{ds/dt}=\frac{d\boldsymbol{T}}{ds}$ by the Chain Rule.

48. For a plane curve, $\boldsymbol{T}=|\boldsymbol{T}|\cos\phi\,\boldsymbol{i}+|\boldsymbol{T}|\sin\phi\,\boldsymbol{j}=\cos\phi\,\boldsymbol{i}+\sin\phi\,\boldsymbol{j}$. Then $\frac{d\boldsymbol{T}}{ds}=\left(\frac{d\boldsymbol{T}}{d\phi}\right)\left(\frac{d\phi}{ds}\right)=(-\sin\phi\,\boldsymbol{i}+\cos\phi\,\boldsymbol{j})\left(\frac{d\phi}{ds}\right)$ and $\left|\frac{d\boldsymbol{T}}{ds}\right|=|-\sin\phi\,\boldsymbol{i}+\cos\phi\,\boldsymbol{j}|\left|\frac{d\phi}{ds}\right|=\left|\frac{d\phi}{ds}\right|$. Hence for a plane curve, the curvature is $\kappa=|d\phi/ds|$.

49. **(a)** $|\boldsymbol{B}|=1\Rightarrow\boldsymbol{B}\cdot\boldsymbol{B}=1\Rightarrow\frac{d}{ds}(\boldsymbol{B}\cdot\boldsymbol{B})=0\Rightarrow 2\frac{d\boldsymbol{B}}{ds}\cdot\boldsymbol{B}=0\Rightarrow\frac{d\boldsymbol{B}}{ds}\perp\boldsymbol{B}$

(b) $\boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}\Rightarrow$

$$\frac{d\boldsymbol{B}}{ds}=\frac{d}{ds}(\boldsymbol{T}\times\boldsymbol{N})=\frac{d}{dt}(\boldsymbol{T}\times\boldsymbol{N})\frac{1}{ds/dt}=\frac{d}{dt}(\boldsymbol{T}\times\boldsymbol{N})\frac{1}{|\boldsymbol{r}'(t)|}$$

$$=\left[(\boldsymbol{T}'\times\boldsymbol{N})+(\boldsymbol{T}\times\boldsymbol{N}')\right]\frac{1}{|\boldsymbol{r}'(t)|}=\left[\left(\boldsymbol{T}'\times\frac{\boldsymbol{T}'}{|\boldsymbol{T}'|}\right)+(\boldsymbol{T}\times\boldsymbol{N}')\right]\frac{1}{|\boldsymbol{r}'(t)|}=\frac{\boldsymbol{T}\times\boldsymbol{N}'}{|\boldsymbol{r}'(t)|}$$

$\Rightarrow\frac{d\boldsymbol{B}}{ds}\perp\boldsymbol{T}$

(c) $\boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}\Rightarrow\boldsymbol{T}\perp\boldsymbol{N}$, $\boldsymbol{B}\perp\boldsymbol{T}$ and $\boldsymbol{B}\perp\boldsymbol{N}$. So $\boldsymbol{B}$, $\boldsymbol{T}$ and $\boldsymbol{N}$ form an orthogonal set of vectors in the three–dimensional space R^3 . From parts (a) and (b), $d\boldsymbol{B}/ds$ is perpendicular to both $\boldsymbol{B}$ and $\boldsymbol{T}$, so $d\boldsymbol{B}/ds$ is parallel to $\boldsymbol{N}$. Therefore, $d\boldsymbol{B}/ds=-\tau(s)\boldsymbol{N}$, where $\tau(s)$ is a scalar.

(d) Since $\boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}$, $\boldsymbol{T}\perp\boldsymbol{N}$ and both $\boldsymbol{T}$ and $\boldsymbol{N}$ are unit vectors, $\boldsymbol{B}$ is a unit vector mutually perpendicular to both $\boldsymbol{T}$ and $\boldsymbol{N}$. For a plane curve, $\boldsymbol{T}$ and $\boldsymbol{N}$ always lie in the plane of the curve, so that $\boldsymbol{B}$ is a constant unit vector always perpendicular to the plane. Thus $d\boldsymbol{B}/ds=\boldsymbol{0}$, but $d\boldsymbol{B}/ds=-\tau(s)\boldsymbol{N}$ and $\boldsymbol{N}\neq\boldsymbol{0}$, so $\tau(s)=0$.

50. $\boldsymbol{N}=\boldsymbol{B}\times\boldsymbol{T}\Rightarrow$

$$\frac{d\boldsymbol{N}}{ds} = \frac{d}{ds}(\boldsymbol{B}\times\boldsymbol{T}) = \frac{d\boldsymbol{B}}{ds}\times\boldsymbol{T}+\boldsymbol{B}\times\frac{d\boldsymbol{T}}{ds} \quad \text{[by Theorem 14.2.3 [ET 13.2.3]\#5]}$$
$$= -\tau\boldsymbol{N}\times\boldsymbol{T}+\boldsymbol{B}\times\kappa\boldsymbol{N} \quad \text{[by Formulas 3 and 1]}$$
$$= -\tau(\boldsymbol{N}\times\boldsymbol{T})+\kappa(\boldsymbol{B}\times\boldsymbol{N}) \quad \text{[by Theorem 13.4.8 [ET 12.4.8]\#2]}$$

But $\boldsymbol{B}\times\boldsymbol{N}=\boldsymbol{B}\times(\boldsymbol{B}\times\boldsymbol{T})=(\boldsymbol{B}\cdot\boldsymbol{T})\boldsymbol{B}-(\boldsymbol{B}\cdot\boldsymbol{B})\boldsymbol{T}$ [by Theorem 13.4.8 [ET 12.4.8]#6] $=-\boldsymbol{T}\Rightarrow$ $d\boldsymbol{N}/ds=\tau(\boldsymbol{T}\times\boldsymbol{N})-\kappa\boldsymbol{T}=-\kappa\boldsymbol{T}+\tau\boldsymbol{B}$.

51. **(a)** $\boldsymbol{r}'=s'\boldsymbol{T}\Rightarrow\boldsymbol{r}''=s''\boldsymbol{T}+s'\boldsymbol{T}'=s''\boldsymbol{T}+s'\frac{d\boldsymbol{T}}{ds}s'=s''\boldsymbol{T}+\kappa(s')^2\boldsymbol{N}$ by the first Serret–Frenet formula.

(b) Using part (a), we have

$$\boldsymbol{r}'\times\boldsymbol{r}'' = (s'\boldsymbol{T})\times[s''\boldsymbol{T}+\kappa(s')^2\boldsymbol{N}]$$
$$= \left[(s'\boldsymbol{T})\times(s''\boldsymbol{T})\right]+\left[(s'\boldsymbol{T})\times(\kappa(s')^2\boldsymbol{N})\right]3]$$
$$= (s's'')(\boldsymbol{T}\times\boldsymbol{T})+\kappa(s')^3(\boldsymbol{T}\times\boldsymbol{N})=\boldsymbol{0}+\kappa(s')^3\boldsymbol{B}=\kappa(s')^3\boldsymbol{B}$$

(c) Using part (a), we have

$$\boldsymbol{r}''' = [s''\boldsymbol{T}+\kappa(s')^2\boldsymbol{N}]'=s'''\boldsymbol{T}+s''\boldsymbol{T}'+\kappa'(s')^2\boldsymbol{N}+2\kappa s's''\boldsymbol{N}+\kappa(s')^2\boldsymbol{N}'$$
$$= s'''\boldsymbol{T}+s''\frac{d\boldsymbol{T}}{ds}s'+\kappa'(s')^2\boldsymbol{N}+2\kappa s's''\boldsymbol{N}+\kappa(s')^2\frac{d\boldsymbol{N}}{ds}s'$$
$$= s'''\boldsymbol{T}+s''s'\kappa\boldsymbol{N}+\kappa'(s')^2\boldsymbol{N}+2\kappa s's''\boldsymbol{N}+\kappa(s')^3(-\kappa\boldsymbol{T}+\tau\boldsymbol{B}) \quad \text{[by the second formula]}$$
$$= [s'''-\kappa^2(s')^3]\boldsymbol{T}+[3\kappa s's''+\kappa'(s')^2]\boldsymbol{N}+\kappa\tau(s')^3\boldsymbol{B}$$

(d) Using parts (b) and (c) and the facts that $\boldsymbol{B}\cdot\boldsymbol{T}=0$, $\boldsymbol{B}\cdot\boldsymbol{N}=0$, and $\boldsymbol{B}\cdot\boldsymbol{B}=1$, we get

$$\frac{(\boldsymbol{r}'\times\boldsymbol{r}'')\cdot\boldsymbol{r}'''}{\left|\boldsymbol{r}'\times\boldsymbol{r}''\right|^2} = \frac{\kappa(s')^3\boldsymbol{B}\cdot\left\{[s'''-\kappa^2(s')^3]\boldsymbol{T}+[3\kappa s's''+\kappa'(s')^2]\boldsymbol{N}+\kappa\tau(s')^3\boldsymbol{B}\right\}}{\left|\kappa(s')^3\boldsymbol{B}\right|^2}$$
$$= \frac{\kappa(s')^3\kappa\tau(s')^3}{\left[\kappa(s')^3\right]^2}=\tau$$

52. First we find the quantities required to compute κ :

$$\boldsymbol{r}'(t)=\langle -a\sin t, a\cos t, b\rangle\Rightarrow\boldsymbol{r}''(t)=\langle -a\cos t,-a\sin t,0\rangle\Rightarrow\boldsymbol{r}'''(t)=\langle a\sin t,-a\cos t,0\rangle$$

$|\boldsymbol{r}'(t)| = \sqrt{(-a\sin t)^2+(a\cos t)^2+b^2} = \sqrt{a^2+b^2}$

$\boldsymbol{r}'(t)\times\boldsymbol{r}''(t) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$

$= ab\sin t\,\boldsymbol{i} - ab\cos t\,\boldsymbol{j} + a^2\boldsymbol{k}$

$|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)| = \sqrt{(ab\sin t)^2+(-ab\cos t)^2+(a^2)^2} = \sqrt{a^2b^2+a^4}$

$\left(\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right)\cdot\boldsymbol{r}'''(t) = (ab\sin t)(a\sin t)+(-ab\cos t)(-a\cos t)+(a^2)(0)=a^2b$

Then by Theorem 10,

$$\kappa(t)=\frac{|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|^3}=\frac{\sqrt{a^2b^2+a^4}}{\left(\sqrt{a^2+b^2}\right)^3}=\frac{a\sqrt{a^2+b^2}}{\left(\sqrt{a^2+b^2}\right)^3}=\frac{a}{a^2+b^2}$$

which is a constant.

From Exercise 51(d), the torsion τ is given by

$$\tau=\frac{(\boldsymbol{r}'\times\boldsymbol{r}'')\cdot\boldsymbol{r}'''}{|\boldsymbol{r}'\times\boldsymbol{r}''|^2}=\frac{a^2b}{\left(\sqrt{a^2b^2+a^4}\right)^2}=\frac{b}{a^2+b^2}$$

which is also a constant.

53. $\boldsymbol{r}=\left\langle t,\frac{1}{2}t^2,\frac{1}{3}t^3\right\rangle \Rightarrow \boldsymbol{r}'=\left\langle 1,t,t^2\right\rangle$, $\boldsymbol{r}''=\langle 0,1,2t\rangle$, $\boldsymbol{r}'''=\langle 0,0,2\rangle \Rightarrow \boldsymbol{r}'\times\boldsymbol{r}''=\left\langle t^2,-2t,1\right\rangle \Rightarrow$

$$\tau=\frac{\left(\boldsymbol{r}'\times\boldsymbol{r}''\right)\cdot\boldsymbol{r}'''}{|\boldsymbol{r}'\times\boldsymbol{r}''|^2}=\frac{\left\langle t^2,-2t,1\right\rangle\cdot\langle 0,0,2\rangle}{t^4+4t^2+1}=\frac{2}{t^4+4t^2+1}$$

54. $\boldsymbol{r}=\langle \sinh t,\cosh t,t\rangle \Rightarrow \boldsymbol{r}'=\langle \cosh t,\sinh t,1\rangle$, $\boldsymbol{r}''=\langle \sinh t,\cosh t,0\rangle$, $\boldsymbol{r}'''=\langle \cosh t,\sinh t,0\rangle \Rightarrow$

$\boldsymbol{r}'\times\boldsymbol{r}''=\left\langle -\cosh t,\sinh t,\cosh^2 t-\sinh^2 t\right\rangle=\langle -\cosh t,\sinh t,1\rangle \Rightarrow$

$$\kappa=\frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|^3}=\frac{|\langle -\cosh t,\sinh t,1\rangle|}{|\langle \cosh t,\sinh t,1\rangle|^3}=\frac{\sqrt{\cosh^2 t+\sinh^2 t+1}}{\left(\cosh^2 t+\sinh^2 t+1\right)^{3/2}}=\frac{1}{\cosh^2 t+\sinh^2 t+1}=\frac{1}{2\cosh^2 t},$$

$$\tau=\frac{\left(\boldsymbol{r}'\times\boldsymbol{r}''\right)\cdot\boldsymbol{r}'''}{\left|\boldsymbol{r}'\times\boldsymbol{r}''\right|^2}=\frac{\langle -\cosh t,\sinh t,1\rangle\cdot\langle\cosh t,\sinh t,0\rangle}{\cosh^2 t+\sinh^2 t+1}=\frac{-\cosh^2 t+\sinh^2 t}{2\cosh^2 t}=\frac{-1}{2\cosh^2 t}$$

So at the point $(0,1,0)$, $t=0$, and $\kappa=\frac{1}{2}$ and $\tau=-\frac{1}{2}$.

55. For one helix, the vector equation is $\boldsymbol{r}(t)=\langle 10\cos t,10\sin t,34t/(2\pi)\rangle$ (measuring in angstroms), because the radius of each helix is 10 angstroms, and z increases by 34 angstroms for each increase of 2π in t . Using the arc length formula, letting t go from 0 to $2.9\times 10^8\times 2\pi$, we find the approximate length of each helix to be

$$L=\int_0^{2.9\times 10^8\times 2\pi}\left|\boldsymbol{r}'(t)\right|dt=\int_0^{2.9\times 10^8\times 2\pi}\sqrt{(-10\sin t)^2+(10\cos t)^2+\left(\frac{34}{2\pi}\right)^2}\,dt$$

$$=\sqrt{100+\left(\frac{34}{2\pi}\right)^2}\,t\,\Bigg]_0^{2.9\times 10^8\times 2\pi}=2.9\times 10^8\times 2\pi\sqrt{100+\left(\frac{34}{2\pi}\right)^2}$$

$\approx 2.07\times 10^{10}$——more than two meters!

56. **(a)** For the function $F(x)=\begin{cases}0 & \text{if } x<0\\ P(x) & \text{if } 0<x<1\\ 1 & \text{if } x\geq 1\end{cases}$ to be continuous, we must have $P(0)=0$ and $P(1)=1$.

For F' to be continuous, we must have $P'(0)=P'(1)=0$. The curvature of the curve $y=F(x)$ at the point $(x,F(x))$ is $\kappa(x)=\dfrac{\left|F''(x)\right|}{\left(1+\left[F'(x)\right]^2\right)^{3/2}}$. For $\kappa(x)$ to be continuous, we must have $P''(0)=P''(1)=0$.

Write $P(x)=ax^5+bx^4+cx^3+dx^2+ex+f$. Then $P'(x)=5ax^4+4bx^3+3cx^2+2dx+e$ and $P''(x)=20ax^3+12bx^2+6cx+2d$. Our six conditions are:

$P(0)=0\Rightarrow f=0$ (1)

$P(1)=1\Rightarrow a+b+c+d+e+f=1$ (2)

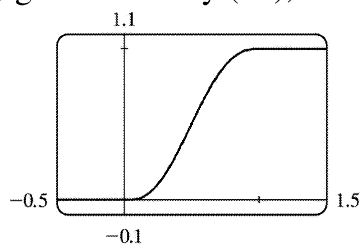
$P'(0)=0\Rightarrow e=0$ (3)

$P'(1)=0\Rightarrow 5a+4b+3c+2d+e=0$ (4)

$P''(0)=0\Rightarrow d=0$ (5)

$P''(1)=0\Rightarrow 20a+12b+6c+2d=0$ (6)

From (1), (3), and (5), we have $d=e=f=0$. Thus (2), (4) and (6) become (7) $a+b+c=1$, (8) $5a+4b+3c=0$, and (9) $10a+6b+3c=0$. Subtracting (8) from (9) gives (10) $5a+2b=0$. Multiplying (7) by 3 and subtracting from (8) gives (11) $2a+b=-3$. Multiplying (11) by 2 and subtracting from (10) gives $a=6$. By (10), $b=-15$. By (7), $c=10$. Thus, $P(x)=6x^5-15x^4+10x^3$.

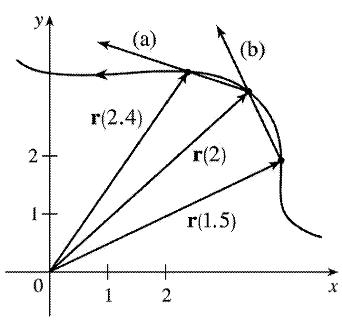
(b) 

1. **(a)** If $\boldsymbol{r}(t)=x(t)\boldsymbol{i}+y(t)\,\boldsymbol{j}+z(t)\boldsymbol{k}$ is the position vector of the particle at time t , then the average velocity over the time interval $[0,1]$ is
$\boldsymbol{v}_{\text{ave}}=\dfrac{\boldsymbol{r}(1)-\boldsymbol{r}(0)}{1-0}=\dfrac{(4.5\boldsymbol{i}+6.0\,\boldsymbol{j}+3.0\boldsymbol{k})-(2.7\boldsymbol{i}+9.8\,\boldsymbol{j}+3.7\boldsymbol{k})}{1}=1.8\boldsymbol{i}-3.8\,\boldsymbol{j}-0.7\boldsymbol{k}$. Similarly, over the other intervals we have

$$\begin{aligned}
[0.5,1]:\boldsymbol{v}_{\text{ave}} &= \frac{\boldsymbol{r}(1)-\boldsymbol{r}(0.5)}{1-0.5}=\frac{(4.5\boldsymbol{i}+6.0\,\boldsymbol{j}+3.0\boldsymbol{k})-(3.5\boldsymbol{i}+7.2\,\boldsymbol{j}+3.3\boldsymbol{k})}{0.5}\\
&= 2.0\boldsymbol{i}-2.4\,\boldsymbol{j}-0.6\boldsymbol{k}\\
[1,2]:\boldsymbol{v}_{\text{ave}} &= \frac{\boldsymbol{r}(2)-\boldsymbol{r}(1)}{2-1}=\frac{(7.3\boldsymbol{i}+7.8\,\boldsymbol{j}+2.7\boldsymbol{k})-(4.5\boldsymbol{i}+6.0\,\boldsymbol{j}+3.0\boldsymbol{k})}{1}\\
&= 2.8\boldsymbol{i}+1.8\,\boldsymbol{j}-0.3\boldsymbol{k}\\
[1,1.5]:\boldsymbol{v}_{\text{ave}} &= \frac{\boldsymbol{r}(1.5)-\boldsymbol{r}(1)}{1.5-1}=\frac{(5.9\boldsymbol{i}+6.4\,\boldsymbol{j}+2.8\boldsymbol{k})-(4.5\boldsymbol{i}+6.0\,\boldsymbol{j}+3.0\boldsymbol{k})}{0.5}\\
&= 2.8\boldsymbol{i}+0.8\,\boldsymbol{j}-0.4\boldsymbol{k}
\end{aligned}$$

(b) We can estimate the velocity at $t=1$ by averaging the average velocities over the time intervals $[0.5,1]$ and $[1,1.5]$: $\boldsymbol{v}(1)\approx\frac{1}{2}\,[(2\boldsymbol{i}-2.4\,\boldsymbol{j}-0.6\boldsymbol{k})+(2.8\boldsymbol{i}+0.8\,\boldsymbol{j}-0.4\boldsymbol{k})]=2.4\boldsymbol{i}-0.8\,\boldsymbol{j}-0.5\boldsymbol{k}$. Then the speed is $|\boldsymbol{v}(1)|\approx\sqrt{(2.4)^2+(-0.8)^2+(-0.5)^2}\approx 2.58$.

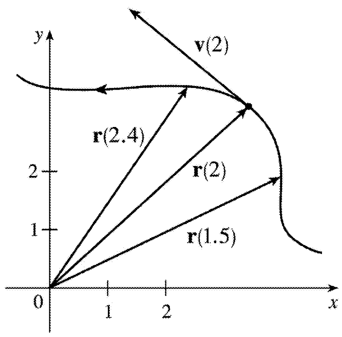
2. **(a)** The average velocity over $2\le t\le 2.4$ is
$\dfrac{\boldsymbol{r}(2.4)-\boldsymbol{r}(2)}{2.4-2}=2.5[\boldsymbol{r}(2.4)-\boldsymbol{r}(2)]$, so we sketch a vector in the same direction but 2.5 times the length of $[\boldsymbol{r}(2.4)-\boldsymbol{r}(2)]$.



(b) The average velocity over $1.5\le t\le 2$ is $\dfrac{\boldsymbol{r}(2)-\boldsymbol{r}(1.5)}{2-1.5}=2[\boldsymbol{r}(2)-\boldsymbol{r}(1.5)]$, so we sketch a vector in the same direction but twice the length of $[\boldsymbol{r}(2)-\boldsymbol{r}(1.5)]$.

(c) Using Equation 2 we have $\boldsymbol{v}(2)=\lim\limits_{h\to 0}\dfrac{\boldsymbol{r}(2+h)-\boldsymbol{r}(2)}{h}$.

(d) $\boldsymbol{v}(2)$ is tangent to the curve at $\boldsymbol{r}(2)$ and points in the direction of increasing t . Its length is the speed of the particle at $t=2$. We can estimate the speed by averaging the lengths of the vectors found in parts (a) and (b) which represent the average speed over $2\le t\le 2.4$ and $1.5\le t\le 2$ respectively. Using the axes scale as a guide, we estimate the vectors to have lengths 2.8 and 2.7 . Thus, we estimate the speed at $t=2$ to be $|\boldsymbol{v}(2)|\approx \frac{1}{2}(2.8+2.7)=2.75$ and we draw the velocity vector $\boldsymbol{v}(2)$ with this length.

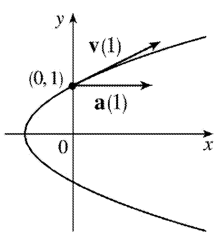


3. $\boldsymbol{r}(t)=\left\langle t^2-1,t\right\rangle \Rightarrow$ At $t=1$:

$\boldsymbol{v}(t)=\boldsymbol{r}'(t)=\langle 2t,1\rangle$, $\boldsymbol{v}(1)=\langle 2,1\rangle$

$\boldsymbol{a}(t)=\boldsymbol{r}''(t)=\langle 2,0\rangle$, $\boldsymbol{a}(1)=\langle 2,0\rangle$

$|\boldsymbol{v}(t)|=\sqrt{4t^2+1}$

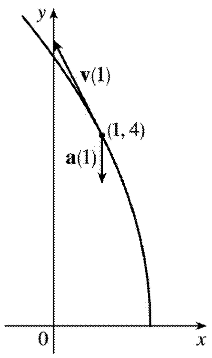


4. $\boldsymbol{r}(t)=\left\langle 2-t,4\sqrt{t}\right\rangle \Rightarrow$ At $t=1$:

$\boldsymbol{v}(t)=\boldsymbol{r}'(t)=\left\langle -1,2/\sqrt{t}\right\rangle$, $\boldsymbol{v}(1)=\langle -1,2\rangle$

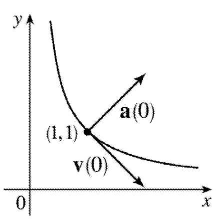
$\boldsymbol{a}(t)=\boldsymbol{r}''(t)=\left\langle 0,-1/t^{3/2}\right\rangle$, $\boldsymbol{a}(1)=\langle 0,-1\rangle$

$|\boldsymbol{v}(t)|=\sqrt{1+4/t}$



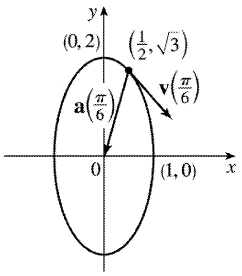
5. $\boldsymbol{r}(t)=e^{t}\boldsymbol{i}+e^{-t}\boldsymbol{j}\Rightarrow$
$\boldsymbol{v}(t)=e^{t}\boldsymbol{i}-e^{-t}\boldsymbol{j}$,
$\boldsymbol{a}(t)=e^{t}\boldsymbol{i}+e^{-t}\boldsymbol{j}$
At $t=0$:
$\boldsymbol{v}(0)=\boldsymbol{i}-\boldsymbol{j}$,
$\boldsymbol{a}(0)=\boldsymbol{i}+\boldsymbol{j}$

$|\boldsymbol{v}(t)|=\sqrt{e^{2t}+e^{-2t}}=e^{-t}\sqrt{e^{4t}+1}$

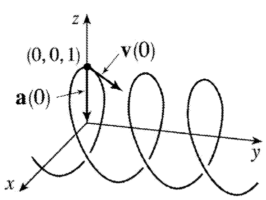


Since $x=e^{t}$, $t=\ln x$ and $y=e^{-t}=e^{-\ln x}=1/x$, and $x>0$, $y>0$.

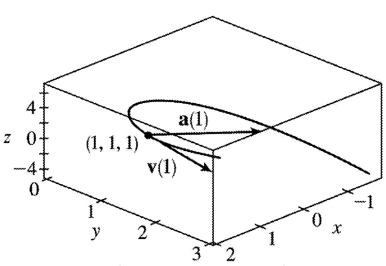
6. $\boldsymbol{r}(t)=\sin t\,\boldsymbol{i}+2\cos t\,\boldsymbol{j}\Rightarrow$
$\boldsymbol{v}(t)=\cos t\,\boldsymbol{i}-2\sin t\,\boldsymbol{j}$, $\boldsymbol{v}\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}\,\boldsymbol{i}-\boldsymbol{j}$
$\boldsymbol{a}(t)=-\sin t\,\boldsymbol{i}-2\cos t\,\boldsymbol{j}$, $\boldsymbol{a}\left(\frac{\pi}{6}\right)=-\frac{1}{2}\,\boldsymbol{i}-\sqrt{3}\,\boldsymbol{j}$
$|\boldsymbol{v}(t)|=\sqrt{\cos^{2}t+4\sin^{2}t}=\sqrt{1+3\sin^{2}t}$
And $x^{2}+y^{2}/4=\sin^{2}t+\cos^{2}t=1$, an ellipse.



7. $\boldsymbol{r}(t)=\sin t\,\boldsymbol{i}+t\,\boldsymbol{j}+\cos t\,\boldsymbol{k}\Rightarrow$
$\boldsymbol{v}(t)=\cos t\,\boldsymbol{i}+\boldsymbol{j}-\sin t\,\boldsymbol{k}$, $\boldsymbol{v}(0)=\boldsymbol{i}+\boldsymbol{j}$
$\boldsymbol{a}(t)=-\sin t\,\boldsymbol{i}-\cos t\,\boldsymbol{k}$, $\boldsymbol{a}(0)=-\boldsymbol{k}$
$|\boldsymbol{v}(t)|=\sqrt{\cos^2 t+1+\sin^2 t}=\sqrt{2}$
Since $x^2+z^2=1$, $y=t$, the path of the particle is a helix about the y –axis.



8. $\boldsymbol{r}(t)=t\,\boldsymbol{i}+t^2\,\boldsymbol{j}+t^3\,\boldsymbol{k}\Rightarrow$
$\boldsymbol{v}(t)=\boldsymbol{i}+2t\,\boldsymbol{j}+3t^2\,\boldsymbol{k}$, $\boldsymbol{v}(1)=\boldsymbol{i}+2\,\boldsymbol{j}+3\,\boldsymbol{k}\boldsymbol{a}(t)=2\,\boldsymbol{j}+6t\,\boldsymbol{k}$, $\boldsymbol{a}(1)=2\,\boldsymbol{j}+6\,\boldsymbol{k}$
$|\boldsymbol{v}(t)|=\sqrt{1+4t^2+9t^4}$
The path is a ‘‘twisted cubic’’ (see Example 14.1.7).



9. $\boldsymbol{r}(t)=\left\langle t^2+1,t^3,t^2-1\right\rangle\Rightarrow\boldsymbol{v}(t)=\boldsymbol{r}'(t)=\left\langle 2t,3t^2,2t\right\rangle$, $\boldsymbol{a}(t)=\boldsymbol{v}'(t)=\left\langle 2,6t,2\right\rangle$,
$|\boldsymbol{v}(t)|=\sqrt{(2t)^2+(3t^2)^2+(2t)^2}=\sqrt{9t^4+8t^2}=|t|\sqrt{9t^2+8}$.
10. $\boldsymbol{r}(t)=\left\langle 2\cos t,3t,2\sin t\right\rangle\Rightarrow\boldsymbol{v}(t)=\boldsymbol{r}'(t)=\left\langle -2\sin t,3,2\cos t\right\rangle$, $\boldsymbol{a}(t)=\boldsymbol{v}'(t)=\left\langle -2\cos t,0,-2\sin t\right\rangle$,
$|\boldsymbol{v}(t)|=\sqrt{4\sin^2 t+9+4\cos^2 t}=\sqrt{13}$.

11. $\boldsymbol{r}(t)=\sqrt{2}\,t\,\boldsymbol{i}+e^{t}\boldsymbol{j}+e^{-t}\boldsymbol{k}\Rightarrow\boldsymbol{v}(t)=\boldsymbol{r}'(t)=\sqrt{2}\,\boldsymbol{i}+e^{t}\boldsymbol{j}-e^{-t}\boldsymbol{k}$, $\boldsymbol{a}(t)=\boldsymbol{v}'(t)=e^{t}\boldsymbol{j}+e^{-t}\boldsymbol{k}$,
$|\boldsymbol{v}(t)|=\sqrt{2+e^{2t}+e^{-2t}}=\sqrt{(e^{t}+e^{-t})^{2}}=e^{t}+e^{-t}$.

12. $\boldsymbol{r}(t)=t^{2}\boldsymbol{i}+\ln t\,\boldsymbol{j}+t\,\boldsymbol{k}\Rightarrow\boldsymbol{v}(t)=\boldsymbol{r}'(t)=2t\,\boldsymbol{i}+t^{-1}\boldsymbol{j}+\boldsymbol{k}$, $\boldsymbol{a}(t)=\boldsymbol{v}'(t)=2\boldsymbol{i}-t^{-2}\boldsymbol{j}$, $|\boldsymbol{v}(t)|=\sqrt{4t^{2}+t^{-2}+1}$.

13. $\boldsymbol{r}(t)=e^{t}\langle\cos t,\sin t,t\rangle\Rightarrow\boldsymbol{v}(t)=\boldsymbol{r}'(t)=e^{t}\langle\cos t,\sin t,t\rangle+e^{t}\langle-\sin t,\cos t,1\rangle=e^{t}\langle\cos t-\sin t,\sin t+\cos t,t+1\rangle$

$$\begin{aligned}\boldsymbol{a}(t)&=\boldsymbol{v}'(t)=e^{t}\langle\cos t-\sin t-\sin t-\cos t,\sin t+\cos t+\cos t-\sin t,t+1+1\rangle\\&=e^{t}\langle-2\sin t,2\cos t,t+2\rangle\end{aligned}$$

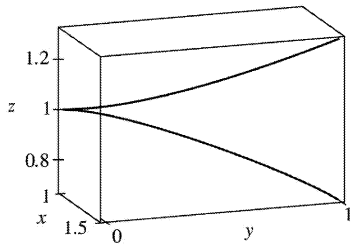
$$\begin{aligned}|\boldsymbol{v}(t)|&=e^{t}\sqrt{\cos^{2}t+\sin^{2}t-2\cos t\sin t+\sin^{2}t+\cos^{2}t+2\sin t\cos t+t^{2}+2t+1}\\&=e^{t}\sqrt{t^{2}+2t+3}\end{aligned}$$

14. $\boldsymbol{r}(t)=t\sin t\,\boldsymbol{i}+t\cos t\,\boldsymbol{j}+t^{2}\boldsymbol{k}\Rightarrow\boldsymbol{v}(t)=\boldsymbol{r}'(t)=(\sin t+t\cos t)\,\boldsymbol{i}+(\cos t-t\sin t)\,\boldsymbol{j}+2t\,\boldsymbol{k}$,
$\boldsymbol{a}(t)=\boldsymbol{v}'(t)=(2\cos t-t\sin t)\,\boldsymbol{i}+(-2\sin t-t\cos t)\,\boldsymbol{j}+2\boldsymbol{k}$,
$|\boldsymbol{v}(t)|=\sqrt{(\sin^{2}t+2t\sin t\cos t+t^{2}\cos^{2}t)+(\cos^{2}t-2t\sin t\cos t+t^{2}\sin^{2}t)+4t^{2}}=\sqrt{5t^{2}+1}$.

15. $\boldsymbol{a}(t)=\boldsymbol{k}\Rightarrow\boldsymbol{v}(t)=\int\boldsymbol{a}(t)\,dt=\int\boldsymbol{k}\,dt=t\,\boldsymbol{k}+\boldsymbol{c}_1$ and $\boldsymbol{i}-\boldsymbol{j}=\boldsymbol{v}(0)=0\,\boldsymbol{k}+\boldsymbol{c}_1$, so $\boldsymbol{c}_1=\boldsymbol{i}-\boldsymbol{j}$ and $\boldsymbol{v}(t)=\boldsymbol{i}-\boldsymbol{j}+t\,\boldsymbol{k}$.
$\boldsymbol{r}(t)=\int\boldsymbol{v}(t)\,dt=\int(\boldsymbol{i}-\boldsymbol{j}+t\,\boldsymbol{k})\,dt=t\,\boldsymbol{i}-t\,\boldsymbol{j}+\frac{1}{2}t^{2}\boldsymbol{k}+\boldsymbol{c}_2$. But $\boldsymbol{0}=\boldsymbol{r}(0)=\boldsymbol{0}+\boldsymbol{c}_2$, so $\boldsymbol{c}_2=\boldsymbol{0}$ and $\boldsymbol{r}(t)=t\,\boldsymbol{i}-t\,\boldsymbol{j}+\frac{1}{2}t^{2}\boldsymbol{k}$.

16. $\boldsymbol{a}(t)=-10\boldsymbol{k}\Rightarrow\boldsymbol{v}(t)=\int(-10\boldsymbol{k})\,dt=-10t\,\boldsymbol{k}+\boldsymbol{c}_1$, and $\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}=\boldsymbol{v}(0)=\boldsymbol{0}+\boldsymbol{c}_1$, so $\boldsymbol{c}_1=\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}$ and
$\boldsymbol{v}(t)=\boldsymbol{i}+\boldsymbol{j}-(10t+1)\,\boldsymbol{k}$.
$\boldsymbol{r}(t)=\int[\boldsymbol{i}+\boldsymbol{j}-(10t+1)\boldsymbol{k}]\,dt=t\,\boldsymbol{i}+t\,\boldsymbol{j}-(5t^{2}+t)\,\boldsymbol{k}+\boldsymbol{c}_2$. But $2\boldsymbol{i}+3\boldsymbol{j}=\boldsymbol{r}(0)=\boldsymbol{0}+\boldsymbol{c}_2$, so $\boldsymbol{c}_2=2\boldsymbol{i}+3\boldsymbol{j}$ and
$\boldsymbol{r}(t)=(t+2)\,\boldsymbol{i}+(t+3)\,\boldsymbol{j}-(5t^{2}+t)\,\boldsymbol{k}$.

17. **(a)** $\boldsymbol{a}(t)=\boldsymbol{i}+2\boldsymbol{j}+2t\,\boldsymbol{k}\Rightarrow\boldsymbol{v}(t)=\int(\boldsymbol{i}+2\boldsymbol{j}+2t\,\boldsymbol{k})\,dt=t\,\boldsymbol{i}+2t\,\boldsymbol{j}+t^{2}\boldsymbol{k}+\boldsymbol{c}_1$, and $\boldsymbol{0}=\boldsymbol{v}(0)=\boldsymbol{0}+\boldsymbol{c}_1$, so $\boldsymbol{c}_1=\boldsymbol{0}$ and
$\boldsymbol{v}(t)=\boldsymbol{i}+2t\,\boldsymbol{j}+t^{2}\boldsymbol{k}$. $\boldsymbol{r}(t)=\int(t\,\boldsymbol{i}+2t\,\boldsymbol{j}+t^{2}\boldsymbol{k})\,dt=\frac{1}{2}t^{2}\boldsymbol{i}+t^{2}\boldsymbol{j}+\frac{1}{3}t^{3}\boldsymbol{k}+\boldsymbol{c}_2$. But $\boldsymbol{i}+\boldsymbol{k}=\boldsymbol{r}(0)=\boldsymbol{0}+\boldsymbol{c}_2$, so $\boldsymbol{c}_2=\boldsymbol{i}+\boldsymbol{k}$ and
$\boldsymbol{r}(t)=\left(1+\frac{1}{2}t^{2}\right)\boldsymbol{i}+t^{2}\boldsymbol{j}+\left(1+\frac{1}{3}t^{3}\right)\boldsymbol{k}$.

(b) 

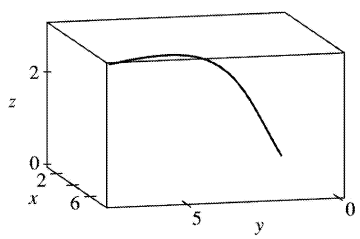
18. **(a)** $\boldsymbol{a}(t)=t\,\boldsymbol{i}+t^2\,\boldsymbol{j}+\cos 2t\,\boldsymbol{k}\Rightarrow$

$\boldsymbol{v}(t)=\int(t\,\boldsymbol{i}+t^2\,\boldsymbol{j}+\cos 2t\,\boldsymbol{k})\,dt$

$=\frac{t^2}{2}\,\boldsymbol{i}+\frac{t^3}{3}\,\boldsymbol{j}+\frac{\sin 2t}{2}\,\boldsymbol{k}+\boldsymbol{c}_1$

and $\boldsymbol{i}+\boldsymbol{k}=\boldsymbol{v}(0)=\boldsymbol{0}+\boldsymbol{c}_1$, so $\boldsymbol{c}_1=\boldsymbol{i}+\boldsymbol{k}$ and

$\boldsymbol{v}(t)=\left(\frac{1}{2}t^2+1\right)\boldsymbol{i}+\frac{1}{3}t^3\boldsymbol{j}+\left(1+\frac{1}{2}\sin 2t\right)\boldsymbol{k}$.



(b)

$\boldsymbol{r}(t)=\int\left[\left(\frac{1}{2}t^2+1\right)\boldsymbol{i}+\frac{1}{3}t^3\boldsymbol{j}+\left(1+\frac{1}{2}\sin 2t\right)\boldsymbol{k}\right]dt=\left(\frac{1}{6}t^3+t\right)\boldsymbol{i}+\frac{1}{12}t^4\boldsymbol{j}+\left(t-\frac{1}{4}\cos 2t\right)\boldsymbol{k}+\boldsymbol{c}_2$

But $\boldsymbol{j}=\boldsymbol{r}(0)=-\frac{1}{4}\,\boldsymbol{k}+\boldsymbol{c}_2$, so $\boldsymbol{c}_2=\boldsymbol{j}+\frac{1}{4}\,\boldsymbol{k}$ and $\boldsymbol{r}(t)=\left(\frac{1}{6}t^3+t\right)\boldsymbol{i}+\left(1+\frac{1}{12}t^4\right)\boldsymbol{j}+\left(\frac{1}{4}+t-\frac{1}{4}\cos 2t\right)\boldsymbol{k}$.

19. $\boldsymbol{r}(t)=\left\langle t^2,5t,t^2-16t\right\rangle\Rightarrow\boldsymbol{v}(t)=\langle 2t,5,2t-16\rangle$, $|\boldsymbol{v}(t)|=\sqrt{4t^2+25+4t^2-64t+256}=\sqrt{8t^2-64t+281}$ and $\frac{d}{dt}\,|\boldsymbol{v}(t)|=\frac{1}{2}\,(8t^2-64t+281)^{-1/2}(16t-64)$. This is zero if and only if the numerator is zero, that is, $16t-64=0$ or $t=4$. Since $\frac{d}{dt}\,|\boldsymbol{v}(t)|<0$ for $t<4$ and $\frac{d}{dt}\,|\boldsymbol{v}(t)|>0$ for $t>4$, the minimum speed of $\sqrt{153}$ is attained at $t=4$ units of time.

20. Since $\boldsymbol{r}(t)=t^3\,\boldsymbol{i}+t^2\,\boldsymbol{j}+t^3\,\boldsymbol{k}$, $\boldsymbol{a}(t)=\boldsymbol{r}''(t)=6t\,\boldsymbol{i}+2\,\boldsymbol{j}+6t\,\boldsymbol{k}$. By Newton’s Second Law, $\boldsymbol{F}(t)=m\,\boldsymbol{a}(t)=6mt\,\boldsymbol{i}+2m\,\boldsymbol{j}+6mt\,\boldsymbol{k}$ is the required force.

21. $|\boldsymbol{F}(t)|=20$ N in the direction of the positive z –axis, so $\boldsymbol{F}(t)=20\boldsymbol{k}$. Also $m=4$ kg, $\boldsymbol{r}(0)=\boldsymbol{0}$ and

$\boldsymbol{v}(0)=\boldsymbol{i}-\boldsymbol{j}$. Since $20\boldsymbol{k}=\boldsymbol{F}(t)=4\,\boldsymbol{a}(t)$, $\boldsymbol{a}(t)=5\,\boldsymbol{k}$. Then $\boldsymbol{v}(t)=5t\,\boldsymbol{k}+\boldsymbol{c}_1$ where $\boldsymbol{c}_1=\boldsymbol{i}-\boldsymbol{j}$ so $\boldsymbol{v}(t)=\boldsymbol{i}-\boldsymbol{j}+5t\,\boldsymbol{k}$ and the speed is $|\boldsymbol{v}(t)|=\sqrt{1+1+25t^2}=\sqrt{25t^2+2}$. Also $\boldsymbol{r}(t)=t\,\boldsymbol{i}-t\,\boldsymbol{j}+\frac{5}{2}t^2\boldsymbol{k}+\boldsymbol{c}_2$ and $\boldsymbol{0}=\boldsymbol{r}(0)$, so $\boldsymbol{c}_2=\boldsymbol{0}$ and $\boldsymbol{r}(t)=t\,\boldsymbol{i}-t\,\boldsymbol{j}+\frac{5}{2}t^2\boldsymbol{k}$.

22. The argument here is the same as that in Example 14.2.5 with $\boldsymbol{r}(t)$ replaced by $\boldsymbol{v}(t)$ and $\boldsymbol{r}'(t)$ replaced
by $\boldsymbol{a}(t)$.

23. $|\boldsymbol{v}(0)|=500$ m / s and since the angle of elevation is $30°$, the direction of the velocity is $\frac{1}{2}\left(\sqrt{3}\,\boldsymbol{i}+\boldsymbol{j}\right)$. Thus $\boldsymbol{v}(0)=250\left(\sqrt{3}\,\boldsymbol{i}+\boldsymbol{j}\right)$ and if we set up the axes so the projectile starts at the origin, then $\boldsymbol{r}(0)=\boldsymbol{0}$. Ignoring air resistance, the only force is that due to gravity, so $\boldsymbol{F}(t)=-mg\,\boldsymbol{j}$ where $g\approx 9.8$ m / s^2 . Thus $\boldsymbol{a}(t)=-g\,\boldsymbol{j}$ and $\boldsymbol{v}(t)=-gt\,\boldsymbol{j}+\boldsymbol{c}_1$. But $250\left(\sqrt{3}\,\boldsymbol{i}+\boldsymbol{j}\right)=\boldsymbol{v}(0)=\boldsymbol{c}_1$, so $\boldsymbol{v}(t)=250\sqrt{3}\,\boldsymbol{i}+(250-gt)\,\boldsymbol{j}$ and $\boldsymbol{r}(t)=250\sqrt{3}\,t\,\boldsymbol{i}+\left(250t-\frac{1}{2}gt^2\right)\boldsymbol{j}+\boldsymbol{c}_2$ where $\boldsymbol{0}=\boldsymbol{r}(0)=\boldsymbol{c}_2$. Thus $\boldsymbol{r}(t)=250\sqrt{3}\,t\,\boldsymbol{i}+\left(250t-\frac{1}{2}gt^2\right)\boldsymbol{j}$.

(a) Setting $250t-\frac{1}{2}gt^2=0$ gives $t=0$ or $t=\frac{500}{g}\approx 51.0$ s. So the range is $250\sqrt{3}\cdot\frac{500}{g}\approx 22$ km.

(b) $0=\frac{d}{dt}\left(250t-\frac{1}{2}gt^2\right)=250-gt$ implies that the maximum height is attained when $t=250/g\approx 25.5$ s. Thus, the maximum height is $(250)(250/g)-g(250/g)^2\frac{1}{2}=(250)^2/(2g)\approx 3.2$ km.

(c) From part (a), impact occurs at $t=500/g\approx 51.0$. Thus, the velocity at impact is $\boldsymbol{v}(500/g)=250\sqrt{3}\,\boldsymbol{i}+[250-g(500/g)]\,\boldsymbol{j}=250\sqrt{3}\,\boldsymbol{i}-250\,\boldsymbol{j}$ and the speed is $|\boldsymbol{v}(500/g)|=250\sqrt{3+1}=500$ m / s.

24. As in Exercise 23, $\boldsymbol{v}(t)=250\sqrt{3}\,\boldsymbol{i}+(250-gt)\,\boldsymbol{j}$ and $\boldsymbol{r}(t)=250\sqrt{3}\,t\,\boldsymbol{i}+\left(250t-\frac{1}{2}gt^2\right)\boldsymbol{j}+\boldsymbol{c}_2$. But $\boldsymbol{r}(0)=200\,\boldsymbol{j}$, so $\boldsymbol{c}_2=200\,\boldsymbol{j}$ and $\boldsymbol{r}(t)=250\sqrt{3}\,t\,\boldsymbol{i}+\left(200+250t-\frac{1}{2}gt^2\right)\boldsymbol{j}$.

(a) $200+250t-\frac{1}{2}gt^2=0$ implies that $gt^2-500t-400=0$ or $t=\frac{500\pm\sqrt{500^2+1600g}}{2g}$. Taking the positive t -value gives $t=\frac{500+\sqrt{250{,}000+1600g}}{2g}\approx 51.8$ s. Thus the range is

$\left(250\sqrt{3}\right)\frac{500+\sqrt{250{,}000+1600g}}{2g}\approx 22.4$ km.

(b)

$0=\frac{d}{dt}\left(200+250t-\frac{1}{2}gt^2\right)=250-gt$ implies that the maximum height is attained when $t=250/g\approx 25.5$ s and thus the maximum height is $\left[200+(250)\left(\frac{250}{g}\right)-\frac{g}{2}\left(\frac{250}{g}\right)^2\right]=200+\frac{(250)^2}{2g}\approx 3.4$ km.
Alternate solution: Because the projectile is fired in the same direction and with the same velocity as in Exercise 23, but from a point 200 m higher, the maximum height reached is 200 m higher than that found in Exercise 23, that is, 3.2 km +200 m =3.4 km.
(c) From part (a), impact occurs at $t=\frac{500+\sqrt{250{,}000+1600g}}{2g}$. Thus the velocity at impact is $250\sqrt{3}\,\boldsymbol{i}+\left[250-g\,\frac{500+\sqrt{250{,}000+1600g}}{2g}\right]\boldsymbol{j}$, so $|\boldsymbol{v}|\approx\sqrt{(250)^2(3)+(250-51.8g)^2}\approx 504$ m / s.

25. As in Example 5, $\boldsymbol{r}(t)=(v_0\cos 45^\circ)t\,\boldsymbol{i}+\left[(v_0\sin 45^\circ)t-\frac{1}{2}gt^2\right]\boldsymbol{j}=\frac{1}{2}\left[v_0\sqrt{2}\,t\,\boldsymbol{i}+\left(v_0\sqrt{2}\,t-gt^2\right)\boldsymbol{j}\right]$.

Then the ball lands at $t=\frac{v_0\sqrt{2}}{g}$ s. Now since it lands 90 m away, $90=\frac{1}{2}v_0\sqrt{2}\,\frac{v_0\sqrt{2}}{g}$ or $v_0^2=90g$ and the initial velocity is $v_0=\sqrt{90g}\approx 30$ m / s.

26. As in Example 5, $\boldsymbol{r}(t)=(v_0\cos 30^\circ)t\,\boldsymbol{i}+\left[(v_0\sin 30^\circ)t-\frac{1}{2}gt^2\right]\boldsymbol{j}=\frac{1}{2}\left[v_0\sqrt{3}\,t\,\boldsymbol{i}+(v_0t-gt^2)\boldsymbol{j}\right]$ and then $\boldsymbol{v}(t)=\boldsymbol{r}'(t)=\frac{1}{2}\left[v_0\sqrt{3}\,\boldsymbol{i}+(v_0-2gt)\boldsymbol{j}\right]$. The shell reaches its maximum height when the vertical component of velocity is zero, so $\frac{1}{2}(v_0-2gt)=0\Rightarrow t=\frac{v_0}{2g}$. The vertical height of the shell at that time is 500 m, so $\frac{1}{2}\left[v_0\left(\frac{v_0}{2g}\right)-g\left(\frac{v_0}{2g}\right)^2\right]=500\Rightarrow\frac{v_0^2}{8g}=500\Rightarrow v_0=\sqrt{4000g}=\sqrt{4000(9.8)}\approx 198$ m / s.

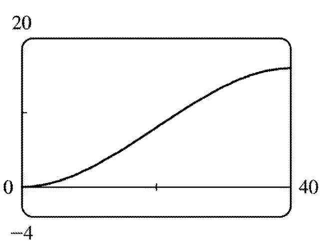
27. Let α be the angle of elevation. Then $v_0=150$ m / s and from Example 5, the horizontal distance traveled by the projectile is $d=\frac{v_0^2\sin 2\alpha}{g}$. Thus $\frac{150^2\sin 2\alpha}{g}=800\Rightarrow\sin 2\alpha=\frac{800g}{150^2}\approx 0.3484\Rightarrow$ $2\alpha\approx 20.4^\circ$ or $180-20.4=159.6^\circ$. Two angles of elevation then are $\alpha\approx 10.2^\circ$ and $\alpha\approx 79.8^\circ$.

28. Here $v_0=115$ ft / s, the angle of elevation is $\alpha=50^\circ$, and if we place the origin at home plate, then $\boldsymbol{r}(0)=3\,\boldsymbol{j}$. As in Example 5, we have

$\boldsymbol{r}(t)=-\frac{1}{2}gt^2\boldsymbol{j}+t\boldsymbol{v}_0+\boldsymbol{D}$ where $\boldsymbol{D}=\boldsymbol{r}(0)=3\boldsymbol{j}$ and $\boldsymbol{v}_0=v_0\cos\alpha\,\boldsymbol{i}+v_0\sin\alpha\,\boldsymbol{j}$, so $\boldsymbol{r}(t)=(v_0\cos\alpha)t\,\boldsymbol{i}+\left[(v_0\sin\alpha)t-\frac{1}{2}gt^2+3\right]\boldsymbol{j}$. Thus, parametric equations for the trajectory of the ball are $x=(v_0\cos\alpha)t$, $y=(v_0\sin\alpha)t-\frac{1}{2}gt^2+3$. The ball reaches the fence when $x=400\Rightarrow(v_0\cos\alpha)t=400\Rightarrow t=\frac{400}{v_0\cos\alpha}=\frac{400}{115\cos 50^\circ}\approx 5.41$ s. At this time, the height of the ball is

$y=(v_0\sin\alpha)t-\frac{1}{2}gt^2+3\approx(115\sin 50^\circ)(5.41)-\frac{1}{2}(32)(5.41)^2+3\approx 11.2$ ft. Since the fence is 10 ft high, the ball clears the fence.

29. **(a)** After t seconds, the boat will be $5t$ meters west of point A . The velocity of the water at that location is $\frac{3}{400}(5t)(40-5t)\boldsymbol{j}$. The velocity of the boat in still
water is $5\boldsymbol{i}$, so the resultant velocity of the boat is $\boldsymbol{v}(t)=5\boldsymbol{i}+\frac{3}{400}(5t)(40-5t)\boldsymbol{j}=5\boldsymbol{i}+\left(\frac{3}{2}t-\frac{3}{16}t^2\right)\boldsymbol{j}$.
Integrating, we obtain $\boldsymbol{r}(t)=5t\,\boldsymbol{i}+\left(\frac{3}{4}t^2-\frac{1}{16}t^3\right)\boldsymbol{j}+\boldsymbol{C}$.



If we place the origin at A (and consider $\boldsymbol{j}$ to coincide with the northern direction) then $\boldsymbol{r}(0)=\boldsymbol{0}\Rightarrow\boldsymbol{C}=\boldsymbol{0}$ and we have $\boldsymbol{r}(t)=5t\,\boldsymbol{i}+\left(\frac{3}{4}t^2-\frac{1}{16}t^3\right)\boldsymbol{j}$. The boat reaches the east bank after 8 s, and it is located at $\boldsymbol{r}(8)=5(8)\boldsymbol{i}+\left(\frac{3}{4}(8)^2-\frac{1}{16}(8)^3\right)\boldsymbol{j}=40\boldsymbol{i}+16\boldsymbol{j}$. Thus the boat is 16 m downstream.

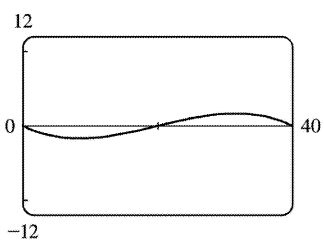
(b) Let α be the angle north of east that the boat heads. Then the velocity of the boat in still water is given by $5(\cos\alpha)\boldsymbol{i}+5(\sin\alpha)\boldsymbol{j}$. At t seconds, the boat is $5(\cos\alpha)t$ meters from the west bank, at which point the velocity of the water is $\frac{3}{400}[5(\cos\alpha)t][40-5(\cos\alpha)t]\,\boldsymbol{j}$. The resultant velocity of the boat is given by

$$\mathbf{v}(\mathrm{t})=5(\cos\alpha)\mathbf{i}+\left[5\sin\alpha+\frac{3}{400}(5t\cos\alpha)(40-5t\cos\alpha)\right]\mathbf{j}$$
$$=(5\cos\alpha)\mathbf{i}+\left(5\sin\alpha+\frac{3}{2}t\cos\alpha-\frac{3}{16}t^2\cos^2\alpha\right)\mathbf{j}$$

Integrating, $\mathbf{r}(t)=(5t\cos\alpha)\mathbf{i}+\left(5t\sin\alpha+\frac{3}{4}t^2\cos\alpha-\frac{1}{16}t^3\cos^2\alpha\right)\mathbf{j}$ (where we have again placed the origin at A). The boat will reach the east bank when $5t\cos\alpha=40\Rightarrow t=\frac{40}{5\cos\alpha}=\frac{8}{\cos\alpha}$.

In order to land at point $B(40,0)$ we need $5t\sin\alpha+\frac{3}{4}t^2\cos\alpha-\frac{1}{16}t^3\cos^2\alpha=0\Rightarrow$
$5\left(\frac{8}{\cos\alpha}\right)\sin\alpha+\frac{3}{4}\left(\frac{8}{\cos\alpha}\right)^2\cos\alpha-\frac{1}{16}\left(\frac{8}{\cos\alpha}\right)^3\cos^2\alpha=0\Rightarrow\frac{1}{\cos\alpha}(40\sin\alpha+48-32)=0\Rightarrow$
$40\sin\alpha+16=0\Rightarrow\sin\alpha=-\frac{2}{5}$. Thus $\alpha=\sin^{-1}\left(-\frac{2}{5}\right)\approx-23.6°$, so the boat should head $23.6°$ south of east (upstream).

The path does seem realistic. The boat initially heads upstream to counteract the effect of the current. Near the center of the river, the current is stronger and the boat is pushed downstream. When the boat nears the eastern bank, the current is slower and the boat is able to progress upstream to arrive at point B .



30. As in Exercise 29(b), let α be the angle north of east that the boat heads, so the velocity of the boat in still water is given by $5(\cos\alpha)\mathbf{i}+5(\sin\alpha)\mathbf{j}$. At t seconds, the boat is $5(\cos\alpha)t$ meters from the west bank, at which point the velocity of the water is
$3\sin(\pi x/40)\mathbf{j}=3\sin[\pi\cdot5(\cos\alpha)t/40]\mathbf{j}=3\sin\left(\frac{\pi}{8}t\cos\alpha\right)\mathbf{j}$. The resultant velocity of the boat then is given by $\mathbf{v}(t)=5(\cos\alpha)\mathbf{i}+\left[5\sin\alpha+3\sin\left(\frac{\pi}{8}t\cos\alpha\right)\right]\mathbf{j}$. Integrating,
$\mathbf{r}(t)=(5t\cos\alpha)\mathbf{i}+\left[5t\sin\alpha-\frac{24}{\pi\cos\alpha}\cos\left(\frac{\pi}{8}t\cos\alpha\right)\right]\mathbf{j}+\mathbf{C}$.

If we place the origin at A then $\mathbf{r}(0)=\mathbf{0}\Rightarrow-\frac{24}{\pi\cos\alpha}\mathbf{j}+\mathbf{C}=\mathbf{0}\Rightarrow\mathbf{C}=\frac{24}{\pi\cos\alpha}\mathbf{j}$ and
$\mathbf{r}(t)=(5t\cos\alpha)\mathbf{i}+\left[5t\sin\alpha-\frac{24}{\pi\cos\alpha}\cos\left(\frac{\pi}{8}t\cos\alpha\right)+\frac{24}{\pi\cos\alpha}\right]\mathbf{j}$.

The boat will reach the east bank when $5t\cos\alpha = 40 \Rightarrow t = \dfrac{8}{\cos\alpha}$. In order to land at point $B(40,0)$ we need $5t\sin\alpha - \dfrac{24}{\pi\cos\alpha}\cos\left(\dfrac{\pi}{8}t\cos\alpha\right) + \dfrac{24}{\pi\cos\alpha} = 0 \Rightarrow$

$5\left(\dfrac{8}{\cos\alpha}\right)\sin\alpha - \dfrac{24}{\pi\cos\alpha}\cos\left[\dfrac{\pi}{8}\left(\dfrac{8}{\cos\alpha}\right)\cos\alpha\right] + \dfrac{24}{\pi\cos\alpha} = 0 \Rightarrow$

$\dfrac{1}{\cos\alpha}\left(40\sin\alpha - \dfrac{24}{\pi}\cos\pi + \dfrac{24}{\pi}\right) = 0 \Rightarrow 40\sin\alpha + \dfrac{48}{\pi} = 0 \Rightarrow \sin\alpha = -\dfrac{6}{5\pi}$. Thus

$\alpha = \sin^{-1}\left(-\dfrac{6}{5\pi}\right) \approx -22.5°$, so the boat should head $22.5°$ south of east.

31. $\boldsymbol{r}(t) = (3t - t^3)\boldsymbol{i} + 3t^2\boldsymbol{j} \Rightarrow \boldsymbol{r}'(t) = (3 - 3t^2)\boldsymbol{i} + 6t\,\boldsymbol{j}$,
$|\boldsymbol{r}'(t)| = \sqrt{(3-3t^2)^2 + (6t)^2} = \sqrt{9 + 18t^2 + 9t^4} = \sqrt{(3-3t^2)^2} = 3 + 3t^2$, $\boldsymbol{r}''(t) = -6t\,\boldsymbol{i} + 6\boldsymbol{j}$,
$\boldsymbol{r}'(t) \times \boldsymbol{r}''(t) = (18 + 18t^2)\boldsymbol{k}$. Then Equation 9 gives

$a_T = \dfrac{\boldsymbol{r}'(t)\cdot\boldsymbol{r}''(t)}{|\boldsymbol{r}'(t)|} = \dfrac{(3-3t^2)(-6t) + (6t)(6)}{3+3t^2} = \dfrac{18t + 18t^3}{3+3t^2} = \dfrac{18t(1+t^2)}{3(1+t^2)} = 6t$ [or by Equation 8,

$a_T = v' = \dfrac{d}{dt}\left[3 + 3t^2\right] = 6t$] and Equation 10 gives $a_N = \dfrac{|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|} = \dfrac{18+18t^2}{3+3t^2} = \dfrac{18(1+t^2)}{3(1+t^2)} = 6$.

32. $\boldsymbol{r}(t) = (1+t)\boldsymbol{i} + (t^2 - 2t)\boldsymbol{j} \Rightarrow \boldsymbol{r}'(t) = \boldsymbol{i} + (2t-2)\boldsymbol{j}$, $|\boldsymbol{r}'(t)| = \sqrt{1^2 + (2t-2)^2} = \sqrt{4t^2 - 8t + 5}$, $\boldsymbol{r}''(t) = 2\boldsymbol{j}$,
$\boldsymbol{r}'(t)\times\boldsymbol{r}''(t) = 2\boldsymbol{k}$. Then Equation 9 gives $a_T = \dfrac{\boldsymbol{r}'(t)\cdot\boldsymbol{r}''(t)}{|\boldsymbol{r}'(t)|} = \dfrac{2(2t-2)}{\sqrt{4t^2-8t+5}}$ and Equation 10 gives

$a_N = \dfrac{|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|} = \dfrac{2}{\sqrt{4t^2-8t+5}}$.

33. $\boldsymbol{r}(t) = \cos t\,\boldsymbol{i} + \sin t\,\boldsymbol{j} + t\,\boldsymbol{k} \Rightarrow \boldsymbol{r}'(t) = -\sin t\,\boldsymbol{i} + \cos t\,\boldsymbol{j} + \boldsymbol{k}$, $|\boldsymbol{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$,
$\boldsymbol{r}''(t) = -\cos t\,\boldsymbol{i} - \sin t\,\boldsymbol{j}$, $\boldsymbol{r}'(t)\times\boldsymbol{r}''(t) = \sin t\,\boldsymbol{i} - \cos t\,\boldsymbol{j} + \boldsymbol{k}$. Then

$a_T = \dfrac{\boldsymbol{r}'(t)\cdot\boldsymbol{r}''(t)}{|\boldsymbol{r}'(t)|} = \dfrac{\sin t\cos t - \sin t\cos t}{\sqrt{2}} = 0$ and $a_N = \dfrac{|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|} = \dfrac{\sqrt{\sin^2 t + \cos^2 t + 1}}{\sqrt{2}} = \dfrac{\sqrt{2}}{\sqrt{2}} = 1$.

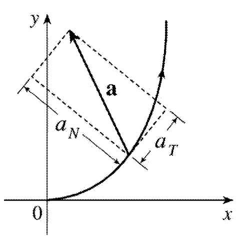
34. $\boldsymbol{r}(t) = t\,\boldsymbol{i} + t^2\boldsymbol{j} + 3t\,\boldsymbol{k} \Rightarrow \boldsymbol{r}'(t) = \boldsymbol{i} + 2t\,\boldsymbol{j} + 3\boldsymbol{k}$, $|\boldsymbol{r}'(t)| = \sqrt{1^2 + (2t)^2 + 3^2} = \sqrt{4t^2 + 10}$, $\boldsymbol{r}''(t) = 2\boldsymbol{j}$,

$\mathbf{r}'(t)\times\mathbf{r}''(t)=-6\mathbf{i}+2\mathbf{k}$. Then $a_T=\dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}=\dfrac{4t}{\sqrt{4t^2+10}}$ and $a_N=\dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}=\dfrac{2\sqrt{10}}{\sqrt{4t^2+10}}$.

35. $\mathbf{r}(t)=e^t\mathbf{i}+\sqrt{2}\,t\,\mathbf{j}+e^{-t}\mathbf{k}\Rightarrow\mathbf{r}'(t)=e^t\mathbf{i}+\sqrt{2}\,\mathbf{j}-e^{-t}\mathbf{k}$, $|\mathbf{r}(t)|=\sqrt{e^{2t}+2+e^{-2t}}=\sqrt{(e^t+e^{-t})^2}=e^t+e^{-t}$,
$\mathbf{r}''(t)=e^t\mathbf{i}+e^{-t}\mathbf{k}$. Then $a_T=\dfrac{e^{2t}-e^{-2t}}{e^t+e^{-t}}=\dfrac{(e^t+e^{-t})(e^t-e^{-t})}{e^t+e^{-t}}=e^t-e^{-t}=2\sinh t$ and
$a_N=\dfrac{|\sqrt{2}\,e^{-t}\mathbf{i}-2\mathbf{j}-\sqrt{2}\,e^t\mathbf{k}|}{e^t+e^{-t}}=\dfrac{\sqrt{2(e^{-2t}+2+e^{2t})}}{e^t+e^{-t}}=\sqrt{2}\ \dfrac{e^t+e^{-t}}{e^t+e^{-t}}=\sqrt{2}$.

36. $\mathbf{r}(t)=t\,\mathbf{i}+\cos^2t\,\mathbf{j}+\sin^2t\,\mathbf{k}\Rightarrow\mathbf{r}'(t)=\mathbf{i}-2\cos t\sin t\,\mathbf{j}+2\sin t\cos t\,\mathbf{k}=\mathbf{i}-\sin 2t\,\mathbf{j}+\sin 2t\,\mathbf{k}$,
$|\mathbf{r}'(t)|=\sqrt{1+2\sin^2 2t}$, $\mathbf{r}''(t)=2(\sin^2t-\cos^2t)\,\mathbf{j}+2(\cos^2t-\sin^2t)\,\mathbf{k}=-2\cos 2t\,\mathbf{j}+2\cos 2t\,\mathbf{k}$. So
$a_T=\dfrac{2\sin 2t\cos 2t+2\sin 2t\cos 2t}{\sqrt{1+2\sin^2 2t}}=\dfrac{4\sin 2t\cos 2t}{\sqrt{1+2\sin^2 2t}}$ and $a_N=\dfrac{|-2\cos 2t\,\mathbf{j}-2\cos 2t\,\mathbf{k}|}{\sqrt{1+2\sin^2 2t}}=\dfrac{2\sqrt{2}\,|\cos 2t|}{\sqrt{1+2\sin^2 t}}$.

37. The tangential component of $\mathbf{a}$ is the length of the projection of $\mathbf{a}$ onto $\mathbf{T}$, so we sketch the scalar projection of $\mathbf{a}$ in the tangential direction to the curve and estimate its length to be 4.5 (using the fact that $\mathbf{a}$ has length 10 as a guide). Similarly, the normal component of $\mathbf{a}$ is the length of the projection of $\mathbf{a}$ onto $\mathbf{N}$, so we sketch the scalar projection of $\mathbf{a}$ in the normal direction to the curve and estimate its length to be 9.0 . Thus $a_T\approx4.5\text{ cm / s}^2$ and $a_N\approx9.0\text{ cm / s}^2$.



38. $\mathbf{L}(t)=m\mathbf{r}(t)\times\mathbf{v}(t)\Rightarrow$

$$\mathbf{L}'(t)=m[\mathbf{r}'(t)\times\mathbf{v}(t)+\mathbf{r}(t)\times\mathbf{v}'(t)]\text{[by Theorem 14.2.3 [ET 13.2.3]\#5]}$$
$$=m[\mathbf{v}(t)\times\mathbf{v}(t)+\mathbf{r}(t)\times\mathbf{v}'(t)]=m[\mathbf{0}+\mathbf{r}(t)\times\mathbf{a}(t)]=\boldsymbol{\tau}(t)$$

So if the torque is always $\mathbf{0}$, then $\mathbf{L}'(t)=\mathbf{0}$ for all t , and so $\mathbf{L}(t)$ is constant.

39. If the engines are turned off at time t , then the spacecraft will continue to travel in the direction of

$\boldsymbol{v}(t)$, so we need a t such that for some scalar $s>0$, $\boldsymbol{r}(t)+s\,\boldsymbol{v}(t)=\langle 6,4,9\rangle$. $\boldsymbol{v}(t)=\boldsymbol{r}'(t)=\boldsymbol{i}+\frac{1}{t}\,\boldsymbol{j}+\frac{8t}{(t^2+1)^2}\,\boldsymbol{k}$

$\Rightarrow \boldsymbol{r}(t)+s\,\boldsymbol{v}(t)=\left\langle 3+t+s, 2+\ln t+\frac{s}{t}, 7-\frac{4}{t^2+1}+\frac{8st}{(t^2+1)^2}\right\rangle \Rightarrow 3+t+s=6 \Rightarrow s=3-t$, so $7-\frac{4}{t^2+1}+\frac{8(3-t)t}{(t^2+1)^2}=9 \Leftrightarrow$

$\frac{24t-12t^2-4}{(t^2+1)^2}=2 \Leftrightarrow t^4+8t^2-12t+3=0$. It is easily seen that $t=1$ is a root of this polynomial. Also

$2+\ln 1+\frac{3-1}{1}=4$, so $t=1$ is the desired solution.

40. **(a)** $m\frac{d\boldsymbol{v}}{dt}=\frac{dm}{dt}\boldsymbol{v}_e \Leftrightarrow \frac{d\boldsymbol{v}}{dt}=\frac{1}{m}\frac{dm}{dt}\boldsymbol{v}_e$. Integrating both sides of this equation with respect to t gives

$\int_0^t \frac{d\boldsymbol{v}}{du}\,du=\boldsymbol{v}_e\int_0^t \frac{1}{m}\frac{dm}{du}\,du \Rightarrow \int_{\boldsymbol{v}(0)}^{\boldsymbol{v}(t)} d\boldsymbol{v}=\boldsymbol{v}_e\int_{m(0)}^{m(t)}\frac{dm}{m} \Rightarrow \boldsymbol{v}(t)-\boldsymbol{v}(0)=\ln\left(\frac{m(t)}{m(0)}\right)\boldsymbol{v}_e \Rightarrow$

$\boldsymbol{v}(t)=\boldsymbol{v}(0)-\ln\left(\frac{m(0)}{m(t)}\right)\boldsymbol{v}_e$.

(b) $|\boldsymbol{v}(t)|=2\left|\boldsymbol{v}_e\right|$, and $|\boldsymbol{v}(0)|=0$. Therefore, by part (a), $2\left|\boldsymbol{v}_e\right|=\left|-\ln\left(\frac{m(0)}{m(t)}\right)\boldsymbol{v}_e\right| \Rightarrow$

$2\left|\boldsymbol{v}_e\right|=\ln\left(\frac{m(0)}{m(t)}\right)\left|\boldsymbol{v}_e\right|$. $\left[\text{Note: } m(0)>m(t) \text{ so that } \ln\left(\frac{m(0)}{m(t)}\right)>0\right] \Rightarrow m(t)=e^{-2}m(0)$.

Thus $\frac{m(0)-e^{-2}m(0)}{m(0)}=1-e^{-2}$ is the fraction of the initial mass that is burned as fuel.

1. **(a)** From Table 1, $f(-15,40)=-27$, which means that if the temperature is -15° C and the wind speed is 40 km / h, then the air would feel equivalent to approximately -27° C without wind.
(b) The question is asking: when the temperature is -20° C , what wind speed gives a wind–chill index of -30° C ? From Table 1, the speed is 20 km / h.
(c) The question is asking: when the wind speed is 20 km / h, what temperature gives a wind–chill index of -49° C ? From Table 1, the temperature is -35° C .
(d) The function $W=f(-5,v)$ means that we fix T at -5 and allow v to vary, resulting in a function of one variable. In other words, the function gives wind–chill index values for different wind speeds when the temperature is -5° C . From Table 1 (look at the row corresponding to $T=-5$), the function decreases and appears to approach a constant value as v increases.
(e) The function $W=f(T,50)$ means that we fix v at 50 and allow T to vary, again giving a function of one variable. In other words, the function gives wind–chill index values for different temperatures when the wind speed is 50 km / h . From Table 1 (look at the column corresponding to $v=50$), the function increases almost linearly as T increases.

2. **(a)** From the table, $f(95,70)=124$, which means that when the actual temperature is 95 and the relative humidity is 70% , the perceived air temperature is approximately 124° F .
(b) Looking at the row corresponding to $T=90$, we see that $f(90,h)=100$ when $h=60$.
(c) Looking at the column corresponding to $h=50$, we see that $f(T,50)=88$ when $T=85$.
(d) $I=f(80,h)$ means that T is fixed at 80 and h is allowed to vary, resulting in a function of h that gives the humidex values for different relative humidities when the actual temperature is 80° F . Similarly, $I=f(100,h)$ is a function of one variable that gives the humidex values for different relative humidities when the actual temperature is 100° F . Looking at the rows of the table corresponding to $T=80$ and $T=100$, we see that $f(80,h)$ increases at a relatively constant rate of approximately 1° F per 10% relative humidity, while $f(100,h)$ increases more quickly (at first with an average rate of change of 5° F per 10% relative humidity) and at an increasing rate (approximately 12° F per 10% relative humidity for larger values of h).

3. If the amounts of labor and capital are both doubled, we replace L,K in the function with $2L,2K$, giving

$$P(2L,2K)=1.01(2L)^{0.75}(2K)^{0.25}=1.01(2^{0.75})(2^{0.25})L^{0.75}K^{0.25}=(2^{1})1.01L^{0.75}K^{0.25}$$
$$=2P(L,K)$$

Thus, the production is doubled. It is also true for the general case $P(L,K)=bL^{\alpha}K^{1-\alpha}$:
$P(2L,2K)=b(2L)^{\alpha}(2K)^{1-\alpha}=b(2^{\alpha})(2^{1-\alpha})L^{\alpha}K^{1-\alpha}=(2^{\alpha+1-\alpha})bL^{\alpha}K^{1-\alpha}=2P(L,K)$.

4. We compare the values for the wind–chill index given by Table 1 with those given by the model function:

Modeled Wind–Chill Index Values $W(T,v)$

Wind Speed (km/h)

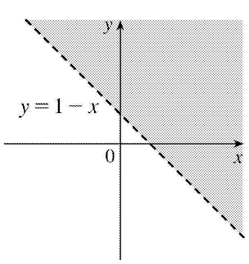
T \ V	5	10	15	20	25	30	40	50	60	70	80
5	4.08	2.66	1.74	1.07	0.52	0.05	−0.71	−1.33	−1.85	−2.30	−2.70
0	−1.59	−3.31	−4.42	−5.24	−5.91	−6.47	−7.40	−8.14	−8.77	−9.32	−9.80
−5	−7.26	−9.29	−10.58	−11.55	−12.34	−13.00	−14.08	−14.96	−15.70	−16.34	−16.91
−10	−12.93	−15.26	−16.75	−17.86	−18.76	−19.52	−20.77	−21.77	−22.62	−23.36	−24.01
−15	−18.61	−21.23	−22.91	−24.17	−25.19	−26.04	−27.45	−28.59	−29.54	−30.38	−31.11
−20	−24.28	−27.21	−29.08	−30.48	−31.61	−32.57	−34.13	−35.40	−36.47	−37.40	−38.22
−25	−29.95	−33.18	−35.24	−36.79	−38.04	−39.09	−40.82	−42.22	−43.39	−44.42	−45.32
−30	−35.62	−39.15	−41.41	−43.10	−44.46	−45.62	−47.50	−49.03	−50.32	−51.44	−52.43
−35	−41.30	−45.13	−47.57	−49.41	−50.89	−52.14	−54.19	−55.84	−57.24	−58.46	−59.53
−40	−46.97	−51.10	−53.74	−55.72	−57.31	−58.66	−60.87	−62.66	−64.17	−65.48	−66.64

Actual temperature (°C)

The values given by the function appear to be fairly close (within 0.5) to the values in Table 1.

5. **(a)** According to the table, $f(40,15)=25$, which means that if a 40 –knot wind has been blowing in the open sea for 15 hours, it will create waves with estimated heights of 25 feet.
(b) $h=f(30,t)$ means we fix v at 30 and allow t to vary, resulting in a function of one variable. Thus here, $h=f(30,t)$ gives the wave heights produced by 30 –knot winds blowing for t hours. From the table (look at the row corresponding to $v=30$), the function increases but at a declining rate as t increases. In fact, the function values appear to be approaching a limiting value of approximately 19 , which suggests that 30 –knot winds cannot produce waves higher than about 19 feet.
(c) $h=f(v,30)$ means we fix t at 30 , again giving a function of one variable. So, $h=f(v,30)$ gives the wave heights produced by winds of speed v blowing for 30 hours. From the table (look at the column corresponding to $t=30$), the function appears to increase at an increasing rate, with no apparent limiting value. This suggests that faster winds (lasting 30 hours) always create higher waves.

6. **(a)** $f(1,1)=\ln(1+1-1)=\ln 1=0$
(b) $f(e,1)=\ln(e+1-1)=\ln e=1$
(c) $\ln(x+y-1)$ is defined only when $x+y-1>0$, that is, $y>1-x$. So the domain of f is $\{(x,y)\mid y>1-x\}$.



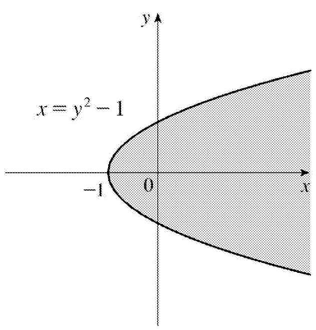
(d) Since $\ln(x+y-1)$ can be any real number, the range is R .

7. **(a)** $f(2,0)=2^2e^{3(2)(0)}=4(1)=4$

(b) Since both x^2 and the exponential function are defined everywhere, x^2e^{3xy} is defined for all choices of values for x and y. Thus, the domain of f is R^2.

(c) Because the range of $g(x,y)=3xy$ is R, and the range of e^x is $(0,\infty)$, the range of $e^{g(x,y)}=e^{3xy}$ is $(0,\infty)$. The range of x^2 is $[0,\infty)$, so the range of the product x^2e^{3xy} is $[0,\infty)$.

8. $\sqrt{1+x-y^2}$ is defined only when $1+x-y^2\geq 0\Rightarrow x\geq y^2-1$, so the domain of f is $\{(x,y)\mid x\geq y^2-1\}$, all those points on or to the right of the parabola $x=y^2-1$.
The range of f is $[0,\infty)$.



9. **(a)** $f(2,-1,6)=e^{\sqrt{6-2^2-(-1)^2}}=e^{\sqrt{1}}=e$.

(b) $e^{\sqrt{z-x^2-y^2}}$ is defined when $z-x^2-y^2\geq 0\Rightarrow z\geq x^2+y^2$. Thus the domain of f is $\{(x,y,z)\mid z\geq x^2+y^2\}$.

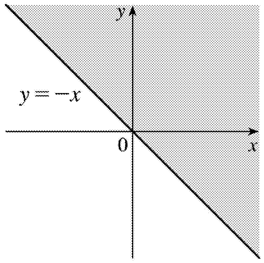
(c) Since $\sqrt{z-x^2-y^2}\geq 0$, we have $e^{\sqrt{z-x^2-y^2}}\geq 1$. Thus the range of f is $[1,\infty)$.

10. **(a)** $g(2,-2,4)=\ln\left(25-2^2-(-2)^2-4^2\right)=\ln 1=0.$

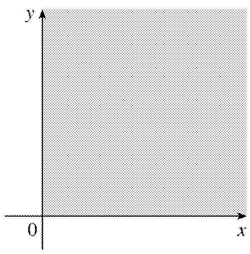
(b) For the logarithmic function to be defined, we need $25-x^2-y^2-z^2>0$. Thus the domain of g is $\{(x,y,z)\mid x^2+y^2+z^2<25\}$, the interior of the sphere $x^2+y^2+z^2=25$.

(c) Since $0<25-x^2-y^2-z^2\leq 25$ for (x,y,z) in the domain of g, $\ln\left(25-x^2-y^2-z^2\right)\leq \ln 25$. Thus the range of g is $(-\infty,\ln 25]$.

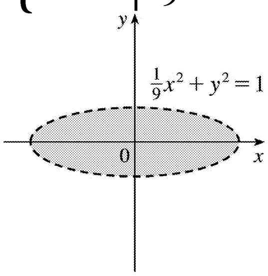
11. $\sqrt{x+y}$ is defined only when $x+y\geq 0$, or $y\geq -x$. So the domain of f is $\{(x,y)\mid y\geq -x\}$.



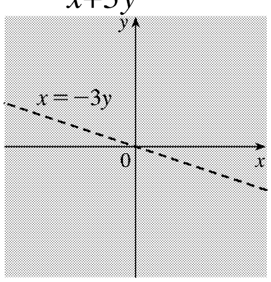
12. We need $x \geq 0$ and $y \geq 0$, so $D = \{ (x,y) \mid x \geq 0 \text{ and } y \geq 0 \}$, the first quadrant.



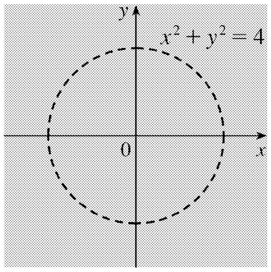
13. $\ln \left(9 - x^2 - 9y^2\right)$ is defined only when $9 - x^2 - 9y^2 > 0$, or $\frac{1}{9} x^2 + y^2 < 1$. So the domain of f is $\left\{ (x,y) \,\middle|\, \frac{1}{9} x^2 + y^2 < 1 \right\}$, the interior of an ellipse.



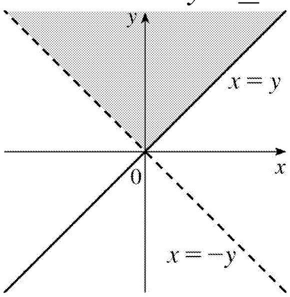
14. $\dfrac{x - 3y}{x + 3y}$ is defined only when $x + 3y \neq 0$, or $x \neq -3y$. So the domain of f is $\{ (x,y) \mid x \neq -3y \}$.



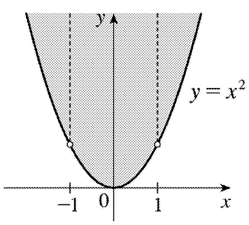
15. $\dfrac{3x + 5y}{x^2 + y^2 - 4}$ is defined only when $x^2 + y^2 - 4 \neq 0$, or $x^2 + y^2 \neq 4$. So the domain of f is $\left\{ (x,y) \mid x^2 + y^2 \neq 4 \right\}$.



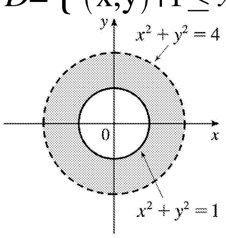
16. We need $y-x\geq 0$ or $y\geq x$ and $y+x>0$ or $x>-y$. Thus $D=\{(x,y)\mid -y<x\leq y, y>0\}$.



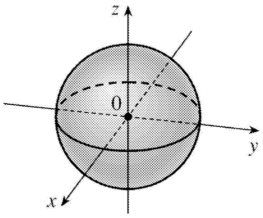
17. $\sqrt{y-x^2}$ is defined only when $y-x^2\geq 0$, or $y\geq x^2$. In addition, f is not defined if $1-x^2=0\Rightarrow x=\pm 1$. Thus the domain of f is $\left\{(x,y)\mid y\geq x^2, x\neq \pm 1\right\}$.



18. f is defined only when $x^2+y^2-1\geq 0\Rightarrow x^2+y^2\geq 1$ and $4-x^2-y^2>0\Rightarrow x^2+y^2<4$. Thus $D=\left\{(x,y)\mid 1\leq x^2+y^2<4\right\}$.

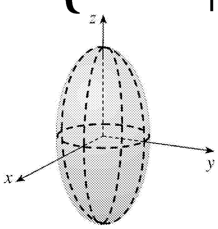


19. We need $1-x^2-y^2-z^2\geq 0$ or $x^2+y^2+z^2\leq 1$, so $D=\left\{(x,y,z)\mid x^2+y^2+z^2\leq 1\right\}$ (the points inside or on the sphere of radius 1 , center the origin).

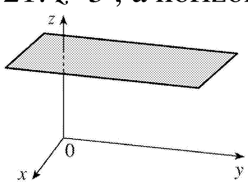


20. f is defined only when $16-4x^2-4y^2-z^2>0 \Rightarrow \frac{x^2}{4}+\frac{y^2}{4}+\frac{z^2}{16}<1$. Thus,

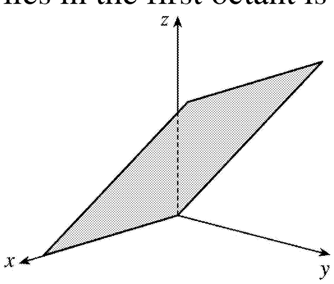
$D=\left\{ (x,y,z) \;\middle|\; \frac{x^2}{4}+\frac{y^2}{4}+\frac{z^2}{16}<1 \right\}$, that is, the points inside the ellipsoid $\frac{x^2}{4}+\frac{y^2}{4}+\frac{z^2}{16}=1$.



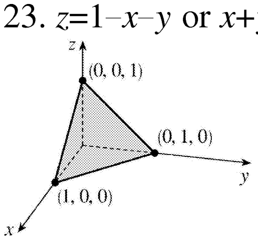
21. $z=3$, a horizontal plane through the point $(0,0,3)$.



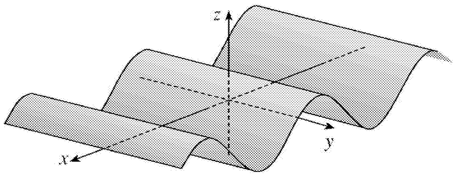
22. $z=y$, a plane which intersects the yz –plane in the line $z=y$, $x=0$. The portion of this plane that lies in the first octant is shown.



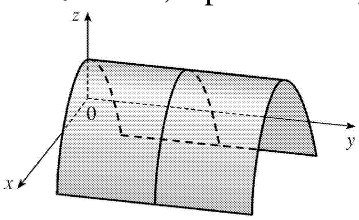
23. $z=1-x-y$ or $x+y+z=1$, a plane with intercepts 1 , 1 , and 1 .



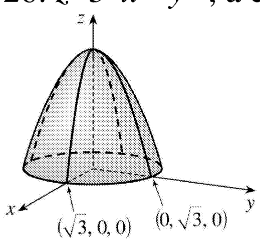
24. $z=\cos x$, a "wave."



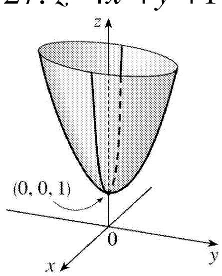
25. $z=1-x^2$, a parabolic cylinder.



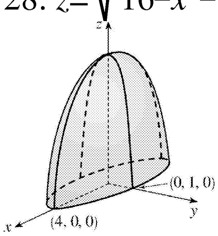
26. $z=3-x^2-y^2$, a circular paraboloid with vertex at $(0,0,3)$.



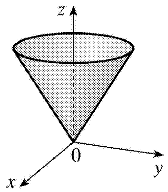
27. $z=4x^2+y^2+1$, an elliptic paraboloid with vertex at $(0,0,1)$.



28. $z=\sqrt{16-x^2-16y^2}$ so $z\geq 0$ and $z^2+x^2+16y^2=16$, the top half of an ellipsoid.



29. $z=\sqrt{x^2+y^2}$ so $x^2+y^2=z^2$ and $z\geq 0$, the top half of a right circular cone.



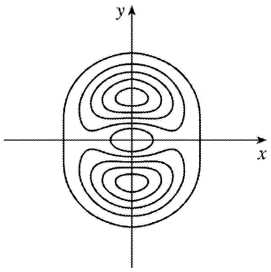
30. All six graphs have different traces in the planes $x=0$ and $y=0$, so we investigate these for each function.
(a) $f(x,y)=|x|+|y|$. The trace in $x=0$ is $z=|y|$, and in $y=0$ is $z=|x|$, so it must be graph VI.
(b) $f(x,y)=|xy|$. The trace in $x=0$ is $z=0$, and in $y=0$ is $z=0$, so it must be graph V.
(c) $f(x,y)=\dfrac{1}{1+x^2+y^2}$. The trace in $x=0$ is $z=\dfrac{1}{1+y^2}$, and in $y=0$ is $z=\dfrac{1}{1+x^2}$. In addition, we can see that f is close to 0 for large values of x and y , so this is graph I.
(d) $f(x,y)=(x^2-y^2)^2$. The trace in $x=0$ is $z=y^4$, and in $y=0$ is $z=x^4$. Both graph II and graph IV seem plausible; notice the trace in $z=0$ is $0=(x^2-y^2)^2 \Rightarrow y=\pm x$, so it must be graph IV.
(e) $f(x,y)=(x-y)^2$. The trace in $x=0$ is $z=y^2$, and in $y=0$ is $z=x^2$. Both graph II and graph IV seem plausible; notice the trace in $z=0$ is $0=(x-y)^2 \Rightarrow y=x$, so it must be graph II.
(f) $f(x,y)=\sin(|x|+|y|)$. The trace in $x=0$ is $z=\sin|y|$, and in $y=0$ is $z=\sin|x|$. In addition, notice that the oscillating nature of the graph is characteristic of trigonometric functions. So this is graph III.

31. The point $(-3,3)$ lies between the level curves with z –values 50 and 60 . Since the point is a little closer to the level curve with $z=60$, we estimate that $f(-3,3)\approx 56$. The point $(3,-2)$ appears to be just about halfway between the level curves with z –values 30 and 40 , so we estimate $f(3,-2)\approx 35$. The graph rises as we approach the origin, gradually from above, steeply from below.

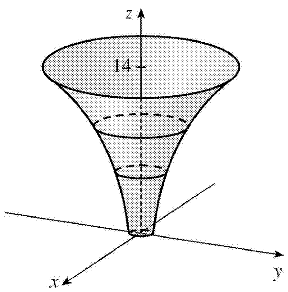
32. If we start at the origin and move along the x –axis, for example, the z –values of a cone centered at the origin increase at a constant rate, so we would expect its level curves to be equally spaced. A paraboloid with vertex the origin, on the other hand, has z –values which change slowly near the origin and more quickly as we move farther away. Thus, we would expect its level curves near the origin to be spaced more widely apart than those farther from the origin. Therefore contour map I must correspond to the paraboloid, and contour map II the cone.

33. Near A , the level curves are very close together, indicating that the terrain is quite steep. At B , the level curves are much farther apart, so we would expect the terrain to be much less steep than near A , perhaps almost flat.

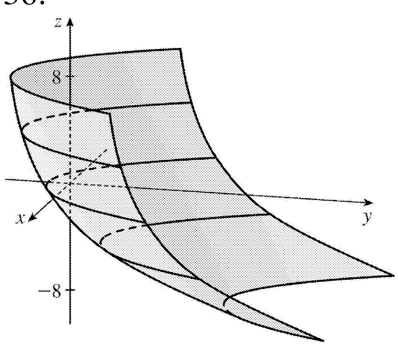
34.



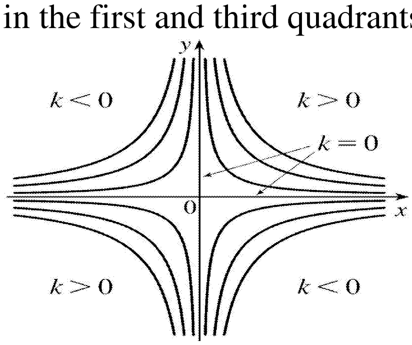
35.



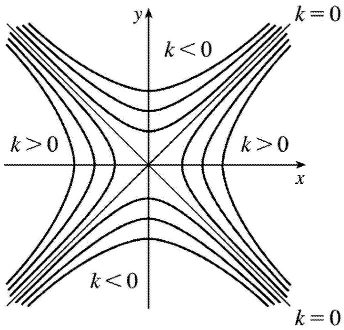
36.



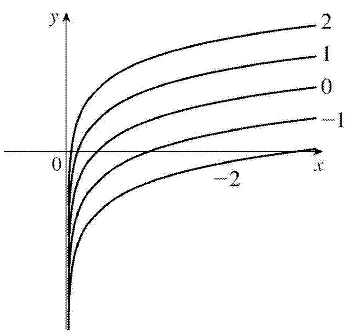
37. The level curves are $xy=k$. For $k=0$ the curves are the coordinate axis; if $k>0$, they are hyperbolas in the first and third quadrants; if $k<0$, they are hyperbolas in the second and fourth quadrants.



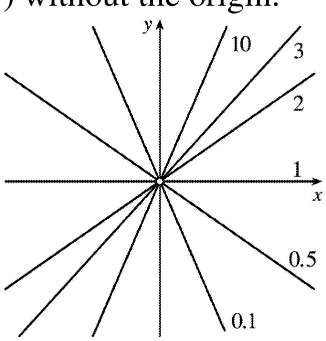
38. The level curves are $k=x^2-y^2$. When $k=0$, these are the lines $y=\pm x$. When $k>0$, the curves are hyperbolas with axis the x –axis and when $k<0$, they are hyperbolas with axis the y –axis.



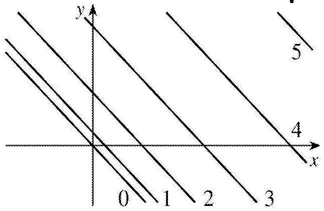
39. The level curves are $y-\ln x=k$ or $y=\ln x+k$.



40. The level curves are $e^{y/x}=k$ or equivalently $y=x\ln k$ ($x\neq 0$), a family of lines with slope $\ln k$ ($k>0$) without the origin.

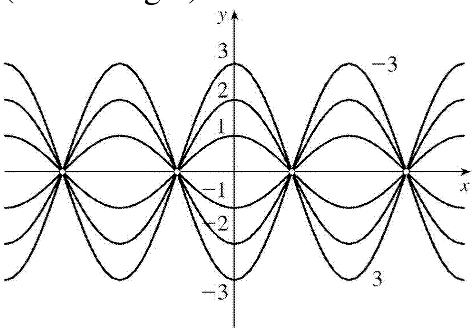


41. $k=\sqrt{x+y}$ or for $x+y\geq 0$, $k^2=x+y$, or $y=-x+k^2$.
Note: $k\geq 0$ since $k=\sqrt{x+y}$.

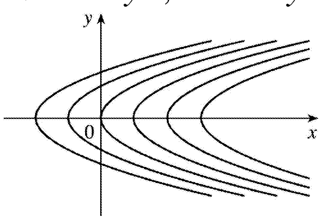


42. $k=y\sec x$ or $y=k\cos x$, $x\neq \dfrac{\pi}{2}+n\pi$

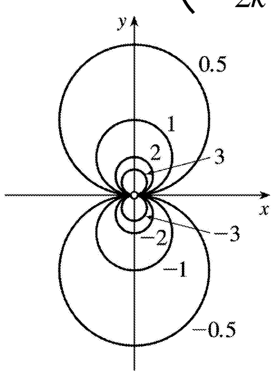
(n an integer).



43. $k = x - y^2$, or $x - k = y^2$, a family of parabolas with vertex $(k,0)$.

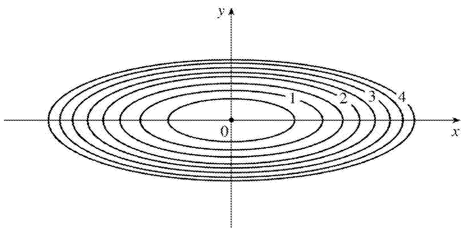


44. For $k \neq 0$ and $(x,y) \neq (0,0)$, $k = \frac{y}{x^2 + y^2} \Leftrightarrow x^2 + y^2 - \frac{y}{k} = 0 \Leftrightarrow x^2 + \left(y - \frac{1}{2k} \right)^2 = \frac{1}{4k^2}$, a family of circles with center $\left(0, \frac{1}{2k} \right)$ and radius $\frac{1}{2k}$ (without the origin). If $k = 0$, the level curve is the x –axis.

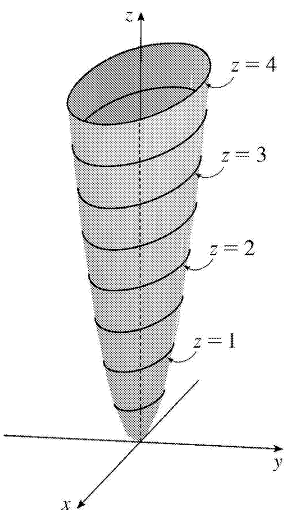


45. The contour map consists of the level curves $k = x^2 + 9y^2$, a family of ellipses with major axis the x –axis. (Or, if $k = 0$, the origin.)

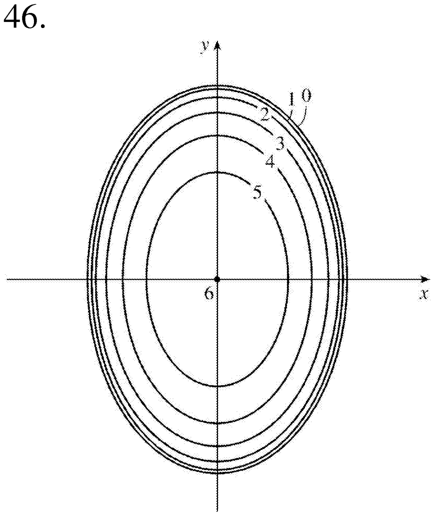
The graph of $f(x,y)$ is the surface $z = x^2 + 9y^2$, an elliptic paraboloid.



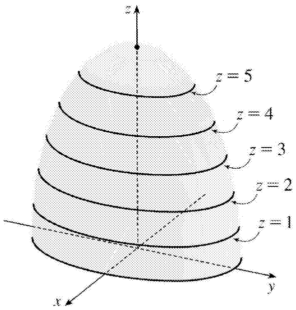
If we visualize lifting each ellipse $k=x^2+9y^2$ of the contour map to the plane $z=k$, we have horizontal traces that indicate the shape of the graph of f .



46.

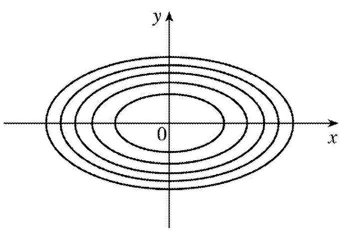


The contour map consists of the level curves $k=\sqrt{36-9x^2-4y^2} \Rightarrow 9x^2+4y^2=36-k^2$, $k\geq 0$, a family of ellipses with major axis the y –axis. (Or, if $k=6$, the origin.)



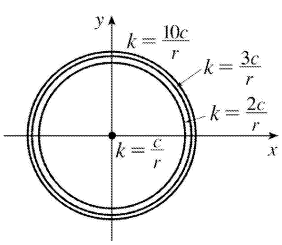
The graph of $f(x,y)$ is the surface $z=\sqrt{36-9x^2-4y^2}$, or equivalently the upper half of the ellipsoid $9x^2+4y^2+z^2=36$. If we visualize lifting each ellipse $k=\sqrt{36-9x^2-4y^2}$ of the contour map to the plane $z=k$, we have horizontal traces that indicate the shape of the graph of f .

47. The isothermals are given by $k=100/\left(1+x^2+2y^2\right)$ or $x^2+2y^2=(100-k)/k$ ($0<k\leq 100$), a family of ellipses.

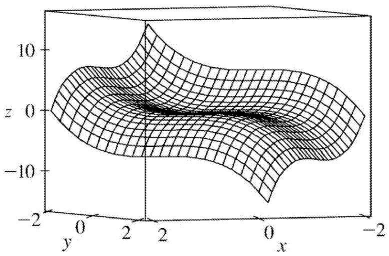


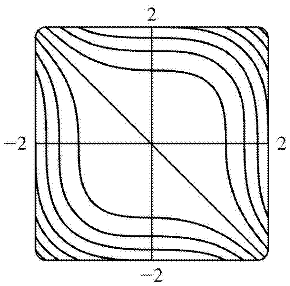
48. The equipotential curves are $k=\dfrac{c}{\sqrt{r^2-x^2-y^2}}$ or $x^2+y^2=r^2-\left(\dfrac{c}{k}\right)^2$, a family of circles ($k\geq c/r$).

Note: As $k\to\infty$, the radius of the circle approaches r .



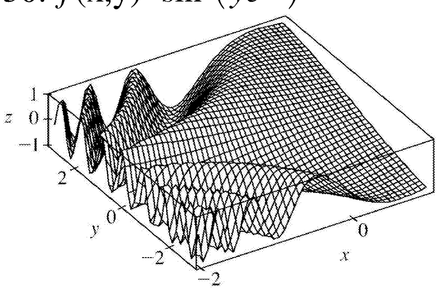
49. $f(x,y)=x^3+y^3$

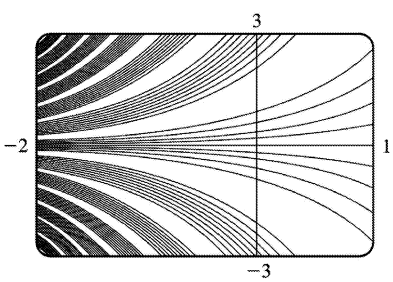




Note that the function is 0 along the line $y=-x$.

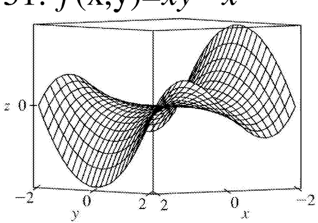
50. $f(\text{x,y})=\sin\left(ye^{-x}\right)$

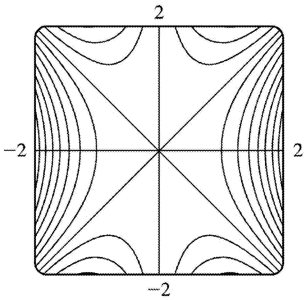




Cross–sections parallel to the yz –plane (such as the left–front trace in the first graph above) are sine–like curves. The periods of these curves decrease as x decreases.

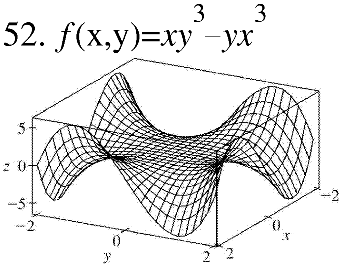
51. $f(\text{x,y})=xy^2-x^3$

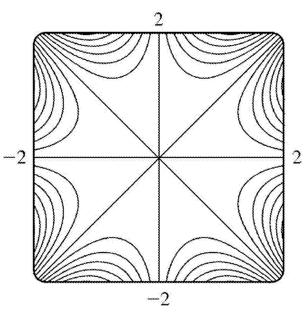




The traces parallel to the yz –plane (such as the left–front trace in the graph above) are parabolas; those parallel to
the xz –plane (such as the right–front trace) are cubic curves. The surface is called a monkey saddle because a monkey sitting on the surface near the origin has places for both legs and tail to rest.

52. $f(x,y)=xy^3-yx^3$





The traces parallel to either the yz –plane or the xz –plane are cubic curves.

53. **(a)** B
(b) III
Reasons: This function is constant on any circle centered at the origin, a description which matches only B and III.

54. **(a)** C
(b) II
Reasons: This function is the same if x is interchanged with y , so its graph is symmetric about the plane $x=y$. Also, $z(0,0)=0$ and the values of z approach 0 as we use points farther from the origin. These conditions are satisfied only by C and II.

55. **(a)** F
(b) V
Reasons: z increases without bound as we use points closer to the origin, a condition satisfied only by

F and V.

56. **(a)** A
(b) VI
Reasons: Along the lines $y=\pm\frac{1}{\sqrt{3}}x$ and $x=0$, this function is 0 .

57. **(a)** D
(b) IV
Reasons: This function is periodic in both x and y , with period 2π in each variable.

58. **(a)** E
(b) I
Reasons: This function is periodic along the x –axis, and increases as $|y|$ increases.

59. $k=x+3y+5z$ is a family of parallel planes with normal vector $\langle 1,3,5\rangle$.

60. $k=x^2+3y^2+5z^2$ is a family of ellipsoids for $k>0$ and the origin for $k=0$.

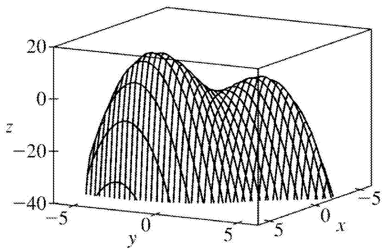
61. $k=x^2-y^2+z^2$ are the equations of the level surfaces. For $k=0$, the surface is a right circular cone with vertex the origin and axis the y– axis. For $k>0$, we have a family of hyperboloids of one sheet with axis the y –axis. For $k<0$, we have a family of hyperboloids of two sheets with axis the y –axis.

62. $k=x^2-y^2$ is a family of hyperbolic cylinders. The cross section of this family in the xy –plane has the same graph as the level curves in Exercise 38.

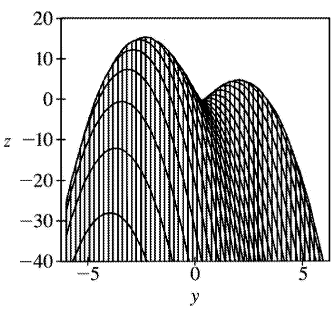
63. **(a)** The graph of g is the graph of f shifted upward 2 units.
(b) The graph of g is the graph of f stretched vertically by a factor of 2.
(c) The graph of g is the graph of f reflected about the xy –plane.
(d) The graph of $g(\mathrm{x,y})=-f(\mathrm{x,y})+2$ is the graph of f reflected about the xy –plane and then shifted upward 2 units.

64. **(a)** The graph of g is the graph of f shifted 2 units in the positive x –direction.
(b) The graph of g is the graph of f shifted 2 units in the negative y –direction.
(c) The graph of g is the graph of f shifted 3 units in the negative x –direction and 4 units in the positive y –direction.

65. $f(\mathrm{x,y})=3x-x^4-4y^2-10xy$



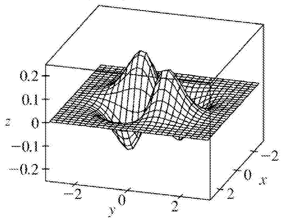
Three–dimensional view



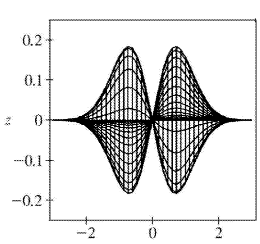
Front view

It does appear that the function has a maximum value, at the higher of the two "hilltops." From the front view graph, the maximum value appears to be approximately 15 . Both hilltops could be considered local maximum points, as
the values of f there are larger than at the neighboring points. There does not appear to be any local minimum point; although the valley shape between the two peaks looks like a minimum of some kind, some neighboring points have lower function values.

66. $f(\mathrm{x,y})=xye^{-x^2-y^2}$



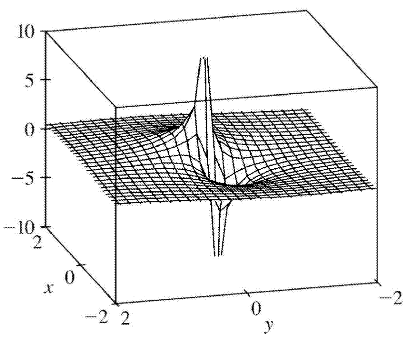
Three–dimensional view



Front view

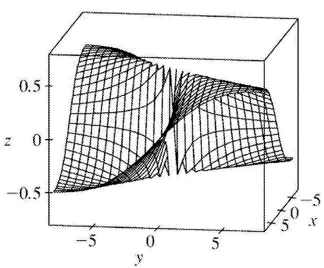
The function does have a maximum value, which it appears to achieve at two different points (the two "hilltops"). From the front view graph, we can estimate the maximum value to be approximately 0.18 . These same two points can also be considered local maximum points. The two "valley bottoms" visible in the graph can be considered local minimum points, as all the neighboring points give greater values of f .

67.



$f(x,y)=\dfrac{x+y}{x^2+y^2}$. As both x and y become large, the function values appear to approach 0 , regardless of which direction is considered. As (x,y) approaches the origin, the graph exhibits asymptotic behavior. From some directions, $f(x,y)\to\infty$, while in others $f(x,y)\to-\infty$. (These are the vertical spikes visible in the graph.) If the graph is examined carefully, however, one can see that $f(x,y)$ approaches 0 along the line $y=-x$.

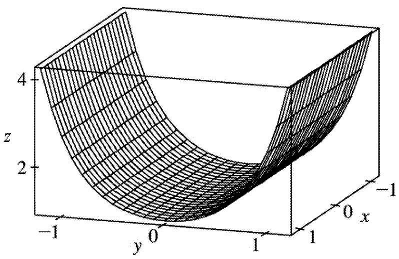
68.



$f(x,y)=\dfrac{xy}{x^2+y^2}$. The graph exhibits different limiting values as x and y become large or as (x,y) approaches the origin, depending on the direction being examined. For example, although f is undefined at the origin, the function values appear to be $\dfrac{1}{2}$ along the line $y=x$, regardless of the distance from the origin. Along the line $y=-x$, the value is always

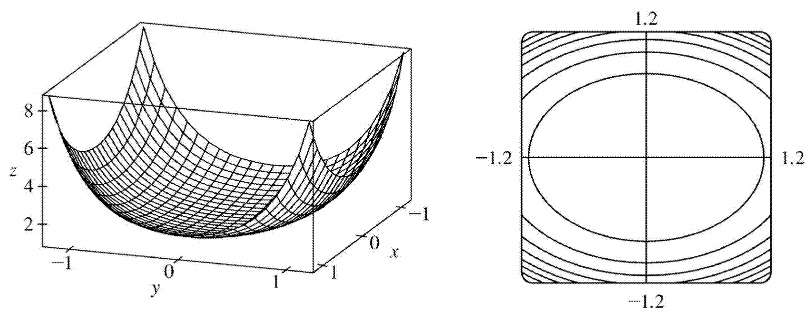
$-\frac{1}{2}$. Along the axes, $f(x,y)=0$ for all values of (x,y) except the origin. Other directions, heading toward the origin or away from the origin, give various limiting values between $-\frac{1}{2}$ and $\frac{1}{2}$.

69. $f(x,y)=e^{cx^2+y^2}$. First, if $c=0$, the graph is the cylindrical surface $z=e^{y^2}$ (whose level curves are parallel lines). When $c>0$, the vertical trace above the y –axis remains fixed while the sides of the surface in the x –direction ‘‘curl’’ upward, giving the graph a shape resembling an elliptic paraboloid. The level curves of the surface are ellipses centered at the origin.



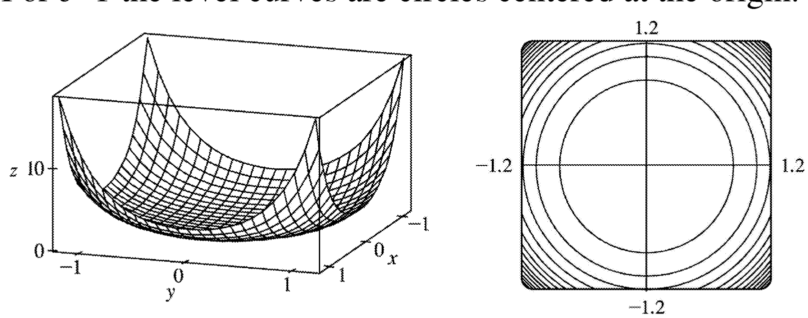
$c=0$

For $0<c<1$, the ellipses have major axis the x –axis and the eccentricity increases as $c\rightarrow 0$.



$c=0.5$ (level curves in increments of 1)

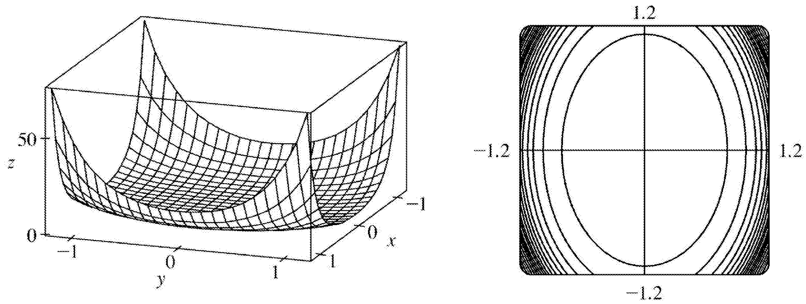
For $c=1$ the level curves are circles centered at the origin.



$c=1$ (level curves in increments of 1)

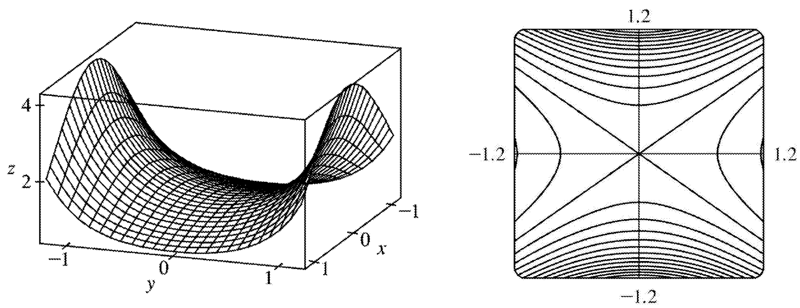
When $c>1$, the level curves are ellipses with major axis the y –axis, and the eccentricity increases as c

increases.

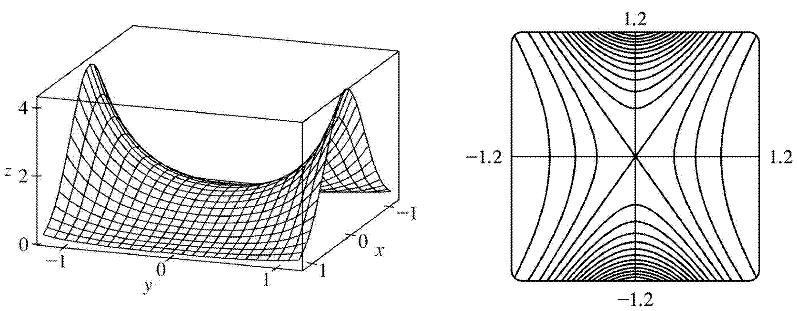


$c=2$ (level curves in increments of 4)

For values of $c<0$, the sides of the surface in the x –direction curl downward and approach the xy –plane (while the vertical trace $x=0$ remains fixed), giving a saddle–shaped appearance to the graph near the point $(0,0,1)$. The level curves consist of a family of hyperbolas. As c decreases, the surface becomes flatter in the x –direction and the surface's approach to the curve in the trace $x=0$ becomes steeper, as the graphs demonstrate.

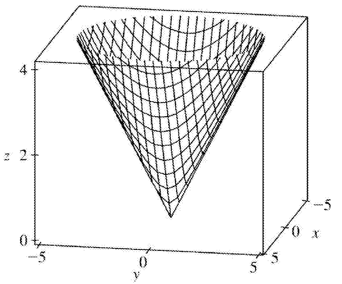


$c=–0.5$ (level curves in increments of 0.25)

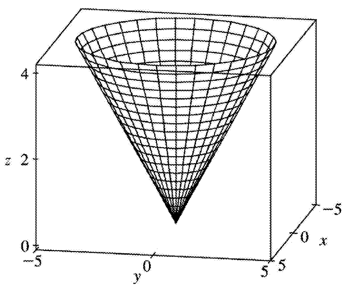


$c=–2$ (level curves in increments of 0.25)

70. First, we graph $f(x,y)=\sqrt{x^2+y^2}$. As an alternative, the x^2+y^2 expression suggests that cylindrical coordinates may be appropriate, giving the equivalent equation $z=\sqrt{r^2}=r$, $r\geq 0$ which we graph as well. Notice that the graph in cylindrical coordinates better demonstrates the symmetry of the surface.

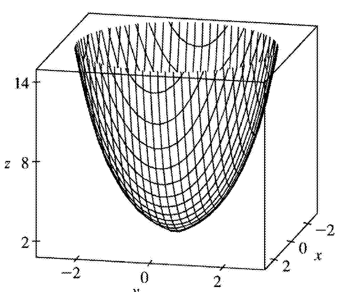


$f(x,y)=\sqrt{x^2+y^2}$

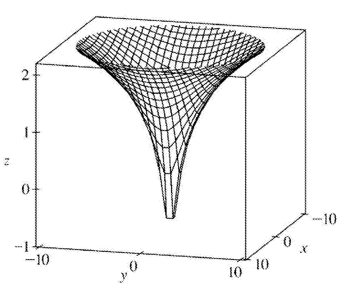


$z = r, r \geq 0$

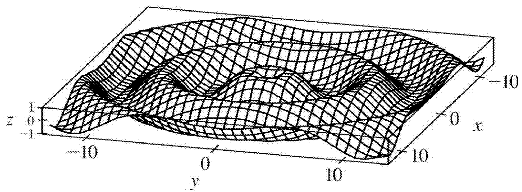
Graphs of the other four functions follow.



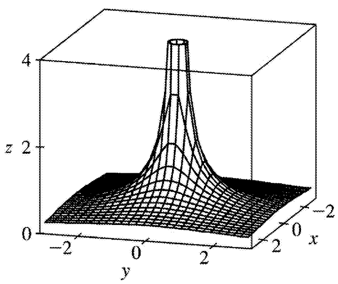
$f(x,y)=e^{\sqrt{x^2+y^2}}$



$f(x,y)=\ln\sqrt{x^2+y^2}$



$f(x,y)=\sin\left(\sqrt{x^2+y^2}\right)$



$f(x,y)=\dfrac{1}{\sqrt{x^2+y^2}}$

Notice that each graph $f(x,y)=g\left(\sqrt{x^2+y^2}\right)$ exhibits radial symmetry about the z –axis and the trace in the xz –plane for $x\geq 0$ is the graph of $z=g(x)$, $x\geq 0$. This suggests that the graph of $f(x,y)=g\left(\sqrt{x^2+y^2}\right)$ is obtained from the graph of g by graphing $z=g(x)$ in the xz –plane and rotating the curve about the z –axis.

71. **(a)** $P=bL^{\alpha}K^{1-\alpha} \Rightarrow \dfrac{P}{K}=bL^{\alpha}K^{-\alpha} \Rightarrow \dfrac{P}{K}=b\left(\dfrac{L}{K}\right)^{\alpha} \Rightarrow \ln\dfrac{P}{K}=\ln\left(b\left(\dfrac{L}{K}\right)^{\alpha}\right) \Rightarrow$
$\ln\dfrac{P}{K}=\ln b+\alpha\ln\left(\dfrac{L}{K}\right)$

(b) We list the values for ln (L/K) and ln (P/K) for the years 1899—1922. (Historically, these values were rounded to 2 decimal places.)

Year	$x=\ln(L/K)$	$y=\ln(P/K)$
1899	0	0
1900	–0.02	–0.06
1901	–0.04	–0.02
1902	–0.04	0
1903	–0.07	–0.05
1904	–0.13	–0.12
1905	–0.18	–0.04

1906	–0.20	–0.07
1907	–0.23	–0.15
1908	–0.41	–0.38
1909	–0.33	–0.24
1910	–0.35	–0.27
1911	–0.38	–0.34
1912	–0.38	–0.24
1913	–0.41	–0.25
1914	–0.47	–0.37
1915	–0.53	–0.34
1915	–0.53	–0.34
1916	–0.49	–0.28
1917	–0.53	–0.39
1918	–0.60	–0.50
1919	–0.68	–0.57
1920	–0.74	–0.57
1921	–1.05	–0.85
1922	–0.98	–0.59

After entering the (x,y) pairs into a calculator or CAS, the resulting least squares regression line through the points is approximately $y=0.75136x+0.01053$, which we round to $y=0.75x+0.01$.
(c) Comparing the regression line from part (b) to the equation $y=\ln b+\alpha x$ with $x=\ln(L/K)$ and $y=\ln(P/K)$, we have $\alpha=0.75$ and $\ln b=0.01\Rightarrow b=e^{0.01}\approx 1.01$. Thus, the Cobb–Douglas production function is $P=bL^{\alpha}K^{1-\alpha}=1.01L^{0.75}K^{0.25}$.

1. In general, we can't say anything about $f(3,1)$! $\lim_{(x,y)\to(3,1)} f(x,y)=6$ means that the values of $f(x,y)$ approach 6 as (x,y) approaches, but is not equal to, (3,1) . If f is continuous, we know that $\lim_{(x,y)\to(a,b)} f(x,y)=f(a,b)$, so $\lim_{(x,y)\to(3,1)} f(x,y)=f(3,1)=6$.

2. **(a)** The outdoor temperature as a function of longitude, latitude, and time is continuous. Small changes in longitude, latitude, or time can produce only small changes in temperature, as the temperature doesn't jump abruptly from one value to another.
(b) Elevation is not necessarily continuous. If we think of a cliff with a sudden drop–off, a very small change in longitude or latitude can produce a comparatively large change in elevation, without all the intermediate values being attained. Elevation *can* jump from one value to another.
(c) The cost of a taxi ride is usually discontinuous. The cost normally increases in jumps, so small changes in distance traveled or time can produce a jump in cost. A graph of the function would show breaks in the surface.

3. We make a table of values of $f(x,y)=\dfrac{x^2y^3+x^3y^2-5}{2-xy}$ for a set of (x,y) points near the origin.

x \ y	−0.2	−0.1	−0.05	0	0.05	0.1	0.2
−0.2	−2.551	−2.525	−2.513	−2.500	−2.488	−2.475	−2.451
−0.1	−2.525	−2.513	−2.506	−2.500	−2.494	−2.488	−2.475
−0.05	−2.513	−2.506	−2.503	−2.500	−2.497	−2.494	−2.488
0	−2.500	−2.500	−2.500		−2.500	−2.500	−2.500
0.05	−2.488	−2.494	−2.497	−2.500	−2.503	−2.506	−2.513
0.1	−2.475	−2.488	−2.494	−2.500	−2.506	−2.513	−2.525
0.2	−2.451	−2.475	−2.488	−2.500	−2.513	−2.525	−2.551

As the table shows, the values of $f(x,y)$ seem to approach −2.5 as (x,y) approaches the origin from a variety of different directions. This suggests that $\lim_{(x,y)\to(0,0)} f(x,y)=-2.5$.
Since f is a rational function, it is continuous on its domain. f is defined at (0,0) , so we can use direct substitution to establish that $\lim_{(x,y)\to(0,0)} f(x,y)=\dfrac{0^2 0^3+0^3 0^2-5}{2-0\cdot 0}=-\dfrac{5}{2}$, verifying our guess.

4. We make a table of values of $f(x,y)=\dfrac{2xy}{x^2+2y^2}$ for a set of (x,y) points near the origin.

x \ y	−0.3	−0.2	−0.1	0	0.1	0.2	0.3
−0.3	0.667	0.706	0.545	0.000	−0.545	−0.706	−0.667
−0.2	0.545	0.667	0.667	0.000	−0.667	−0.667	−0.545
−0.1	0.316	0.444	0.667	0.000	−0.667	−0.444	−0.316
0	0.000	0.000	0.000		0.000	0.000	0.000
0.1	−0.316	−0.444	−0.667	0.000	0.667	0.444	0.316
0.2	−0.545	−0.667	−0.667	0.000	0.667	0.667	0.545
0.3	−0.667	−0.706	−0.545	0.000	0.545	0.706	0.667

It appears from the table that the values of $f(x,y)$ are not approaching a single value as (x,y) approaches the origin. For verification, if we first approach (0,0) along the x –axis, we have $f(x,0)=0$, so $f(x,y)\to 0$. But if we approach (0,0) along the line $y=x$, $f(x,x)=\dfrac{2x^2}{x^2+2x^2}=\dfrac{2}{3}$ ($x\neq 0$), so $f(x,y)\to \dfrac{2}{3}$. Since f approaches different values along different paths to the origin, this limit does not exist.

5. $f(x,y)=x^5+4x^3y-5xy^2$ is a polynomial, and hence continuous, so $\lim_{(x,y)\to(5,-2)} f(x,y)=f(5,-2)=5^5+4(5)^3(-2)-5(5)(-2)^2=2025$.

6. $x-2y$ is a polynomial and therefore continuous. Since $\cos t$ is a continuous function, the composition $\cos(x-2y)$ is also continuous. xy is also a polynomial, and hence continuous, so the product $f(x,y)=xy\cos(x-2y)$ is a continuous function. Then $\lim_{(x,y)\to(6,3)} f(x,y)=f(6,3)=(6)(3)\cos(6-2\cdot 3)=18$.

7. $f(x,y)=x^2/\left(x^2+y^2\right)$. First approach (0,0) along the x –axis. Then $f(x,0)=x^2/x^2=1$ for $x\neq 0$, so $f(x,y)\to 1$. Now approach (0,0) along the y –axis. Then for $y\neq 0$, $f(0,y)=0$, so $f(x,y)\to 0$. Since f has two different limits along two different lines, the limit does not exist.

8. $f(x,y)=\left(x^2+\sin^2 y\right)/\left(2x^2+y^2\right)$. First approach $(0,0)$ along the x –axis. Then $f(x,0)=x^2/2x^2=\dfrac{1}{2}$ for $x\neq 0$, so $f(x,y)\to \dfrac{1}{2}$. Next approach $(0,0)$ along the y –axis. For $y\neq 0$, $f(0,y)=\dfrac{\sin^2 y}{y^2}=\left(\dfrac{\sin y}{y}\right)^2$ and $\lim_{y\to 0}\dfrac{\sin y}{y}=1$, so $f(x,y)\to 1$. Since f has two different limits along two different lines, the limit does not exist.

9. $f(x,y)=(xy\cos y)/(3x^2+y^2)$. On the x –axis, $f(x,0)=0$ for $x\neq 0$, so $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the x –axis. Approaching $(0,0)$ along the line $y=x$, $f(x,x)=(x^2\cos x)/4x^2=\frac{1}{4}\cos x$ for $x\neq 0$, so $f(x,y)\to \frac{1}{4}$ along this line. Thus the limit does not exist.

10. $f(x,y)=6x^3y/\left(2x^4+y^4\right)$. On the x –axis, $f(x,0)=0$ for $x\neq 0$, so $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the x –axis. Approaching $(0,0)$ along the line $y=x$ gives $f(x,x)=6x^4/\left(3x^4\right)=2$ for $x\neq 0$, so along this line $f(x,y)\to 2$ as $(x,y)\to(0,0)$. Thus the limit does not exist.

11. $f(x,y)=\dfrac{xy}{\sqrt{x^2+y^2}}$. We can see that the limit along any line through $(0,0)$ is 0 , as well as along other paths through $(0,0)$ such as $x=y^2$ and $y=x^2$. So we suspect that the limit exists and equals 0 ; we use the Squeeze Theorem to prove our assertion. $0\leq \left|\dfrac{xy}{\sqrt{x^2+y^2}}\right|\leq |x|$ since $|y|\leq\sqrt{x^2+y^2}$, and $|x|\to 0$ as $(x,y)\to(0,0)$. So $\lim\limits_{(x,y)\to(0,0)} f(x,y)=0$.

12. $f(x,y)=\left(x^4-y^4\right)/\left(x^2+y^2\right)=\left(x^2+y^2\right)\left(x^2-y^2\right)/(x^2+y^2\big)=x^2-y^2$ for $(x,y)\neq(0,0)$. Thus the limit as $(x,y)\to(0,0)$ is 0 .

13. Let $f(x,y)=\dfrac{2x^2y}{x^4+y^2}$. Then $f(x,0)=0$ for $x\neq 0$, so $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the x –axis. But $f\left(x,x^2\right)=\dfrac{2x^4}{2x^4}=1$ for $x\neq 0$, so $f(x,y)\to 1$ as $(x,y)\to(0,0)$ along the parabola $y=x^2$. Thus the limit doesn't exist.

14. We can use the Squeeze Theorem to show that $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2\sin^2 y}{x^2+2y^2}=0$:

$0\leq\dfrac{x^2\sin^2 y}{x^2+2y^2}\leq\sin^2 y$ since $\dfrac{x^2}{x^2+2y^2}\leq 1$, and $\sin^2 y\to 0$ as $(x,y)\to(0,0)$, so $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2\sin^2 y}{x^2+2y^2}=0$.

15.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}\cdot\frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\left(x^2+y^2\right)\left(\sqrt{x^2+y^2+1}+1\right)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \left(\sqrt{x^2+y^2+1}+1\right)=2$$

16. $f(x,y)=xy^4/\left(x^2+y^8\right)$. On the x –axis, $f(x,0)=0$ for $x\neq 0$, so $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the x –axis. Approaching $(0,0)$ along the curve $x=y^4$ gives $f(y^4,y)=y^8/2y^8=\frac{1}{2}$ for $y\neq 0$, so along this path $f(x,y)\to \frac{1}{2}$ as $(x,y)\to(0,0)$. Thus the limit does not exist.

17. e^{-xy} and $\sin(\pi z/2)$ are each compositions of continuous functions, and hence continuous, so their product $f(x,y,z)=e^{-xy}\sin(\pi z/2)$ is a continuous function. Then $\lim_{(x,y,z)\to(3,0,1)} f(x,y,z)=f(3,0,1)=e^{-(3)(0)}\sin(\pi\cdot 1/2)=1$.

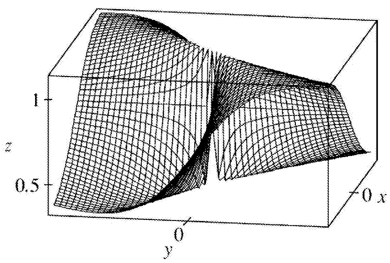
18. $f(x,y,z)=\frac{x^2+2y^2+3z^2}{x^2+y^2+z^2}$. Then $f(x,0,0)=\frac{x^2+0+0}{x^2+0+0}=1$ for $x\neq 0$, so $f(x,y,z)\to 1$ as $(x,y,z)\to(0,0,0)$ along the x –axis. But $f(0,y,0)=\frac{0+2y^2+0}{0+y^2+0}=2$ for $y\neq 0$, so $f(x,y,z)\to 2$ as $(x,y,z)\to(0,0,0)$ along the y –axis. Thus, the limit doesn't exist.

19. $f(x,y,z)=\frac{xy+yz^2+xz^2}{x^2+y^2+z^4}$. Then $f(x,0,0)=0/x^2=0$ for $x\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the x –axis, $f(x,y,z)\to 0$. But $f(x,x,0)=x^2/\left(2x^2\right)=\frac{1}{2}$ for $x\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the line $y=x$, $z=0$, $f(x,y,z)\to \frac{1}{2}$. Thus the limit doesn't exist.

20. $f(x,y,z)=\frac{xy+yz+zx}{x^2+y^2+z^2}$. Then $f(x,0,0)=0$ for $x\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the x –axis,

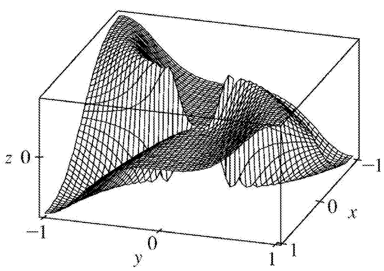
$f(x,y,z)\to 0$. But $f(x,x,0)=x^2/(2x^2)=\frac{1}{2}$ for $x\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the line $y=x$, $z=0$, $f(x,y,z)\to \frac{1}{2}$. Thus the limit doesn't exist.

21.



From the ridges on the graph, we see that as $(x,y)\to(0,0)$ along the lines under the two ridges, $f(x,y)$ approaches different values. So the limit does not exist.

22.



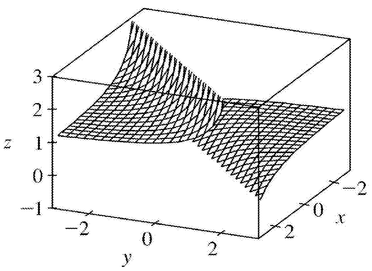
From the graph, it appears that as we approach the origin along the lines $x=0$ or $y=0$, the function is everywhere 0 , whereas if we approach the origin along a certain curve it has a constant value of about $\frac{1}{2}$. Since the function approaches different values depending on the path of approach, the limit does not exist.

23. $h(x,y)=g(f(x,y))=(2x+3y-6)^2+\sqrt{2x+3y-6}$. Since f is a polynomial, it is continuous on R^2 and g is continuous on its domain $\{t\mid t\geq 0\}$. Thus h is continuous on its domain $D=\{(x,y)\mid 2x+3y-6\geq 0\}=\left\{(x,y)\mid y\geq -\frac{2}{3}x+2\right\}$, which consists of all points on or above the line $y=-\frac{2}{3}x+2$.

24. $h(x,y)=g(f(x,y))=\left(\sqrt{x^2-y}-1\right)/\left(\sqrt{x^2-y}+1\right)$. Since f is a polynomial, it is continuous on R^2 and g is continuous on its domain $\{t\mid t\geq 0\}$. Thus h is continuous on its domain

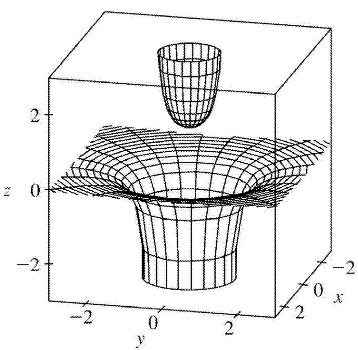
$D=\{(x,y) \mid x^2-y\geq 0\}=\{(x,y) \mid y\leq x^2\}$ which consists of all points below or on the parabola $y=x^2$.

25.



From the graph, it appears that f is discontinuous along the line $y=x$. If we consider $f(x,y)=e^{1/(x-y)}$ as a composition of functions, $g(x,y)=1/(x-y)$ is a rational function and therefore continuous except where $x-y=0 \Rightarrow y=x$. Since the function $h(t)=e^t$ is continuous everywhere, the composition $h(g(x,y))=e^{1/(x-y)}=f(x,y)$ is continuous except along the line $y=x$, as we suspected.

26.



We can see a circular break in the graph, corresponding approximately to the unit circle, where f is discontinuous.
[*Note:* For a more accurate graph, try converting to cylindrical coordinates first.] Since $f(x,y)=\dfrac{1}{1-x^2-y^2}$ is a rational function, it is continuous except where $1-x^2-y^2=0 \Rightarrow x^2+y^2=1$, confirming our observation that f is discontinuous on the circle $x^2+y^2=1$.

27. The functions $\sin(xy)$ and e^x-y^2 are continuous everywhere, so $F(x,y)=\dfrac{\sin(xy)}{e^x-y^2}$ is continuous except where $e^x-y^2=0 \Rightarrow y^2=e^x \Rightarrow y=\pm\sqrt{e^x}=\pm e^{\frac{1}{2}x}$. Thus F is continuous on its domain $\{(x,y) \mid y\neq \pm e^{x/2}\}$.

28. $F(x,y)=\dfrac{x-y}{1+x^2+y^2}$ is a rational function and thus is continuous on its domain R^2 (since the denominator is never zero).

29. $F(x,y)=\arctan\left(x+\sqrt{y}\right)=g(f(x,y))$ where $f(x,y)=x+\sqrt{y}$, continuous on its domain $\{(x,y)\mid y\geq 0\}$, and $g(t)=\arctan t$ is continuous everywhere. Thus F is continuous on its domain $\{(x,y)\mid y\geq 0\}$.

30. e^{x^2y} is continuous on R^2 and $\sqrt{x+y^2}$ is continuous on its domain $\left\{(x,y)\mid x+y^2\geq 0\right\}=\left\{(x,y)\mid x\geq -y^2\right\}$, so $F(x,y)=e^{x^2y}+\sqrt{x+y^2}$ is continuous on the set $\left\{(x,y)\mid x\geq -y^2\right\}$.

31. $G(x,y)=\ln\left(x^2+y^2-4\right)=g(f(x,y))$ where $f(x,y)=x^2+y^2-4$, continuous on R^2 , and $g(t)=\ln t$, continuous on its domain $\{t\mid t>0\}$. Thus G is continuous on its domain $\left\{(x,y)\mid x^2+y^2-4>0\right\}=\left\{(x,y)\mid x^2+y^2>4\right\}$, the exterior of the circle $x^2+y^2=4$.

32. $G(x,y)=g\left(f(x,y)\right)$ where $f(x,y)=x^2+y^2$, continuous on R^2 , and $g(t)=\sin^{-1}t$, continuous on its domain $\{t\mid -1\leq t\leq 1\}$. Thus G is continuous on its domain $D=\left\{(x,y)\mid -1\leq x^2+y^2\leq 1\right\}=\left\{(x,y)\mid x^2+y^2\leq 1\right\}$, inside and on the circle $x^2+y^2=1$.

33. $\sqrt{y}$ is continuous on its domain $\{y\mid y\geq 0\}$ and $x^2-y^2+z^2$ is continuous everywhere, so $f(x,y,z)=\dfrac{\sqrt{y}}{x^2-y^2+z^2}$ is continuous for $y\geq 0$ and $x^2-y^2+z^2\neq 0\Rightarrow y^2\neq x^2+z^2$, that is, $\left\{(x,y,z)\mid y\geq 0, y\neq\sqrt{x^2+z^2}\right\}$.

34. $f(x,y,z)=\sqrt{x+y+z}=h(g(x,y,z))$ where $g(x,y,z)=x+y+z$, continuous everywhere, and $h(t)=\sqrt{t}$ is continuous on its domain $\{t\mid t\geq 0\}$. Thus f is continuous on its domain $\{(x,y,z)\mid x+y+z\geq 0\}$, so f is continuous on and above the plane $z=-x-y$.

35. $f(x,y)=\begin{cases}\dfrac{x^2y^3}{2x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 1 & \text{if } (x,y)=(0,0)\end{cases}$ The first piece of f is a rational function defined everywhere except at the origin, so f is continuous on R^2 except possibly at the origin. Since $x^2\leq 2x^2+y^2$, we have $\left|x^2y^3/(2x^2+y^2)\right|\leq\left|y^3\right|$. We know that $\left|y^3\right|\to 0$ as $(x,y)\to(0,0)$. So, by the

Squeeze Theorem, $\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+y^2}=0$. But $f(0,0)=1$, so f is discontinuous at (0,0) . Therefore, f is continuous on the set $\{(x,y)|(x,y)\neq(0,0)\}$.

36. $f(x,y)=\begin{cases} \frac{xy}{x^2+xy+y^2} & \text{if}(x,y)\neq(0,0) \\ 0 & \text{if}(x,y)=(0,0) \end{cases}$ The first piece of f is a rational function defined everywhere except at the origin, so f is continuous on R^2 except possibly at the origin. $f(x,0)=0/x^2=0$ for $x\neq 0$, so $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the x –axis. But $f(x,x)=x^2/\left(3x^2\right)=\frac{1}{3}$ for $x\neq 0$, so $f(x,y)\to\frac{1}{3}$ as $(x,y)\to(0,0)$ along the line $y=x$. Thus $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so f is not continuous at (0,0) and the largest set on which f is continuous is $\{(x,y)|(x,y)\neq(0,0)\}$.

37. $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}=\lim_{r\to 0^+} \frac{(r\cos\theta)^3+(r\sin\theta)^3}{r^2}=\lim_{r\to 0^+} \left(r\cos^3\theta+r\sin^3\theta\right)=0$

38.

$$\lim_{(x,y)\to(0,0)} \left(x^2+y^2\right)\ln\left(x^2+y^2\right)=\lim_{r\to 0^+} r^2\ln r^2=\lim_{r\to 0^+} \frac{\ln r^2}{1/r^2}$$

$$=\lim_{r\to 0^+} \frac{\left(1/r^2\right)(2r)}{-2/r^3} \text{ [using l'Hospital's Rule] } =\lim_{r\to 0^+} \left(-r^2\right)=0$$

39.

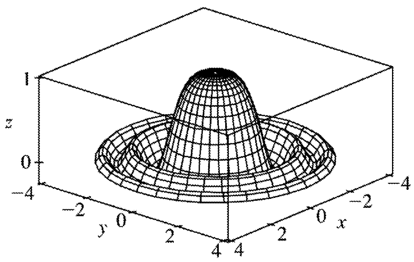
$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}=\lim_{\rho\to 0^+} \frac{(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)}{\rho^2}$$

$$=\lim_{\rho\to 0^+} \left(\rho\sin^2\phi\cos\phi\sin\theta\cos\theta\right)=0$$

40. $\lim_{(x,y)\to(0,0)} \frac{\sin\left(x^2+y^2\right)}{x^2+y^2}=\lim_{r\to 0^+} \frac{\sin\left(r^2\right)}{r^2}$, which is an indeterminate form of type 0/0 . Using l'Hospital's Rule, we get

$$\lim_{r\to 0^+} \frac{\sin\left(r^2\right)}{r^2} = \lim_{r\to 0^+} \frac{2r\cos\left(r^2\right)}{2r} = \lim_{r\to 0^+} \cos\left(r^2\right) = 1 .$$

Or: Use the fact that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.



41. Since $|\mathbf{x}-\mathbf{a}|^2 = |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2|\mathbf{x}|\,|\mathbf{a}|\cos\theta \geq |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2|\mathbf{x}|\,|\mathbf{a}| = (|\mathbf{x}| - |\mathbf{a}|)^2$, we have $||\mathbf{x}| - |\mathbf{a}|| \leq |\mathbf{x}-\mathbf{a}|$. Let $\varepsilon>0$ be given and set $\delta=\varepsilon$. Then whenever $0<|\mathbf{x}-\mathbf{a}|<\delta$, $||\mathbf{x}| - |\mathbf{a}|| \leq |\mathbf{x}-\mathbf{a}| < \delta = \varepsilon$. Hence $|\mathbf{x}| = |\mathbf{a}|$ and $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on R^n .

42. Let $\varepsilon>0$ be given. We need to find $\delta>0$ such that $|f(\mathbf{x}) - f(\mathbf{a})| < \varepsilon$ whenever $|\mathbf{x}-\mathbf{a}|<\delta$ or $|\mathbf{c}\cdot\mathbf{x} - \mathbf{c}\cdot\mathbf{a}| < \varepsilon$ whenever $|\mathbf{x}-\mathbf{a}|<\delta$. But $|\mathbf{c}\cdot\mathbf{x} - \mathbf{c}\cdot\mathbf{a}| = |\mathbf{c}\cdot(\mathbf{x}-\mathbf{a})|$ and $|\mathbf{c}\cdot(\mathbf{x}-\mathbf{a})| \leq |\mathbf{c}|\,|\mathbf{x}-\mathbf{a}|$ by Exercise 13.3.57 [ET 12.3.57] (the Cauchy–Schwartz Inequality). Let $\varepsilon>0$ be given and set $\delta=\varepsilon/|\mathbf{c}|$. Then whenever $0<|\mathbf{ax}-\mathbf{a}|<\delta$, $|f(\mathbf{x}) - f(\mathbf{a})| = |\mathbf{c}\cdot\mathbf{x} - \mathbf{c}\cdot\mathbf{a}| \leq |\mathbf{c}|\,|\mathbf{x}-\mathbf{a}| < |\mathbf{c}|\delta = |\mathbf{c}|\,(\varepsilon/|\mathbf{c}|) = \varepsilon$. So f is continuous on R^n .

1. **(a)** $\partial T/\partial x$ represents the rate of change of T when we fix y and t and consider T as a function of the single variable x, which describes how quickly the temperature changes when longitude changes but latitude and time are constant. $\partial T/\partial y$ represents the rate of change of T when we fix x and t and consider T as a function of y, which describes how quickly the temperature changes when latitude changes but longitude and time are constant. $\partial T/\partial t$ represents the rate of change of T when we fix x and y and consider T as a function of t, which describes how quickly the temperature changes over time for a constant longitude and latitude.
(b) $f_x(158,21,9)$ represents the rate of change of temperature at longitude $158°$ W, latitude $21°$ N at 9:00 A.M. when only longitude varies. Since the air is warmer to the west than to the east, increasing longitude results in an increased air temperature, so we would expect $f_x(158,21,9)$ to be positive. $f_y(158,21,9)$ represents the rate of change of temperature at the same time and location when only latitude varies. Since the air is warmer to the south and cooler to the north, increasing latitude results in a decreased air temperature, so we would expect $f_y(158,21,9)$ to be negative. $f_t(158,21,9)$ represents the rate of change of temperature at the same time and location when only time varies. Since typically air temperature increases from the morning to the afternoon as the sun warms it, we would expect $f_t(158,21,9)$ to be positive.

2. By Definition 4, $f_T(92,60)=\lim_{h\to 0}\dfrac{f(92+h,60)-f(92,60)}{h}$, which we can approximate by considering $h=2$ and $h=-2$ and using the values given in Table 1:
$f_T(92,60)\approx\dfrac{f(94,60)-f(92,60)}{2}=\dfrac{111-105}{2}=3$, $f_T(92,60)\approx\dfrac{f(90,60)-f(92,60)}{-2}=\dfrac{100-105}{-2}=2.5$.
Averaging these values, we estimate $f_T(92,60)$ to be approximately 2.75. Thus, when the actual temperature is $92°$ F and the relative humidity is 60%, the apparent temperature rises by about $2.75°$ F for every degree that the actual temperature rises.
Similarly, $f_H(92,60)=\lim_{h\to 0}\dfrac{f(92,60+h)-f(92,60)}{h}$ which we can approximate by considering $h=5$ and $h=-5$: $f_H(92,60)\approx\dfrac{f(92,65)-f(92,60)}{5}=\dfrac{108-105}{5}=0.6$,
$f_H(92,60)\approx\dfrac{f(92,55)-f(92,60)}{-5}=\dfrac{103-105}{-5}=0.4$. Averaging these values, we estimate $f_H(92,60)$ to be approximately 0.5. Thus, when the actual temperature is $92°$ F and the relative humidity is 60%, the apparent temperature rises by about $0.5°$ F for every percent that the relative humidity increases.

3. **(a)** By Definition 4, $f_T(-15,30)=\lim_{h\to 0}\dfrac{f(-15+h,30)-f(-15,30)}{h}$, which we can approximate by considering $h=5$ and $h=-5$ and using the values given in the table:

$f_T(-15,30) \approx \dfrac{f(-10,30)-f(-15,30)}{5} = \dfrac{-20-(-26)}{5} = \dfrac{6}{5} = 1.2$,

$f_T(-15,30) \approx \dfrac{f(-20,30)-f(-15,30)}{-5} = \dfrac{-33-(-26)}{-5} = \dfrac{-7}{-5} = 1.4$. Averaging these values, we estimate $f_T(-15,30)$ to be approximately 1.3 . Thus, when the actual temperature is $-15°$ F and the wind speed is 30 km / h, the apparent temperature rises by about $1.3°$ F for every degree that the actual temperature rises.

Similarly, $f_v(-15,30) = \lim_{h\to 0} \dfrac{f(-15,30+h)-f(-15,30)}{h}$ which we can approximate by considering $h=10$ and $h=-10$: $f_v(-15,30) \approx \dfrac{f(-15,40)-f(-15,30)}{10} = \dfrac{-27-(-26)}{10} = \dfrac{-1}{10} = -0.1$,

$f_v(-15,30) \approx \dfrac{f(-15,20)-f(-15,30)}{-10} = \dfrac{-24-(-26)}{-10} = \dfrac{2}{-10} = -0.2$. Averaging these values, we estimate $f_v(-15,30)$ to be approximately -0.15 . Thus, when the actual temperature is $-15°$ F and the wind speed is 30 km / h, the apparent temperature decreases by about $0.15°$ F for every km / h that the wind speed increases.

(b) For a fixed wind speed v , the values of the wind–chill index W increase as temperature T increases (look at a column of the table), so $\dfrac{\partial W}{\partial T}$ is positive. For a fixed temperature T , the values of W decrease (or remain constant) as v increases (look at a row of the table), so $\dfrac{\partial W}{\partial v}$ is negative (or perhaps 0).

(c) For fixed values of T , the function values $f(\mathrm{T,v})$ appear to become constant (or nearly constant) as v increases, so the corresponding rate of change is 0 or near 0 as v increases. This suggests that $\lim_{v\to\infty} (\partial W/\partial v) = 0$.

4. **(a)** $\partial h/\partial v$ represents the rate of change of h when we fix t and consider h as a function of v , which describes how quickly the wave heights change when the wind speed changes for a fixed time duration. $\partial h/\partial t$ represents the rate of change of h when we fix v and consider h as a function of t , which describes how quickly the wave heights change when the duration of time changes, but the wind speed is constant.

(b) By Definition 4, $f_v(40,15) = \lim_{h\to 0} \dfrac{f(40+\mathrm{h},15)-f(40,15)}{h}$ which we can approximate by considering $h=10$ and $h=-10$ and using the values given in the table: $f_v(40,15) \approx \dfrac{f(50,15)-f(40,15)}{10} = \dfrac{36-25}{10} = 1.1$, $f_v(40,15) \approx \dfrac{f(30,15)-f(40,15)}{-10} = \dfrac{16-25}{-10} = 0.9$. Averaging these values, we have $f_v(40,15) \approx 1.0$. Thus, when a 40 –knot wind has been blowing for 15 hours, the wave heights should increase by about 1 foot for every knot that the wind speed increases (with the same time duration). Similarly,

$f_t(40,15)=\lim_{h\to 0}\dfrac{f(40,15+h)-f(40,15)}{h}$ which we can approximate by considering $h=5$ and $h=-5$:
$f_t(40,15)\approx\dfrac{f(40,20)-f(40,15)}{5}=\dfrac{28-25}{5}=0.6$, $f_t(40,15)\approx\dfrac{f(40,10)-f(40,15)}{-5}=\dfrac{21-25}{-5}=0.8$.
Averaging these values, we have $f_t(40,15)\approx 0.7$. Thus, when a 40 –knot wind has been blowing for 15 hours, the wave heights increase by about 0.7 feet for every additional hour that the wind blows.
(c) For fixed values of v , the function values f(v,t) appear to increase in smaller and smaller increments, becoming nearly constant as t increases. Thus, the corresponding rate of change is nearly 0 as t increases, suggesting that $\lim_{t\to\infty}(\partial h/\partial t)=0$.

5. **(a)** If we start at $(1,2)$ and move in the positive x –direction, the graph of f increases. Thus $f_x(1,2)$ is positive.
(b) If we start at $(1,2)$ and move in the positive y –direction, the graph of f decreases. Thus $f_y(1,2)$ is negative.

6. **(a)** The graph of f decreases if we start at $(-1,2)$ and move in the positive x –direction, so $f_x(-1,2)$ is negative.
(b) The graph of f decreases if we start at $(-1,2)$ and move in the positive y –direction, so $f_y(-1,2)$ is negative.
(c) $f_{xx}=\dfrac{\partial}{\partial x}\left(f_x\right)$, so f_{xx} is the rate of change of f_x in the x –direction. f_x is negative at $(-1,2)$ and if we move in the positive x –direction, the surface becomes less steep. Thus the values of f_x are increasing and $f_{xx}(-1,2)$ is positive.
(d) f_{yy} is the rate of change of f_y in the y –direction. f_y is negative at $(-1,2)$ and if we move in the positive y –direction, the surface becomes steeper. Thus the values of f_y are decreasing, and $f_{yy}(-1,2)$ is negative.

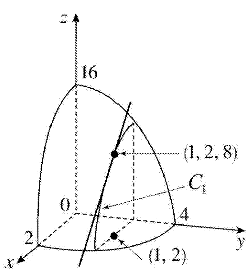
7. First of all, if we start at the point $(3,-3)$ and move in the positive y –direction, we see that both b and c decrease, while a increases. Both b and c have a low point at about $(3,-1.5)$, while a is 0 at this point. So a is definitely the graph of f_y , and one of b and c is the graph of f . To see which is which, we start at the point $(-3,-1.5)$ and move in the positive x –direction. b traces out a line with negative slope, while c traces out a parabola opening downward. This tells us that b is the x –derivative of c . So c is the graph of f , b is the graph of f_x , and a is the graph of f_y .

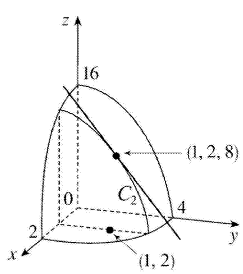
8.

$f_x(2,1)$ is the rate of change of f at $(2,1)$ in the x –direction. If we start at $(2,1)$, where $f(2,1)=10$, and move in the positive x –direction, we reach the next contour line (where $f(x,y)=12$) after approximately 0.6 units. This represents an average rate of change of about $\frac{2}{0.6}$. If we approach the point $(2,1)$ from the left (moving in the positive x –direction) the output values increase from 8 to 10 with an increase in x of approximately 0.9 units, corresponding to an average rate of change of $\frac{2}{0.9}$. A good estimate for $f_x(2,1)$ would be the average of these two, so $f_x(2,1)\approx 2.8$. Similarly, $f_y(2,1)$ is the rate of change of f at $(2,1)$ in the y –direction. If we approach $(2,1)$ from below, the output values decrease from 12 to 10 with a change in y of approximately 1 unit, corresponding to an average rate of change of -2 . If we start at $(2,1)$ and move in the positive y –direction, the output values decrease from 10 to 8 after approximately 0.9 units, a rate of change of $\frac{-2}{0.9}$. Averaging these two results, we estimate $f_y(2,1)\approx -2.1$.

9. $f(x,y)=16-4x^2-y^2 \Rightarrow f_x(x,y)=-8x$ and $f_y(x,y)=-2y \Rightarrow f_x(1,2)=-8$ and $f_y(1,2)=-4$. The graph of f is the paraboloid $z=16-4x^2-y^2$ and the vertical plane $y=2$ intersects it in the parabola $z=12-4x^2$, $y=2$ (the curve C_1 in the first figure).

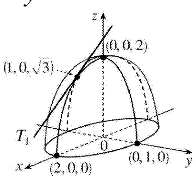
The slope of the tangent line to this parabola at $(1,2,8)$ is $f_x(1,2)=-8$. Similarly the plane $x=1$ intersects the paraboloid in the parabola $z=12-y^2$, $x=1$ (the curve C_2 in the second figure) and the slope of the tangent line at $(1,2,8)$ is $f_y(1,2)=-4$.

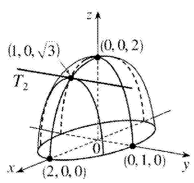




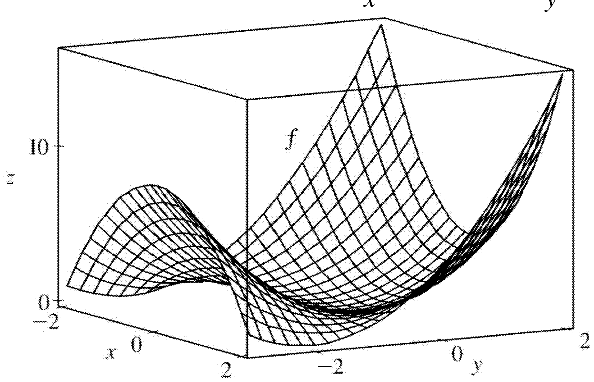
10. $f(x,y)=(4-x^2-4y^2)^{1/2} \Rightarrow f_x(x,y)=-x(4-x^2-4y^2)^{-1/2}$ and $f_y(x,y)=-4y(4-x^2-4y^2)^{-1/2} \Rightarrow$

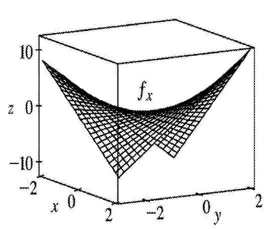
$f_x(1,0)=-\frac{1}{\sqrt{3}}$, $f_y(1,0)=0$. The graph of f is the upper half of the ellipsoid $z^2+x^2+4y^2=4$ and the plane $y=0$ intersects the graph in the semicircle $x^2+z^2=4$, $z\geq 0$ and the slope of the tangent line T_1 to this semicircle at $\left(1,0,\sqrt{3}\right)$ is $f_x(1,0)=-\frac{1}{\sqrt{3}}$. Similarly the plane $x=1$ intersects the graph in the semi–ellipse $z^2+4y^2=3$, $z\geq 0$ and the slope of the tangent line T_2 to this semi–ellipse at $\left(1,0,\sqrt{3}\right)$ is $f_y(1,0)=0$.

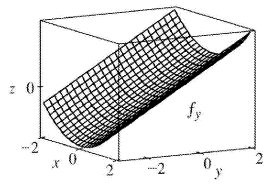




11. $f(x,y)=x^2+y^2+x^2y\Rightarrow f_x=2x+2xy$, $f_y=2y+x^2$

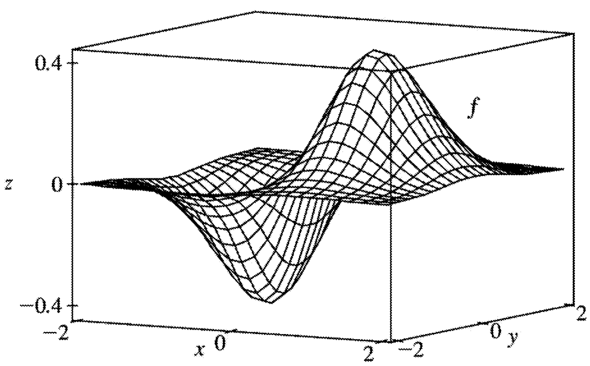


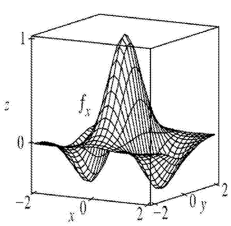


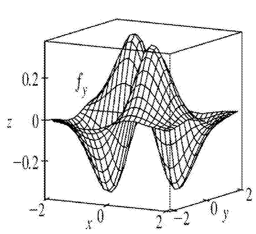


Note that the traces of f in planes parallel to the xz –plane are parabolas which open downward for $y<-1$ and
upward for $y>-1$, and the traces of f_x in these planes are straight lines, which have negative slopes for $y<-1$ and positive slopes for $y>-1$. The traces of f in planes parallel to the yz –plane are parabolas which always open upward, and the traces of f_y in these planes are straight lines with positive slopes.

12. $f(x,y)=xe^{-x^2-y^2} \Rightarrow f_x=x\left(-2xe^{-x^2-y^2}\right)+e^{-x^2-y^2}=e^{-x^2-y^2}(1-2x^2)$, $f_y=-2xye^{-x^2-y^2}$







Note that traces of f in planes parallel to the xz –plane have two extreme values, while traces of f_x in these planes have two zeros. Traces of f in planes parallel to the yz –plane have only one extreme value (a minimum if $x<0$, a maximum if $x>0$), and traces of f_y in these planes have only one zero (going from negative to positive if $x<0$ and from positive to negative if $x>0$).

13. $f(\mathrm{x,y})=3x-2y^4 \Rightarrow f_x(\mathrm{x,y})=3-0=3$, $f_y(\mathrm{x,y})=0-8y^3=-8y^3$

14. $f(\mathrm{x,y})=x^5+3x^3y^2+3xy^4 \Rightarrow f_x(\mathrm{x,y})=5x^4+3\cdot 3x^2\cdot y^2+3\cdot 1\cdot y^4=5x^4+9x^2y^2+3y^4$,
$f_y(\mathrm{x,y})=0+3x^3\cdot 2y+3x\cdot 4y^3=6x^3y+12xy^3$.

15. $z=xe^{3y} \Rightarrow \dfrac{\partial z}{\partial x}=e^{3y}$, $\dfrac{\partial z}{\partial y}=3xe^{3y}$

16. $z=y\ln x \Rightarrow \dfrac{\partial z}{\partial x}=\dfrac{y}{x}$, $\dfrac{\partial z}{\partial y}=\ln x$

17. $f(\mathrm{x,y})=\dfrac{x-y}{x+y} \Rightarrow f_x(\mathrm{x,y})=\dfrac{(1)(x+y)-(x-y)(1)}{(x+y)^2}=\dfrac{2y}{(x+y)^2}$, $f_y(\mathrm{x,y})=\dfrac{(-1)(x+y)-(x-y)(1)}{(x+y)^2}=-\dfrac{2x}{(x+y)^2}$

18. $f(\mathrm{x,y})=x^y \Rightarrow f_x(\mathrm{x,y})=yx^{y-1}$, $f_y(\mathrm{x,y})=x^y\ln x$

19. $w=\sin\alpha\cos\beta \Rightarrow \dfrac{\partial w}{\partial \alpha}=\cos\alpha\cos\beta$, $\dfrac{\partial w}{\partial \beta}=-\sin\alpha\sin\beta$

20. $f(\mathrm{s,t})=\dfrac{st^2}{s^2+t^2} \Rightarrow$
$f_s(\mathrm{s,t})=\dfrac{t^2\left(s^2+t^2\right)-st^2(2s)}{\left(s^2+t^2\right)^2}=\dfrac{t^4-s^2t^2}{\left(s^2+t^2\right)^2}$, $f_t(\mathrm{s,t})=\dfrac{2st\left(s^2+t^2\right)-st^2(2t)}{\left(s^2+t^2\right)^2}=\dfrac{2s^3t}{\left(s^2+t^2\right)^2}$

21. $f(\mathrm{r,s})=r\ln\left(r^2+s^2\right) \Rightarrow f_r(\mathrm{r,s})=r\cdot\dfrac{2r}{r^2+s^2}+\ln\left(r^2+s^2\right)\cdot 1=\dfrac{2r^2}{r^2+s^2}+\ln\left(r^2+s^2\right)$,
$f_s(\mathrm{r,s})=r\cdot\dfrac{2s}{r^2+s^2}+0=\dfrac{2rs}{r^2+s^2}$

22. $f(\mathrm{x,t})=\arctan\left(x\sqrt{t}\right) \Rightarrow f_x(\mathrm{x,t})=\dfrac{1}{1+\left(x\sqrt{t}\right)^2}\cdot\sqrt{t}=\dfrac{\sqrt{t}}{1+x^2t}$,
$f_t(\mathrm{x,t})=\dfrac{1}{1+\left(x\sqrt{t}\right)^2}\cdot x\left(\dfrac{1}{2}t^{-1/2}\right)=\dfrac{x}{2\sqrt{t}\left(1+x^2t\right)}$

23. $u=te^{w/t} \Rightarrow \dfrac{\partial u}{\partial t}=t\cdot e^{w/t}(-wt^{-2})+e^{w/t}\cdot 1=e^{w/t}-\dfrac{w}{t}e^{w/t}=e^{w/t}\left(1-\dfrac{w}{t}\right)$, $\dfrac{\partial u}{\partial w}=te^{w/t}\cdot\dfrac{1}{t}=e^{w/t}$

24. $f(\mathrm{x,y})=\int_y^x \cos(t^2)\,dt \Rightarrow f_x(\mathrm{x,y})=\dfrac{\partial}{\partial x}\int_y^x \cos\left(t^2\right)dt=\cos\left(x^2\right)$ by the Fundamental Theorem of Calculus, Part 1; $f_y(\mathrm{x,y})=\dfrac{\partial}{\partial y}\int_y^x \cos\left(t^2\right)dt=-\dfrac{\partial}{\partial y}\cos\left(t^2\right)dt=-\cos\left(y^2\right)$.

25. $f(\mathrm{x,y,z})=xy^2z^3+3yz \Rightarrow f_x(\mathrm{x,y,z})=y^2z^3$, $f_y(\mathrm{x,y,z})=2xyz^3+3z$, $f_z(\mathrm{x,y,z})=3xy^2z^2+3y$

26. $f(\mathrm{x,y,z})=x^2e^{yz} \Rightarrow f_x(\mathrm{x,y,z})=2xe^{yz}$, $f_y(\mathrm{x,y,z})=x^2e^{yz}(z)=x^2ze^{yz}$, $f_z(\mathrm{x,y,z})=x^2e^{yz}(y)=x^2ye^{yz}$

27. $w=\ln\left(x+2y+3z\right) \Rightarrow \dfrac{\partial w}{\partial x}=\dfrac{1}{x+2y+3z}$, $\dfrac{\partial w}{\partial y}=\dfrac{2}{x+2y+3z}$, $\dfrac{\partial w}{\partial z}=\dfrac{3}{x+2y+3z}$

28. $w=\sqrt{r^2+s^2+t^2} \Rightarrow \dfrac{\partial w}{\partial r}=\dfrac{1}{2}(r^2+s^2+t^2)^{-1/2}(2r)=\dfrac{r}{\sqrt{r^2+s^2+t^2}}$, $\dfrac{\partial w}{\partial s}=\dfrac{s}{\sqrt{r^2+s^2+t^2}}$, $\dfrac{\partial w}{\partial t}=\dfrac{t}{\sqrt{r^2+s^2+t^2}}$

29. $u=xe^{-t}\sin\theta \Rightarrow \dfrac{\partial u}{\partial x}=e^{-t}\sin\theta$, $\dfrac{\partial u}{\partial t}=-xe^{-t}\sin\theta$, $\dfrac{\partial u}{\partial \theta}=xe^{-t}\cos\theta$

30. $u=x^{y/z} \Rightarrow u_x=\dfrac{y}{z}x^{(y/z)-1}$, $u_y=x^{y/z}\ln x\cdot\dfrac{1}{z}=\dfrac{x^{y/z}}{z}\ln x$, $u_z=x^{y/z}\ln x\cdot\dfrac{-y}{z^2}=-\dfrac{yx^{y/z}}{z^2}\ln x$

31. $f(\mathrm{x,y,z,t})=xyz^2\tan(yt) \Rightarrow f_x(\mathrm{x,y,z,t})=yz^2\tan(yt)$,
$f_y(\mathrm{x,y,z,t})=xyz^2\cdot\sec^2(yt)\cdot t+xz^2\tan(yt)=xyz^2t\sec^2(yt)+xz^2\tan(yt)$,
$f_z(\mathrm{x,y,z,t})=2xyz\tan(yt)$, $f_t(\mathrm{x,y,z,t})=xyz^2\sec^2(yt)\cdot y=xy^2z^2\sec^2(yt)$.

32.

$f(x,y,z,t)=\dfrac{xy^2}{t+2z} \Rightarrow$

$f_x(x,y,z,t)=\dfrac{y^2}{t+2z}$, $f_y(x,y,z,t)=\dfrac{2xy}{t+2z}$,

$f_z(x,y,z,t)=xy^2(-1)(t+2z)^{-2}(2)=-\dfrac{2xy^2}{(t+2z)^2}$, $f_t(x,y,z,t)=xy^2(-1)(t+2z)^{-2}(1)=-\dfrac{xy^2}{(t+2z)^2}$.

33. $u=\sqrt{x_1^2+x_2^2+\cdot\cdot\cdot+x_n^2}$. For each $i=1$, … , n ,

$u_{x_i}=\dfrac{1}{2}\left(x_1^2+x_2^2+\cdot\cdot\cdot+x_n^2\right)^{-1/2}\left(2x_i\right)=\dfrac{x_i}{\sqrt{x_1^2+x_2^2+\cdot\cdot\cdot+x_n^2}}$.

34. $u=\sin\left(x_1+2x_2+\cdot\cdot\cdot+nx_n\right)$. For each $i=1$, … , n , $u_{x_i}=i\cos\left(x_1+2x_2+\cdot\cdot\cdot+nx_n\right)$.

35. $f(x,y)=\sqrt{x^2+y^2} \Rightarrow f_x(x,y)=\dfrac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x)=\dfrac{x}{\sqrt{x^2+y^2}}$, so $f_x(3,4)=\dfrac{3}{\sqrt{3^2+4^2}}=\dfrac{3}{5}$.

36. $f(x,y)=\sin(2x+3y) \Rightarrow f_y(x,y)=\cos(2x+3y)\cdot 3=3\cos(2x+3y)$, so
$f_y(-6,4)=3\cos\left[2(-6)+3(4)\right]=3\cos 0=3$.

37. $f(x,y,z)=\dfrac{x}{y+z}=x(y+z)^{-1} \Rightarrow f_z(x,y,z)=x(-1)(y+z)^{-2}=-\dfrac{x}{(y+z)^2}$, so $f_z(3,2,1)=-\dfrac{3}{(2+1)^2}=-\dfrac{1}{3}$.

38. $f(u,v,w)=w\tan(uv) \Rightarrow f_v(u,v,w)=w\sec^2(uv)\cdot u=uw\sec^2(uv)$, so $f_v(2,0,3)=(2)(3)\sec^2(2\cdot 0)=6$.

39. $f(x,y)=x^2-xy+2y^2 \Rightarrow$

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}=\lim_{h\to 0}\frac{(x+h)^2-(x+h)y+2y^2-\left(x^2-xy+2y^2\right)}{h} \\ &= \lim_{h\to 0}\frac{h(2x-y+h)}{h}=\lim_{h\to 0}(2x-y+h)=2x-y \end{aligned}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} = \lim_{h\to 0} \frac{x^2-x(y+h)+2(y+h)^2-\left(x^2-xy+2y^2\right)}{h}$$

$$= \lim_{h\to 0} \frac{h(4y-x+2h)}{h} = \lim_{h\to 0} (4y-x+2h) = 4y-x$$

40. $f(x,y)=\sqrt{3x-y} \Rightarrow$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3(x+h)-y}-\sqrt{3x-y}}{h} \cdot \frac{\sqrt{3(x+h)-y}+\sqrt{3x-y}}{\sqrt{3(x+h)-y}+\sqrt{3x-y}}$$

$$= \lim_{h\to 0} \frac{3}{\sqrt{3(x+h)-y}+\sqrt{3x-y}} = \frac{3}{2\sqrt{3x-y}}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3x-(y+h)}-\sqrt{3x-y}}{h} \cdot \frac{\sqrt{3x-(y+h)}+\sqrt{3x-y}}{\sqrt{3x-(y+h)}+\sqrt{3x-y}}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{3x-(y+h)}+\sqrt{3x-y}} = \frac{-1}{2\sqrt{3x-y}}$$

41. $x^2+y^2+z^2=3xyz \Rightarrow \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)=\frac{\partial}{\partial x}(3xyz) \Rightarrow 2x+0+2z\frac{\partial z}{\partial x}=3y\left(x\frac{\partial z}{\partial x}+z\cdot 1\right) \Leftrightarrow$
$2z\frac{\partial z}{\partial x}-3xy\frac{\partial z}{\partial x}=3yz-2x \Leftrightarrow (2z-3xy)\frac{\partial z}{\partial x}=3yz-2x$, so $\frac{\partial z}{\partial x}=\frac{3yz-2x}{2z-3xy}$.
$\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)=\frac{\partial}{\partial y}(3xyz) \Rightarrow 0+2y+2z\frac{\partial z}{\partial y}=3x\left(y\frac{\partial z}{\partial y}+z\cdot 1\right) \Leftrightarrow 2z\frac{\partial z}{\partial y}-3xy\frac{\partial z}{\partial y}=3xz-2y \Leftrightarrow$
$(2z-3xy)\frac{\partial z}{\partial y}=3xz-2y$, so $\frac{\partial z}{\partial y}=\frac{3xz-2y}{2z-3xy}$.

42. $yz=\ln(x+z) \Rightarrow \frac{\partial}{\partial x}(yz)=\frac{\partial}{\partial x}(\ln(x+z)) \Rightarrow y\frac{\partial z}{\partial x}=\frac{1}{x+z}\left(1+\frac{\partial z}{\partial x}\right) \Leftrightarrow \left(y-\frac{1}{x+z}\right)\frac{\partial z}{\partial x}=\frac{1}{x+z}$, so
$\frac{\partial z}{\partial x}=\frac{1/(x+z)}{y-1/(x+z)}=\frac{1}{y(x+z)-1}$.
$\frac{\partial}{\partial y}(yz)=\frac{\partial}{\partial y}(\ln(x+z)) \Rightarrow y\frac{\partial z}{\partial y}+z\cdot 1=\frac{1}{x+z}\left(0+\frac{\partial z}{\partial y}\right) \Leftrightarrow \left(y-\frac{1}{x+z}\right)\frac{\partial z}{\partial y}=-z$, so
$\frac{\partial z}{\partial y}=\frac{-z}{y-1/(x+z)}=\frac{z(x+z)}{1-y(x+z)}$.

43. $x-z=\arctan(yz) \Rightarrow \frac{\partial}{\partial x}(x-z)=\frac{\partial}{\partial x}(\arctan(yz)) \Rightarrow 1-\frac{\partial z}{\partial x}=\frac{1}{1+(yz)^2}\cdot y\frac{\partial z}{\partial x} \Leftrightarrow 1=\left(\frac{y}{1+y^2z^2}+1\right)\frac{\partial z}{\partial x} \Leftrightarrow$
$1=\left(\frac{y+1+y^2z^2}{1+y^2z^2}\right)\frac{\partial z}{\partial x}$, so $\frac{\partial z}{\partial x}=\frac{1+y^2z^2}{1+y+y^2z^2}$.
$\frac{\partial}{\partial y}(x-z)=\frac{\partial}{\partial y}(\arctan(yz)) \Rightarrow 0-\frac{\partial z}{\partial y}=\frac{1}{1+(yz)^2}\cdot\left(y\frac{\partial z}{\partial y}+z\cdot 1\right) \Leftrightarrow -\frac{z}{1+y^2z^2}=\left(\frac{y}{1+y^2z^2}+1\right)\frac{\partial z}{\partial y} \Leftrightarrow$
$-\frac{z}{1+y^2z^2}=\left(\frac{y+1+y^2z^2}{1+y^2z^2}\right)\frac{\partial z}{\partial y} \Leftrightarrow \frac{\partial z}{\partial y}=-\frac{z}{1+y+y^2z^2}$.
44. $\sin(xyz)=x+2y+3z \Rightarrow \frac{\partial}{\partial x}(\sin(xyz))=\frac{\partial}{\partial x}(x+2y+3z) \Rightarrow \cos(xyz)\cdot y\left(x\frac{\partial z}{\partial x}+z\right)=1+3\frac{\partial z}{\partial x} \Leftrightarrow ($
$xy\cos(xyz)-3)\frac{\partial z}{\partial x}=1-yz\cos(xyz)$, so $\frac{\partial z}{\partial x}=\frac{1-yz\cos(xyz)}{xy\cos(xyz)-3}$.
$\frac{\partial}{\partial y}(\sin(xyz))=\frac{\partial}{\partial y}(x+2y+3z) \Rightarrow \cos(xyz)\cdot x\left(y\frac{\partial z}{\partial y}+z\right)=2+3\frac{\partial z}{\partial y} \Leftrightarrow (xy\cos(xyz)-3)\frac{\partial z}{\partial y}=2-xz\cos(xyz)$
, so $\frac{\partial z}{\partial y}=\frac{2-xz\cos(xyz)}{xy\cos(xyz)-3}$.

45. **(a)** $z=f(x)+g(y) \Rightarrow \frac{\partial z}{\partial x}=f'(x)$, $\frac{\partial z}{\partial y}=g'(y)$
(b) $z=f(x+y)$. Let $u=x+y$. Then $\frac{\partial z}{\partial x}=\frac{df}{du}\frac{\partial u}{\partial x}=\frac{df}{du}(1)=f'(u)=f'(x+y)$,
$\frac{\partial z}{\partial y}=\frac{df}{du}\frac{\partial u}{\partial y}=\frac{df}{du}(1)=f'(u)=f'(x+y)$.

46. **(a)** $z=f(x)g(y) \Rightarrow \frac{\partial z}{\partial x}=f'(x)g(y)$, $\frac{\partial z}{\partial y}=f(x)g'(y)$
(b) $z=f(xy)$. Let $u=xy$. Then $\frac{\partial u}{\partial x}=y$ and $\frac{\partial u}{\partial y}=x$. Hence $\frac{\partial z}{\partial x}=\frac{df}{du}\frac{\partial u}{\partial x}=\frac{df}{du}\cdot y=yf'(u)=yf'(xy)$ and
$\frac{\partial z}{\partial y}=\frac{df}{du}\frac{\partial u}{\partial y}=\frac{df}{du}\cdot x=xf'(u)=xf'(xy)$.
(c) $z=f\left(\frac{x}{y}\right)$. Let $u=\frac{x}{y}$. Then $\frac{\partial u}{\partial x}=\frac{1}{y}$ and $\frac{\partial u}{\partial y}=-\frac{x}{y^2}$. Hence $\frac{\partial z}{\partial x}=\frac{df}{du}\frac{\partial u}{\partial x}=f'(u)\frac{1}{y}=\frac{f'(x/y)}{y}$
and $\frac{\partial z}{\partial y}=\frac{df}{du}\frac{\partial u}{\partial y}=f'(u)\left(-\frac{x}{y^2}\right)=-\frac{xf'(x/y)}{y^2}$.
47. $f(x,y)=x^4-3x^2y^3 \Rightarrow f_x(x,y)=4x^3-6xy^3$, $f_y(x,y)=-9x^2y^2$. Then $f_{xx}(x,y)=12x^2-6y^3$,

$f_{xy}(x,y)=-18xy^2$, $f_{yx}(x,y)=-18xy^2$, and $f_{yy}(x,y)=-18x^2y$.

48. $f(x,y)=\ln(3x+5y) \Rightarrow f_x(x,y)=\dfrac{3}{3x+5y}$, $f_y(x,y)=\dfrac{5}{3x+5y}$. Then
$f_{xx}(x,y)=3(-1)(3x+5y)^{-2}(3)=-\dfrac{9}{(3x+5y)^2}$, $f_{xy}(x,y)=-\dfrac{15}{(3x+5y)^2}$, $f_{yx}(x,y)=-\dfrac{15}{(3x+5y)^2}$, and
$f_{yy}(x,y)=-\dfrac{25}{(3x+5y)^2}$.

49. $z=\dfrac{x}{x+y}=x(x+y)^{-1} \Rightarrow z_x=\dfrac{1(x+y)-1(x)}{(x+y)^2}=\dfrac{y}{(x+y)^2}$, $z_y=x(-1)(x+y)^{-2}=-\dfrac{x}{(x+y)^2}$. Then
$z_{xx}=y(-2)(x+y)^{-3}=-\dfrac{2y}{(x+y)^3}$, $z_{xy}=\dfrac{1(x+y)^2-y(2)(x+y)}{\left[(x+y)^2\right]^2}=\dfrac{x+y-2y}{(x+y)^3}=\dfrac{x-y}{(x+y)^3}$,
$z_{yx}=-\dfrac{1(x+y)^2-x(2)(x+y)}{\left[(x+y)^2\right]^2}=-\dfrac{-x^2+xy+y^2}{(x+y)^2}=\dfrac{(x+y)(x-y)}{(x+y)^2}=\dfrac{x-y}{(x+y)^3}$, and $z_{yy}=-x(-2)(x+y)^{-3}=\dfrac{2x}{(x+y)^3}$.

50. $z=y\tan 2x \Rightarrow z_x=y\sec^2(2x)\cdot 2=2y\sec^2(2x)$, $z_y=\tan 2x$. Then
$z_{xx}=2y(2)\sec(2x)\cdot\sec(2x)\tan(2x)\cdot 2=8y\sec^2(2x)\tan(2x)$, $z_{xy}=2\sec^2(2x)$,
$z_{yx}=\sec^2(2x)\cdot 2=2\sec^2(2x)$, and $z_{yy}=0$.

51. $u=e^{-s}\sin t \Rightarrow u_s=-e^{-s}\sin t$, $u_t=e^{-s}\cos t$. Then $u_{ss}=e^{-s}\sin t$, $u_{st}=-e^{-s}\cos t$, $u_{ts}=-e^{-s}\cos t$, and
$u_{tt}=-e^{-s}\sin t$.

52. $v=\sqrt{x+y^2} \Rightarrow v_x=\dfrac{1}{2}\left(x+y^2\right)^{-1/2}=\dfrac{1}{2\sqrt{x+y^2}}$, $v_y=\dfrac{1}{2}\left(x+y^2\right)^{-1/2}(2y)=\dfrac{y}{\sqrt{x+y^2}}$. Then
$v_{xx}=\dfrac{1}{2}\left(-\dfrac{1}{2}\right)\left(x+y^2\right)^{-3/2}=-\dfrac{1}{4\left(x+y^2\right)^{3/2}}$, $v_{xy}=\dfrac{1}{2}\left(-\dfrac{1}{2}\right)\left(x+y^2\right)^{-3/2}(2y)=-\dfrac{y}{2\left(x+y^2\right)^{3/2}}$,
$v_{yx}=y\left(-\dfrac{1}{2}\right)\left(x+y^2\right)^{-3/2}=-\dfrac{y}{2\left(x+y^2\right)^{3/2}}$, and

$$v_{yy}=\frac{1\sqrt{x+y^2}-y\left(\frac{1}{2}\right)\left(x+y^2\right)^{-1/2}(2y)}{\left(\sqrt{x+y^2}\right)^2}=\frac{\left(x+y^2\right)-y^2}{\left(x+y^2\right)^{3/2}}=\frac{x}{\left(x+y^2\right)^{3/2}}\ .$$

53. $u=x\sin(x+2y)\Rightarrow u_x=x\cdot\cos(x+2y)(1)+\sin(x+2y)\cdot 1=x\cos(x+2y)+\sin(x+2y)$, $u_{xy}=x(-\sin(x+2y)(2))+\cos(x+2y)(2)=2\cos(x+2y)-2x\sin(x+2y)$ and $u_y=x\cos(x+2y)(2)=2x\cos(x+2y)$, $u_{yx}=2x\cdot(-\sin(x+2y)(1))+\cos(x+2y)\cdot 2=2\cos(x+2y)-2x\sin(x+2y)$. Thus $u_{xy}=u_{yx}$.

54. $u=x^4y^2-2xy^5\Rightarrow u_x=4x^3y^2-2y^5$, $u_{xy}=8x^3y-10y^4$ and $u_y=2x^4y-10xy^4$, $u_{yx}=8x^3y-10y^4$. Thus $u_{xy}=u_{yx}$.

55. $u=\ln\sqrt{x^2+y^2}=\ln\left(x^2+y^2\right)^{1/2}=\frac{1}{2}\ln\left(x^2+y^2\right)\Rightarrow u_x=\frac{1}{2}\ \frac{1}{x^2+y^2}\cdot 2x=\frac{x}{x^2+y^2}$,
$u_{xy}=x(-1)\left(x^2+y^2\right)^{-2}(2y)=-\frac{2xy}{\left(x^2+y^2\right)^2}$ and $u_y=\frac{1}{2}\ \frac{1}{x^2+y^2}\cdot 2y=\frac{y}{x^2+y^2}$,
$u_{yx}=y(-1)\left(x^2+y^2\right)^{-2}(2x)=-\frac{2xy}{\left(x^2+y^2\right)^2}$. Thus $u_{xy}=u_{yx}$.

56. $u=xye^y\Rightarrow u_x=ye^y$, $u_{xy}=ye^y+e^y=(y+1)e^y$ and $u_y=x\left(ye^y+e^y\right)=x(y+1)e^y$, $u_{yx}=(y+1)e^y$. Thus $u_{xy}=u_{yx}$.

57. $f(\text{x,y})=3xy^4+x^3y^2\Rightarrow f_x=3y^4+3x^2y^2$, $f_{xx}=6xy^2$, $f_{xxy}=12xy$ and $f_y=12xy^3+2x^3y$, $f_{yy}=36xy^2+2x^3$, $f_{yyy}=72xy$.

58. $f(\text{x,t})=x^2e^{-ct}\Rightarrow f_t=x^2(-ce^{-ct})$, $f_{tt}=x^2(c^2e^{-ct})$, $f_{ttt}=x^2(-c^3e^{-ct})=-c^3x^2e^{-ct}$ and $f_{tx}=2x(-ce^{-ct})$, $f_{txx}=2(-ce^{-ct})=-2ce^{-ct}$.

59. $f(\text{x,y,z})=\cos(4x+3y+2z)\Rightarrow$
$f_x=-\sin(4x+3y+2z)(4)=-4\sin(4x+3y+2z)$,
$f_{xy}=-4\cos(4x+3y+2z)(3)=-12\cos(4x+3y+2z)$,
$f_{xyz}=-12(-\sin(4x+3y+2z))(2)=24\sin(4x+3y+2z)$ and $f_y=-\sin(4x+3y+2z)(3)=-3\sin(4x+3y+2z)$,
$f_{yz}=-3\cos(4x+3y+2z)(2)=-6\cos(4x+3y+2z)$,

$f_{yzz}=-6(-\sin(4x+3y+2z))(2)=12\sin(4x+3y+2z)$.

60. $f(r,s,t)=r\ln(rs^2t^3)\Rightarrow$

$f_r=r\cdot\dfrac{1}{rs^2t^3}(s^2t^3)+\ln(rs^2t^3)\cdot 1=\dfrac{rs^2t^3}{rs^2t^3}+\ln(rs^2t^3)=1+\ln(rs^2t^3)$,

$f_{rs}=\dfrac{1}{rs^2t^3}(2rst^3)=\dfrac{2}{s}=2s^{-1}$, $f_{rss}=-2s^{-2}=-\dfrac{2}{s^2}$ and $f_{rst}=0$.

61. $u=e^{r\theta}\sin\theta\Rightarrow\dfrac{\partial u}{\partial\theta}=e^{r\theta}\cos\theta+\sin\theta\cdot e^{r\theta}(r)=e^{r\theta}(\cos\theta+r\sin\theta)$,

$\dfrac{\partial^2 u}{\partial r\partial\theta}=e^{r\theta}(\sin\theta)+(\cos\theta+r\sin\theta)e^{r\theta}(\theta)=e^{r\theta}(\sin\theta+\theta\cos\theta+r\theta\sin\theta)$,

$\dfrac{\partial^3 u}{\partial r^2\partial\theta}=e^{r\theta}(\theta\sin\theta)+(\sin\theta+\theta\cos\theta+r\theta\sin\theta)\cdot e^{r\theta}(\theta)=\theta e^{r\theta}(2\sin\theta+\theta\cos\theta+r\theta\sin\theta)$.

62. $z=u\sqrt{v-w}=u(v-w)^{1/2}\Rightarrow\dfrac{\partial z}{\partial w}=u\left[\dfrac{1}{2}(v-w)^{-1/2}(-1)\right]=-\dfrac{1}{2}u(v-w)^{-1/2}$,

$\dfrac{\partial^2 z}{\partial v\partial w}=-\dfrac{1}{2}u\left(-\dfrac{1}{2}(v-w)^{-3/2}(1)\right)=\dfrac{1}{4}u(v-w)^{-3/2}$, $\dfrac{\partial^3 z}{\partial u\partial v\partial w}=\dfrac{1}{4}(v-w)^{-3/2}$.

63. $w=\dfrac{x}{y+2z}=x(y+2z)^{-1}\Rightarrow\dfrac{\partial w}{\partial x}=(y+2z)^{-1}$, $\dfrac{\partial^2 w}{\partial y\partial x}=-(y+2z)^{-2}(1)=-(y+2z)^{-2}$,

$\dfrac{\partial^3 w}{\partial z\partial y\partial x}=-(-2)(y+2z)^{-3}(2)=4(y+2z)^{-3}=\dfrac{4}{(y+2z)^3}$ and $\dfrac{\partial w}{\partial y}=x(-1)(y+2z)^{-2}(1)=-x(y+2z)^{-2}$,

$\dfrac{\partial^2 w}{\partial x\partial y}=-(y+2z)^{-2}$, $\dfrac{\partial^3 w}{\partial x^2\partial y}=0$.

64. $u=x^ay^bz^c$. If $a=0$, or if $b=0$ or 1 , or if $c=0$, 1 , or 2 , then $\dfrac{\partial^6 u}{\partial x\partial y^2\partial z^3}=0$. Otherwise

$\dfrac{\partial u}{\partial z}=cx^ay^bz^{c-1}$, $\dfrac{\partial^2 u}{\partial z^2}=c(c-1)x^ay^bz^{c-2}$, $\dfrac{\partial^3 u}{\partial z^3}=c(c-1)(c-2)x^ay^bz^{c-3}$, $\dfrac{\partial^4 u}{\partial y\partial z^3}=bc(c-1)(c-2)x^ay^{b-1}z^{c-3}$,

$\dfrac{\partial^5 u}{\partial y^2 \partial z^3}=b(b-1)c(c-1)(c-2)x^a y^{b-2} z^{c-3}$, and $\dfrac{\partial^6 u}{\partial x \partial y^2 \partial z^3}=ab(b-1)c(c-1)(c-2)x^{a-1} y^{b-2} z^{c-3}$.

65. By Definition 4, $f_x(3,2)=\lim_{h\to 0} \dfrac{f(3+\text{h},2)-f(3,2)}{h}$ which we can approximate by considering $h=0.5$
and $h=-0.5$: $f_x(3,2)\approx \dfrac{f(3.5,2)-f(3,2)}{0.5}=\dfrac{22.4-17.5}{0.5}=9.8$,
$f_x(3,2)\approx \dfrac{f(2.5,2)-f(3,2)}{-0.5}=\dfrac{10.2-17.5}{-0.5}=14.6$. Averaging these values, we estimate $f_x(3,2)$ to be approximately 12.2 . Similarly, $f_x(3,2.2)=\lim_{h\to 0} \dfrac{f(3+\text{h},2.2)-f(3,2.2)}{h}$ which we can approximate by considering $h=0.5$ and $h=-0.5$: $f_x(3,2.2)\approx \dfrac{f(3.5,2.2)-f(3,2.2)}{0.5}=\dfrac{26.1-15.9}{0.5}=20.4$,
$f_x(3,2.2)\approx \dfrac{f(2.5,2.2)-f(3,2.2)}{-0.5}=\dfrac{9.3-15.9}{-0.5}=13.2$. Averaging these values, we have $f_x(3,2.2)\approx 16.8$.
To estimate $f_{xy}(3,2)$, we first need an estimate for $f_x(3,1.8)$:
$f_x(3,1.8)\approx \dfrac{f(3.5,1.8)-f(3,1.8)}{0.5}=\dfrac{20.0-18.1}{0.5}=3.8$,
$f_x(3,1.8)\approx \dfrac{f(2.5,1.8)-f(3,1.8)}{-0.5}=\dfrac{12.5-18.1}{-0.5}=11.2$. Averaging these values, we get $f_x(3,1.8)\approx 7.5$.
Now $f_{xy}(\text{x,y})=\dfrac{\partial}{\partial y}\left[f_x(\text{x,y})\right]$ and $f_x(\text{x,y})$ is itself a function of 2 variables, so Definition 4 says that
$f_{xy}(\text{x,y})=\dfrac{\partial}{\partial y}\left[f_x(\text{x,y})\right]=\lim_{h\to 0} \dfrac{f_x(\text{x,y}+h)-f_x(\text{x,y})}{h} \Rightarrow f_{xy}(3,2)=\lim_{h\to 0} \dfrac{f_x(3,2+h)-f_x(3,2)}{h}$.
We can estimate this value using our previous work with $h=0.2$ and $h=-0.2$:
$f_{xy}(3,2)\approx \dfrac{f_x(3,2.2)-f_x(3,2)}{0.2}=\dfrac{16.8-12.2}{0.2}=23$, $f_{xy}(3,2)\approx \dfrac{f_x(3,1.8)-f_x(3,2)}{-0.2}=\dfrac{7.5-12.2}{-0.2}=23.5$.
Averaging these values, we estimate $f_{xy}(3,2)$ to be approximately 23.25 .

66. **(a)** If we fix y and allow x to vary, the level curves indicate that the value of f decreases as we move through P in the positive x –direction, so f_x is negative at P .

(b) If we fix x and allow y to vary, the level curves indicate that the value of f increases as we move through P in the positive y –direction, so f_y is positive at P .

(c) $f_{xx}=\frac{\partial}{\partial x}\left(f_x\right)$, so if we fix y and allow x to vary, f_{xx} is the rate of change of f_x as x increases. Note that at points to the right of P the level curves are spaced farther apart (in the x –direction) than at points to the left of P , demonstrating that f decreases less quickly with respect to x to the right of P. So as we move through P in the positive x –direction the (negative) value of f_x increases, hence $\frac{\partial}{\partial x}\left(f_x\right)=f_{xx}$ is positive at P .

(d) $f_{xy}=\frac{\partial}{\partial y}\left(f_x\right)$, so if we fix x and allow y to vary, f_{xy} is the rate of change of f_x as y increases. The level curves are closer together (in the x –direction) at points above P than at those below P , demonstrating that f decreases more quickly with respect to x for y –values above P . So as we move through P in the positive y –direction, the (negative) value of f_x decreases, hence f_{xy} is negative.

(e) $f_{yy}=\frac{\partial}{\partial y}\left(f_y\right)$, so if we fix x and allow y to vary, f_{yy} is the rate of change of f_y as y increases. The level curves are closer together (in the y –direction) at points above P than at those below P , demonstrating that f increases more quickly with respect to y above P . So as we move through P in the positive y –direction the (positive) value of f_y increases, hence $\frac{\partial}{\partial y}\left(f_y\right)=f_{yy}$ is positive at P .

67. $u=e^{-\alpha^2k^2t}\sin kx\Rightarrow u_x=ke^{-\alpha^2k^2t}\cos kx$, $u_{xx}=-k^2e^{-\alpha^2k^2t}\sin kx$, and $u_t=-\alpha^2k^2e^{-\alpha^2k^2t}\sin kx$. Thus $\alpha^2u_{xx}=u_t$.

68. **(a)** $u=x^2+y^2\Rightarrow u_x=2x$, $u_{xx}=2$; $u_y=2y$, $u_{yy}=2$. Thus $u_{xx}+u_{yy}\neq 0$ and $u=x^2+y^2$ does not satisfy Laplace's Equation.

(b) $u=x^2-y^2$ is a solution: $u_{xx}=2$, $u_{yy}=-2$ so $u_{xx}+u_{yy}=0$.

(c) $u=x^3+3xy^2$ is not a solution: $u_x=3x^2+3y^2$, $u_{xx}=6x$; $u_y=6xy$, $u_{yy}=6x$.

(d) $u=\ln\sqrt{x^2+y^2}$ is a solution: $u_x=\frac{1}{\sqrt{x^2+y^2}}\left(\frac{1}{2}\right)\left(x^2+y^2\right)^{-1/2}(2x)=\frac{x}{x^2+y^2}$,

$u_{xx}=\frac{\left(x^2+y^2\right)-x(2x)}{\left(x^2+y^2\right)^2}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}$. By symmetry, $u_{yy}=\frac{x^2-y^2}{\left(x^2+y^2\right)^2}$, so $u_{xx}+u_{yy}=0$.

(e) $u=\sin x\cosh y+\cos x\sinh y$ is a solution: $u_x=\cos x\cosh y-\sin x\sinh y$, $u_{xx}=-\sin x\cosh y-\cos x\sinh y$, and $u_y=\sin x\sinh y+\cos x\cosh y$, $u_{yy}=\sin x\cosh y+\cos x\sinh y$.

(f) $u=e^{-x}\cos y-e^{-y}\cos x$ is a solution:

$u_x=-e^{-x}\cos y+e^{-y}\sin x$, $u_{xx}=e^{-x}\cos y+e^{-y}\cos x$, and $u_y=-e^{-x}\sin y+e^{-y}\cos x$, $u_{yy}=-e^{-x}\cos y-e^{-y}\cos x$.

69. $u=\dfrac{1}{\sqrt{x^2+y^2+z^2}} \Rightarrow u_x=\left(-\dfrac{1}{2}\right)\left(x^2+y^2+z^2\right)^{-3/2}(2x)=-x\left(x^2+y^2+z^2\right)^{-3/2}$ and

$u_{xx}=-\left(x^2+y^2+z^2\right)^{-3/2}-x\left(-\dfrac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2}(2x)=\dfrac{2x^2-y^2-z^2}{\left(x^2+y^2+z^2\right)^{5/2}}$.

By symmetry, $u_{yy}=\dfrac{2y^2-x^2-z^2}{\left(x^2+y^2+z^2\right)^{5/2}}$ and $u_{zz}=\dfrac{2z^2-x^2-y^2}{\left(x^2+y^2+z^2\right)^{5/2}}$. Thus

$u_{xx}+u_{yy}+u_{zz}=\dfrac{2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2}{\left(x^2+y^2+z^2\right)^{5/2}}=0$.

70. **(a)** $u=\sin(kx)\sin(akt)\Rightarrow u_t=ak\sin(kx)\cos(akt)$, $u_{tt}=-a^2k^2\sin(kx)\sin(akt)$, $u_x=k\cos(kx)\sin(akt)$,

$u_{xx}=-k^2\sin(kx)\sin(akt)$. Thus $u_{tt}=a^2u_{xx}$.

(b) $u=\dfrac{t}{a^2t^2-x^2} \Rightarrow u_t=\dfrac{\left(a^2t^2-x^2\right)-t\left(2a^2t\right)}{\left(a^2t^2-x^2\right)^2}=-\dfrac{a^2t^2+x^2}{\left(a^2t^2-x^2\right)^2}$,

$u_{tt}=\dfrac{-2a^2t\left(a^2t^2-x^2\right)^2+\left(a^2t^2-x^2\right)(2)\left(a^2t^2-x^2\right)\left(2a^2t\right)}{\left(a^2t^2-x^2\right)^4}=\dfrac{2a^4t^3+6a^2tx^2}{a^2t^2-x^2}$,

$u_x=t(-1)\left(a^2t^2-x^2\right)^{-2}(2x)=\dfrac{2tx}{\left(a^2t^2-x^2\right)^2}$,

$u_{xx}=\dfrac{2t\left(a^2t^2-x^2\right)^2-2tx(2)\left(a^2t^2-x^2\right)(-2x)}{\left(a^2t^2-x^2\right)^4}=\dfrac{2a^2t^3-2tx^2+8tx^2}{\left(a^2t^2-x^2\right)^3}=\dfrac{2a^2t^3+6tx^2}{\left(a^2t^2-x^2\right)^4}$.

Thus $u_{tt}=a^2u_{xx}$.

(c) $u=(x-at)^6+(x+at)^6\Rightarrow u_t=-6a(x-at)^5+6a(x+at)^5$, $u_{tt}=30a^2(x-at)^4+30a^2(x+at)^4$,

$u_x=6(x-at)^5+6(x+at)^5$, $u_{xx}=30(x-at)^4+30(x+at)^4$. Thus $u_{tt}=a^2u_{xx}$.

(d) $u=\sin(x-at)+\ln(x+at)\Rightarrow u_t=-a\cos(x-at)+\dfrac{a}{x+at}$, $u_{tt}=-a^2\sin(x-at)-\dfrac{a^2}{(x+at)^2}$,

$u_x=\cos(x-at)+\dfrac{1}{x+at}$,

$u_{xx}=-\sin(x-at)-\dfrac{1}{(x+at)^2}$. Thus $u_{tt}=a^2u_{xx}$.

71. Let $v=x+at$, $w=x-at$. Then $u_t=\dfrac{\partial[f(v)+g(w)]}{\partial t}=\dfrac{df(v)}{dv}\dfrac{\partial v}{\partial t}+\dfrac{dg(w)}{dw}\dfrac{\partial w}{\partial t}=af^{\prime}(v)-ag^{\prime}(w)$ and $u_{tt}=\dfrac{\partial[af^{\prime}(v)-ag^{\prime}(w)]}{\partial t}=a[af^{\prime\prime}(v)+ag^{\prime\prime}(w)]=a^2[f^{\prime\prime}(v)+g^{\prime\prime}(w)]$. Similarly, by using the Chain Rule we have $u_x=f^{\prime}(v)+g^{\prime}(w)$ and $u_{xx}=f^{\prime\prime}(v)+g^{\prime\prime}(w)$. Thus $u_{tt}=a^2u_{xx}$.

72. For each i , $i=1,\dots,n$, $\partial u/\partial x_i=a_ie^{a_1x_1+a_2x_2+\cdots+a_nx_n}$ and $\partial^2u/\partial x_i^2=a_i^2e^{a_1x_1+a_2x_2+\cdots+a_nx_n}$. Then $\dfrac{\partial^2u}{\partial x_1^2}+\dfrac{\partial^2u}{\partial x_2^2}+\cdots+\dfrac{\partial^2u}{\partial x_n^2}=\left(a_1^2+a_2^2+\cdots+a_n^2\right)e^{a_1x_1+a_2x_2+\cdots+a_nx_n}=e^{a_1x_1+a_2x_2+\cdots+a_nx_n}=u$ since $a_1^2+a_2^2+\cdots+a_n^2=1$.

73. $z_x=e^y+ye^x$, $z_{xx}=ye^x$, $\partial^3z/\partial x^3=ye^x$. By symmetry $z_y=xe^y+e^x$, $z_{yy}=xe^y$ and $\partial^3z/\partial y^3=xe^y$. Then $\partial^3z/\partial x\partial y^2=e^y$ and $\partial^3z/\partial x^2\partial y=e^x$. Thus $z=xe^y+ye^x$ satisfies the given partial differential equation.

74. $P=bL^{\alpha}K^{\beta}$, so $\dfrac{\partial P}{\partial L}=\alpha bL^{\alpha-1}K^{\beta}$ and $\dfrac{\partial P}{\partial K}=\beta bL^{\alpha}K^{\beta-1}$. Then

$$\begin{aligned}L\frac{\partial P}{\partial L}+K\frac{\partial P}{\partial K}&=L\left(\alpha bL^{\alpha-1}K^{\beta}\right)+K\left(\beta bL^{\alpha}K^{\beta-1}\right)=\alpha bL^{1+\alpha-1}K^{\beta}+\beta bL^{\alpha}K^{1+\beta-1}\\&=(\alpha+\beta)\,bL^{\alpha}K^{\beta}=(\alpha+\beta)\,P\end{aligned}$$

75. If we fix $K=K_0, P\left(L,K_0\right)$ is a function of a single variable L , and $\dfrac{dP}{dL}=\alpha\dfrac{P}{L}$ is a separable differential equation. Then $\dfrac{dP}{P}=\alpha\dfrac{dL}{L}\Rightarrow\int\dfrac{dP}{P}=\int\alpha\dfrac{dL}{L}\Rightarrow\ln|P|=\alpha\ln|L|+C\left(K_0\right)$, where $C\left(K_0\right)$ can depend on K_0 . Then $|P|=e^{\alpha\ln|L|+C\left(K_0\right)}$, and since $P>0$ and $L>0$, we have $P=e^{\alpha\ln L}e^{C\left(K_0\right)}=e^{C\left(K_0\right)}e^{\ln L^{\alpha}}=C_1\left(K_0\right)L^{\alpha}$ where $C_1\left(K_0\right)=e^{C\left(K_0\right)}$.

76. **(a)** $\partial T/\partial x=-60(2x)/\left(1+x^2+y^2\right)^2$, so at $(2,1)$, $T_x=-240/(1+4+1)^2=-\dfrac{20}{3}$.

(b) $\partial T/\partial y=-60(2y)/\left(1+x^2+y^2\right)^2$, so at $(2,1)$, $T_y=-120/36=-\dfrac{10}{3}$. Thus from the point $(2,1)$ the temperature is decreasing at a rate of $\dfrac{20}{3}$ $^\circ$ F/ m in the x –direction and is decreasing at a rate of $\dfrac{10}{3}$ $^\circ$ F/ m in the y –direction.

77. By the Chain Rule, taking the partial derivative of both sides with respect to R_1 gives

$$\frac{\partial R^{-1}}{\partial R}\frac{\partial R}{\partial R_1}=\frac{\partial\left[\left(1/R_1\right)+\left(1/R_2\right)+\left(1/R_3\right)\right]}{\partial R_1}$$ or $-R^{-2}\dfrac{\partial R}{\partial R_1}=-R_1^{-2}$. Thus $\dfrac{\partial R}{\partial R_1}=\dfrac{R^2}{R_1^2}$.

78. $P=\dfrac{mRT}{V}$ so $\dfrac{\partial P}{\partial V}=\dfrac{-mRT}{V^2}$; $V=\dfrac{mRT}{P}$, so $\dfrac{\partial V}{\partial T}=\dfrac{mR}{P}$; $T=\dfrac{PV}{mR}$, so $\dfrac{\partial T}{\partial P}=\dfrac{V}{mR}$.

Thus $\dfrac{\partial P}{\partial V}\dfrac{\partial V}{\partial T}\dfrac{\partial T}{\partial P}=\dfrac{-mRT}{V^2}\dfrac{mR}{P}\dfrac{V}{mR}=\dfrac{-mRT}{PV}=-1$, since $PV=mRT$.

79. By Exercise 78, $PV=mRT\Rightarrow P=\dfrac{mRT}{V}$, so $\dfrac{\partial P}{\partial T}=\dfrac{mR}{V}$. Also, $PV=mRT\Rightarrow V=\dfrac{mRT}{P}$ and $\dfrac{\partial V}{\partial T}=\dfrac{mR}{P}$. Since $T=\dfrac{PV}{mR}$, we have $T\dfrac{\partial P}{\partial T}\dfrac{\partial V}{\partial T}=\dfrac{PV}{mR}\cdot\dfrac{mR}{V}\cdot\dfrac{mR}{P}=mR$.

80. $\dfrac{\partial W}{\partial T}=0.6215+0.3965v^{0.16}$. When $T=-15^\circ$ F and $v=30$ km / h,

$\dfrac{\partial W}{\partial T}=0.6215+0.3965(30)^{0.16}\approx 1.3048$, so we would expect the apparent temperature to drop by approximately 1.3° F if the actual temperature decreases by 1° F .

$\dfrac{\partial W}{\partial v}=-11.37(0.16)v^{-0.84}+0.3965T(0.16)v^{-0.84}$ and when $T=-15^\circ$ F and $v=30$ km / h,

$\dfrac{\partial W}{\partial v}=-11.37(0.16)(30)^{-0.84}+0.3965(-15)(0.16)(30)^{-0.84}\approx-0.1592$, so we would expect the apparent temperature to drop by approximately 0.16° F if the wind speed increases by 1 km / h.

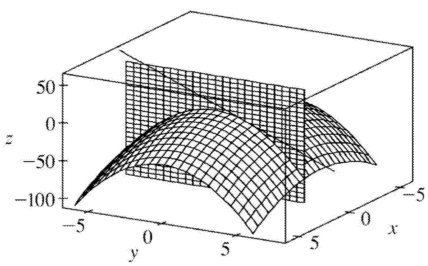
81. $\dfrac{\partial K}{\partial m}=\dfrac{1}{2}V^2$, $\dfrac{\partial K}{\partial V}=mV$, $\dfrac{\partial^2 K}{\partial V^2}=m$. Thus $\dfrac{\partial K}{\partial m}\cdot\dfrac{\partial^2 K}{\partial V^2}=\dfrac{1}{2}V^2m=K$.

82. The Law of Cosines says that $a^2=b^2+c^2-2bc\cos A$. Thus

$\frac{\partial\left(a^2\right)}{\partial a}=\frac{\partial\left(b^2+c^2-2ab\cos A\right)}{\partial a}$ or $2a=-2bc(-\sin A)\,\frac{\partial A}{\partial a}$, implying that $\frac{\partial A}{\partial a}=\frac{a}{bc\sin A}$. Taking the partial derivative of both sides with respect to b gives $0=2b-2c(\cos A)-2bc(-\sin A)\,\frac{\partial A}{\partial b}$. Thus $\frac{\partial A}{\partial b}=\frac{c\cos A-b}{bc\sin A}$. By symmetry $\frac{\partial A}{\partial c}=\frac{b\cos A-c}{bc\sin A}$.

83. $f_x(x,y)=x+4y\Rightarrow f_{xy}(x,y)=4$ and $f_y(x,y)=3x-y\Rightarrow f_{yx}(x,y)=3$. Since f_{xy} and f_{yx} are continuous everywhere but $f_{xy}(x,y)\neq f_{yx}(x,y)$, Clairaut's Theorem implies that such a function $f(x,y)$ does not exist.

84. Setting $x=1$, the equation of the parabola of intersection is $z=6-1-1-2y^2=4-2y^2$. The slope of the tangent is $\partial z/\partial y=-4y$, so at $(1,2,-4)$ the slope is -8 . Parametric equations for the line are therefore $x=1$, $y=2+t$, $z=-4-8t$.



85. By the geometry of partial derivatives, the slope of the tangent line is $f_x(1,2)$. By implicit differentiation of $4x^2+2y^2+z^2=16$, we get $8x+2z(\partial z/\partial x)=0\Rightarrow\partial z/\partial x=-4x/z$, so when $x=1$ and $z=2$ we have $\partial z/\partial x=-2$. So the slope is $f_x(1,2)=-2$. Thus the tangent line is given by $z-2=-2(x-1)$, $y=2$. Taking the parameter to be $t=x-1$, we can write parametric equations for this line: $x=1+t$, $y=2$, $z=2-2t$.

86. $T(x,t)=T_0+T_1e^{-\lambda x}\sin\,(\omega t-\lambda x)$

(a)

$$\begin{aligned}\partial T/\partial x &= T_1e^{-\lambda x}\left[\cos\,(\omega t-\lambda x)(-\lambda)\right]+T_1(-\lambda e^{-\lambda x})\sin\,(\omega t-\lambda x)\\ &= -\lambda T_1e^{-\lambda x}\left[\sin\,(\omega t-\lambda x)+\cos\,(\omega t-\lambda x)\right]\end{aligned}$$

This quantity represents the rate of change of temperature with respect to depth below the surface, at a given time t .

(b) $\partial T/\partial t = T_1 e^{-\lambda x}[\cos(\omega t-\lambda x)(\omega)] = \omega T_1 e^{-\lambda x}\cos(\omega t-\lambda x)$. This quantity represents the rate of change of temperature with respect to time at a fixed depth x .

(c)

$$T_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)$$

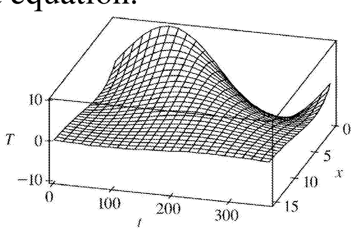
$$= -\lambda T_1 \left(e^{-\lambda x}[\cos(\omega t-\lambda x)(-\lambda) - \sin(\omega t-\lambda x)(-\lambda)]\right.$$

$$\left.\right) + e^{-\lambda x}(-\lambda)[\sin(\omega t-\lambda x)+\cos(\omega t-\lambda x)]$$

$$= 2\lambda^2 T_1 e^{-\lambda x}\cos(\omega t-\lambda x)$$

But from part (b), $T_t = \omega T_1 e^{-\lambda x}\cos(\omega t-\lambda x) = \dfrac{\omega}{2\lambda^2} T_{xx}$. So with $k = \dfrac{\omega}{2\lambda^2}$, the function T satisfies the heat equation.

(d)



Note that near the surface (that is, for small x) the temperature varies greatly as t changes, but deeper (for large x) the temperature is more stable.

(e) The term $-\lambda x$ is a phase shift: it represents the fact that since heat diffuses slowly through soil, it takes time for changes in the surface temperature to affect the temperature at deeper points. As x increases, the phase shift also increases. For example, at the surface the highest temperature is reached at $t \approx 100$, whereas at a depth of 5 feet the peak temperature is attained at $t \approx 150$, and at a depth of 10 feet, at $t \approx 220$.

87. By Clairaut's Theorem, $f_{xyy} = (f_{xy})_y = (f_{yx})_y = f_{yxy} = (f_y)_{xy} = (f_y)_{yx} = f_{yyx}$.

88. **(a)** Since we are differentiating n times, with two choices of variable at each differentiation, there are 2^n n th order partial derivatives.

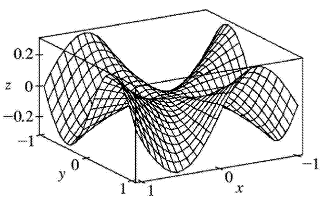
(b) If these partial derivatives are all continuous, then the order in which the partials are taken doesn't affect the value of the result, that is, all n th order partial derivatives with p partials with respect to x and $n-p$ partials with respect to y are equal. Since the number of partials taken with respect to x for an n th order partial derivative can range from 0 to n , a function of two variables has $n+1$ distinct partial derivatives of order n if these partial derivatives are all continuous.

(c) Since n differentiations are to be performed with three choices of variable at each differentiation, there are 3^n n th order partial derivatives of a function of three variables.

89. Let $g(x)=f(x,0)=x\left(x^2\right)^{-3/2}e^0=x|x|^{-3}$. But we are using the point $(1,0)$, so near $(1,0)$, $g(x)=x^{-2}$. Then $g'(x)=-2x^{-3}$ and $g'(1)=-2$, so using (1) we have $f_x(1,0)=g'(1)=-2$.

90. $f_x(0,0)=\lim\limits_{h\to 0}\dfrac{f(0+h,0)-f(0,0)}{h}=\lim\limits_{h\to 0}\dfrac{\left(h^3+0\right)^{1/3}-0}{h}=\lim\limits_{h\to 0}\dfrac{h}{h}=1$.

Or: Let $g(x)=f(x,0)=\sqrt[3]{x^3+0}=x$. Then $g'(x)=1$ and $g'(0)=1$ so, by (1), $f_x(0,0)=g'(0)=1$.

91. **(a)** 

(b) For $(x,y)\neq(0,0)$, $f_x(x,y)=\dfrac{\left(3x^2y-y^3\right)\left(x^2+y^2\right)-\left(x^3y-xy^3\right)(2x)}{\left(x^2+y^2\right)^2}=\dfrac{x^4y+4x^2y^3-y^5}{\left(x^2+y^2\right)^2}$, and by symmetry

$f_y(x,y)=\dfrac{x^5-4x^3y^2-xy^4}{\left(x^2+y^2\right)^2}$.

(c) $f_x(0,0)=\lim\limits_{h\to 0}\dfrac{f(h,0)-f(0,0)}{h}=\lim\limits_{h\to 0}\dfrac{\left(0/h^2\right)-0}{h}=0$ and $f_y(0,0)=\lim\limits_{h\to 0}\dfrac{f(0,h)-f(0,0)}{h}=0$.

(d) By (3), $f_{xy}(0,0)=\dfrac{\partial f_x}{\partial y}=\lim\limits_{h\to 0}\dfrac{f_x(0,h)-f_x(0,0)}{h}=\lim\limits_{h\to 0}\dfrac{\left(-h^5-0\right)/h^4}{h}=-1$ while by (2),

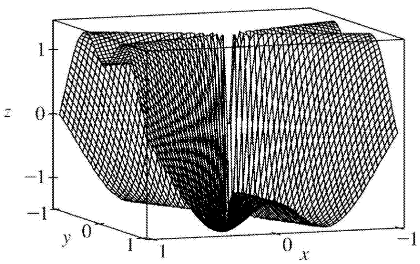
$f_{yx}(0,0)=\dfrac{\partial f_y}{\partial x}=\lim\limits_{h\to 0}\dfrac{f_y(h,0)-f_y(0,0)}{h}=\lim\limits_{h\to 0}\dfrac{h^5/h^4}{h}=1$.

(e) For $(x,y)\neq(0,0)$, we use a CAS to compute

$f_{xy}(x,y)=\dfrac{x^6+9x^4y^2-4x^2y^4+4y^6}{\left(x^2+y^2\right)^3}$.

Now as $(x,y)\to(0,0)$ along the x –axis, $f_{xy}(x,y)\to 1$ while as $(x,y)\to(0,0)$ along the y –axis,

$f_{xy}(x,y) \to 4$. Thus f_{xy} isn't continuous at (0,0) and Clairaut's Theorem doesn't apply, so there is no contradiction. The graphs of f_{xy} and f_{yx} are identical except at the origin, where we observe the discontinuity.



1. $z=f(x,y)=4x^2-y^2+2y \Rightarrow f_x(x,y)=8x$, $f_y(x,y)=-2y+2$, so $f_x(-1,2)=-8$, $f_y(-1,2)=-2$. By Equation 2, an equation of the tangent plane is $z-4=f_x(-1,2)[x-(-1)]+f_y(-1,2)(y-2) \Rightarrow z-4=-8(x+1)-2(y-2)$ or $z=-8x-2y$.

2. $z=f(x,y)=9x^2+y^2+6x-3y+5 \Rightarrow f_x(x,y)=18x+6$, $f_y(x,y)=2y-3$, so $f_x(1,2)=24$ and $f_y(1,2)=1$. By Equation 2, an equation of the tangent plane is $z-18=f_x(1,2)(x-1)+f_y(1,2)(y-2) \Rightarrow$ $z-18=24(x-1)+1(y-2)$ or $z=24x+y-8$.

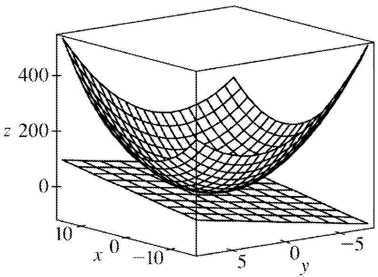
3. $z=f(x,y)=\sqrt{4-x^2-2y^2} \Rightarrow f_x(x,y)=\frac{1}{2}(4-x^2-2y^2)^{-1/2}(-2x)=-\frac{x}{\sqrt{4-x^2-2y^2}}$,
$f_y(x,y)=\frac{1}{2}(4-x^2-2y^2)^{-1/2}(-4y)=-\frac{2y}{\sqrt{4-x^2-2y^2}}$, so $f_x(1,-1)=-1$ and $f_y(1,-1)=2$. Thus, an equation of the tangent plane is $z-1=f_x(1,-1)(x-1)+f_y(1,-1)[y-(-1)] \Rightarrow z-1=-1(x-1)+2(y+1)$ or $x-2y+z=4$.

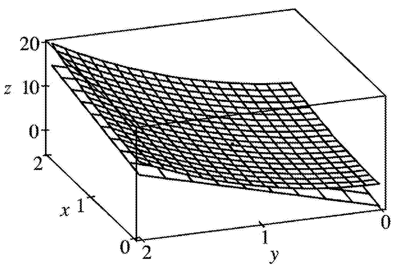
4. $z=f(x,y)=y\ln x \Rightarrow f_x(x,y)=y/x$, $f_y(x,y)=\ln x$, so $f_x(1,4)=4$, $f_y(1,4)=0$, and an equation of the tangent plane is $z-0=f_x(1,4)(x-1)+f_y(1,4)(y-4) \Rightarrow z=4(x-1)+0(y-4)$ or $z=4x-4$.

5. $z=f(x,y)=y\cos(x-y) \Rightarrow f_x=y(-\sin(x-y)(1))=-y\sin(x-y)$,
$f_y=y(-\sin(x-y)(-1))+\cos(x-y)=y\sin(x-y)+\cos(x-y)$, so $f_x(2,2)=-2\sin(0)=0$,
$f_y(2,2)=2\sin(0)+\cos(0)=1$ and an equation of the tangent plane is $z-2=0(x-2)+1(y-2)$ or $z=y$.

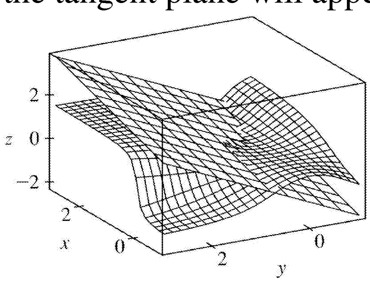
6. $z=f(x,y)=e^{x^2-y^2} \Rightarrow f_x(x,y)=2xe^{x^2-y^2}$, $f_y(x,y)=-2ye^{x^2-y^2}$, so $f_x(1,-1)=2$, $f_y(1,-1)=2$. By Equation 2, an equation of the tangent plane is $z-1=f_x(1,-1)(x-1)+f_y(1,-1)[y-(-1)] \Rightarrow z-1=2(x-1)+2(y+1)$ or $z=2x+2y+1$.

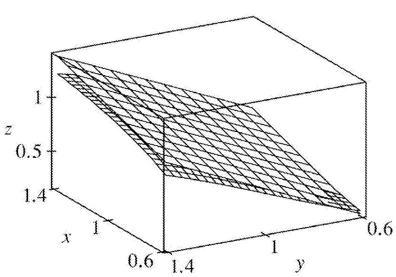
7. $z=f(x,y)=x^2+xy+3y^2$, so $f_x(x,y)=2x+y \Rightarrow f_x(1,1)=3$, $f_y(x,y)=x+6y \Rightarrow f_y(1,1)=7$ and an equation of the tangent plane is $z-5=3(x-1)+7(y-1)$ or $z=3x+7y-5$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is below the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.





8. $z=f(\text{x,y})=\arctan\left(xy^2\right)\Rightarrow f_x=\frac{1}{1+\left(xy^2\right)^2}(y^2)=\frac{y^2}{1+x^2y^4}$, $f_y=\frac{1}{1+\left(xy^2\right)^2}(2xy)=\frac{2xy}{1+x^2y^4}$,
$f_x(1{,}1)=\frac{1}{1+1}=\frac{1}{2}$, $f_y(1{,}1)=\frac{2}{1+1}=1$, so an equation of the tangent plane is $z-\frac{\pi}{4}=\frac{1}{2}(x-1)+1(y-1)$ or $z=\frac{1}{2}x+y-\frac{3}{2}+\frac{\pi}{4}$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here the tangent plane is above the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.

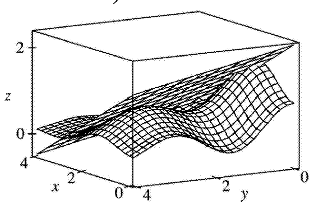


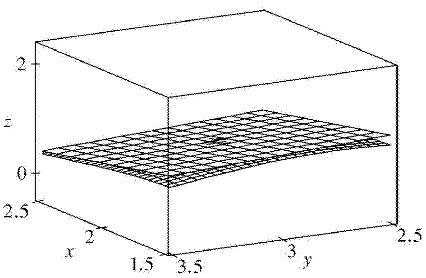


9. $f(\text{x,y})=e^{-\left(x^2+y^2\right)/15}\left(\sin^2 x+\cos^2 y\right)$. A CAS gives

$f_x = -\dfrac{2}{15} e^{-\left(x^2+y^2\right)/15}\left(x\sin^2 x + x\cos^2 y - 15\sin x\cos x\right)$ and

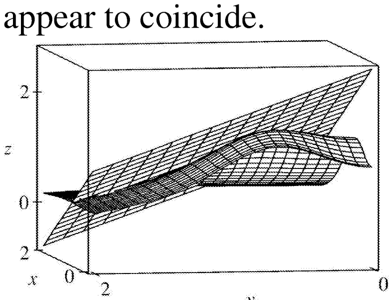
$f_y = -\dfrac{2}{15} e^{-\left(x^2+y^2\right)/15}\left(y\sin^2 x + y\cos^2 y + 15\sin y\cos y\right)$. We use the CAS to evaluate these at $(2,3)$, and then substitute the results into Equation 2 in order to plot the tangent plane. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is above the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.

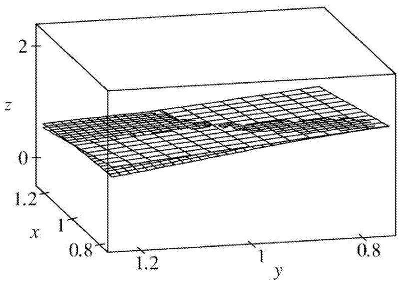




10. $f(x,y) = \dfrac{\sqrt{1+4x^2+4y^2}}{1+x^4+y^4}$. A CAS gives $f_x = \dfrac{4x\left(1-3x^4+y^4-x^2-4x^2y^2\right)}{\sqrt{1+4x^2+4y^2}\left(1+x^4+y^4\right)^2}$ and

$f_y = \dfrac{4y\left(1-3y^4+x^4-y^2-4x^2y^2\right)}{\sqrt{1+4x^2+4y^2}\left(1+x^4+y^4\right)^2}$. We use the CAS to evaluate these at $(1,1)$, and then substitute the results into Equation 2 to get an equation of the tangent plane: $z = \dfrac{25-8x-8y}{9}$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is shown with fewer traces than the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.





11. $f(x,y)=x\sqrt{y}$. The partial derivatives are $f_x(x,y)=\sqrt{y}$ and $f_y(x,y)=\dfrac{x}{2\sqrt{y}}$, so $f_x(1,4)=2$ and $f_y(1,4)=\dfrac{1}{4}$. Both f_x and f_y are continuous functions for $y>0$, so by Theorem 8, f is differentiable at $(1,4)$. By Equation 3, the linearization of f at $(1,4)$ is given by

$$L(x,y)=f(1,4)+f_x(1,4)(x-1)+f_y(1,4)(y-4)=2+2(x-1)+\frac{1}{4}(y-4)=2x+\frac{1}{4}y-1 .$$

12. $f(x,y)=\dfrac{x}{y}$. The partial derivatives are $f_x(x,y)=\dfrac{1}{y}$ and $f_y(x,y)=-\dfrac{x}{y^2}$, so $f_x(6,3)=\dfrac{1}{3}$ and $f_y(6,3)=-\dfrac{2}{3}$. Both f_x and f_y are continuous functions for $y\neq 0$, so f is differentiable at $(6,3)$ by Theorem 8. The linearization of f at $(6,3)$ is given by

$$L(x,y)=f(6,3)+f_x(6,3)(x-6)+f_y(6,3)(y-3)=2+\frac{1}{3}(x-6)-\frac{2}{3}(y-3)=\frac{1}{3}x-\frac{2}{3}y+2 .$$

13. $f(x,y)=e^x\cos xy$. The partial derivatives are $f_x(x,y)=e^x(\cos xy-y\sin xy)$ and $f_y(x,y)=-xe^x\sin xy$, so $f_x(0,0)=1$ and $f_y(0,0)=0$. Both f_x and f_y are continuous functions, so f is differentiable at $(0,0)$ by Theorem 8. The linearization of f at $(0,0)$ is given by

$$L(x,y)=f(0,0)+f_x(0,0)(x-0)+f_y(0,0)(y-0)=1+1(x-0)+0(y-0)=x+1 .$$

14. $f(x,y)=\sqrt{x+e^{4y}}=\left(x+e^{4y}\right)^{1/2}$. The partial derivatives are $f_x(x,y)=\dfrac{1}{2}\left(x+e^{4y}\right)^{-1/2}$ and $f_y(x,y)=\dfrac{1}{2}\left(x+e^{4y}\right)^{-1/2}(4e^{4y})=2e^{4y}\left(x+e^{4y}\right)^{-1/2}$, so $f_x(3,0)=\dfrac{1}{2}\left(3+e^0\right)^{-1/2}=\dfrac{1}{4}$ and $f_y(3,0)=2e^0\left(3+e^0\right)^{-1/2}=1$. Both f_x and f_y are continuous functions near $(3,0)$, so f is differentiable at $(3,0)$ by Theorem 8. The linearization of f at $(3,0)$ is

$$L(x,y)=f(3,0)+f_x(3,0)(x-3)+f_y(3,0)(y-0)=2+\frac{1}{4}(x-3)+1(y-0)=\frac{1}{4}x+y+\frac{5}{4} .$$

15. $f(x,y)=\tan^{-1}(x+2y)$. The partial derivatives are $f_x(x,y)=\dfrac{1}{1+(x+2y)^2}$ and $f_y(x,y)=\dfrac{2}{1+(x+2y)^2}$, so $f_x(1,0)=\dfrac{1}{2}$ and $f_y(1,0)=1$. Both f_x and f_y are continuous functions, so f is differentiable at $(1,0)$, and the linearization of f at $(1,0)$ is
$L(x,y)=f(1,0)+f_x(1,0)(x-1)+f_y(1,0)(y-0)=\dfrac{\pi}{4}+\dfrac{1}{2}(x-1)+1(y)=\dfrac{1}{2}x+y+\dfrac{\pi}{4}-\dfrac{1}{2}$.

16. $f(x,y)=\sin(2x+3y)$. The partial derivatives are $f_x(x,y)=2\cos(2x+3y)$ and $f_y(x,y)=3\cos(2x+3y)$, so $f_x(-3,2)=2$ and $f_y(-3,2)=3$. Both f_x and f_y are continuous functions, so f is differentiable at $(-3,2)$, and the linearization of f at $(-3,2)$ is
$L(x,y)=f(-3,2)+f_x(-3,2)(x+3)+f_y(-3,2)(y-2)=0+2(x+3)+3(y-2)=2x+3y$.

17. $f(x,y)=\sqrt{20-x^2-7y^2}\Rightarrow f_x(x,y)=-\dfrac{x}{\sqrt{20-x^2-7y^2}}$ and $f_y(x,y)=-\dfrac{7y}{\sqrt{20-x^2-7y^2}}$, so $f_x(2,1)=-\dfrac{2}{3}$ and $f_y(2,1)=-\dfrac{7}{3}$. Then the linear approximation of f at $(2,1)$ is given by

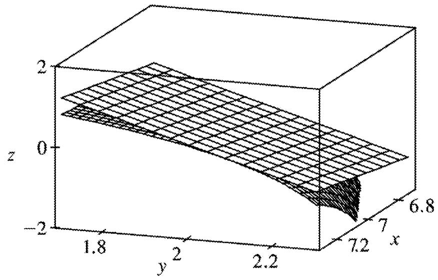
$$\begin{aligned} f(x,y) &\approx f(2,1)+f_x(2,1)(x-2)+f_y(2,1)(y-1)=3-\frac{2}{3}(x-2)-\frac{7}{3}(y-1) \\ &= -\frac{2}{3}x-\frac{7}{3}y+\frac{20}{3} \end{aligned}$$

Thus $f(1.95,1.08)\approx-\dfrac{2}{3}(1.95)-\dfrac{7}{3}(1.08)+\dfrac{20}{3}=2.84\overline{6}$.

18. $f(x,y)=\ln(x-3y)\Rightarrow f_x(x,y)=\dfrac{1}{x-3y}$ and $f_y(x,y)=-\dfrac{3}{x-3y}$, so $f_x(7,2)=1$ and $f_y(7,2)=-3$. Then the linear approximation of f at $(7,2)$ is given by

$$\begin{aligned} f(x,y) &\approx f(7,2)+f_x(7,2)(x-7)+f_y(7,2)(y-2) \\ &= 0+1(x-7)-3(y-2)=x-3y-1 \end{aligned}$$

Thus $f(6.9,2.06)\approx6.9-3(2.06)-1=-0.28$. The graph shows that our approximated value is slightly greater than the actual value.



19. $f(x,y,z)=\sqrt{x^2+y^2+z^2} \Rightarrow f_x(x,y,z)=\dfrac{x}{\sqrt{x^2+y^2+z^2}}$, $f_y(x,y,z)=\dfrac{y}{\sqrt{x^2+y^2+z^2}}$, and $f_z(x,y,z)=\dfrac{z}{\sqrt{x^2+y^2+z^2}}$, so $f_x(3,2,6)=\dfrac{3}{7}$, $f_y(3,2,6)=\dfrac{2}{7}$, and $f_z(3,2,6)=\dfrac{6}{7}$. Then the linear approximation of f at (3,2,6) is given by

$$\begin{aligned} f(x,y,z) &\approx f(3,2,6)+f_x(3,2,6)(x-3)+f_y(3,2,6)(y-2)+f_z(3,2,6)(z-6) \\ &= 7+\frac{3}{7}(x-3)+\frac{2}{7}(y-2)+\frac{6}{7}(z-6)=\frac{3}{7}x+\frac{2}{7}y+\frac{6}{7}z \end{aligned}$$

Thus $\sqrt{(3.02)^2+(1.97)^2+(5.99)^2}=f(3.02,1.97,5.99)\approx\frac{3}{7}(3.02)+\frac{2}{7}(1.97)+\frac{6}{7}(5.99)\approx 6.9914$.

20. From the table, $f(40,20)=28$. To estimate $f_v(40,20)$ and $f_t(40,20)$ we follow the procedure used in Exercise 15.3.4 . Since $f_v(40,20)=\lim_{h\to 0}\dfrac{f(40+h,20)-f(40,20)}{h}$, we approximate this quantity with $h=\pm 10$ and use the values given in the table: $f_v(40,20)\approx\dfrac{f(50,20)-f(40,20)}{10}=\dfrac{40-28}{10}=1.2$, $f_v(40,20)\approx\dfrac{f(30,20)-f(40,20)}{-10}=\dfrac{17-28}{-10}=1.1$. Averaging these values gives $f_v(40,20)\approx 1.15$. Similarly, $f_t(40,20)=\lim_{h\to 0}\dfrac{f(40,20+h)-f(40,20)}{h}$, so we use $h=10$ and $h=-5$: $f_t(40,20)\approx\dfrac{f(40,30)-f(40,20)}{10}=\dfrac{31-28}{10}=0.3$, $f_t(40,20)\approx\dfrac{f(40,15)-f(40,20)}{-5}=\dfrac{25-28}{-5}=0.6$. Averaging these values gives $f_t(40,15)\approx 0.45$. The linear approximation, then, is

$$\begin{aligned} f(v,t) &\approx f(40,20)+f_v(40,20)(v-40)+f_t(40,20)(t-20) \\ &\approx 28+1.15(v-40)+0.45(t-20) \end{aligned}$$

When $v=43$ and $t=24$, we estimate $f(43,24)\approx 28+1.15(43-40)+0.45(24-20)=33.25$, so we would expect the wave heights to be approximately 33.25 ft.

21. From the table, $f(94,80)=127$. To estimate $f_T(94,80)$ and $f_H(94,80)$ we follow the procedure used in Section 15.3 [ET 14.3]. Since $f_T(94,80)=\lim_{h\to 0}\dfrac{f(94+h,80)-f(94,80)}{h}$, we approximate this quantity

with $h=\pm 2$ and use the values given in the table: $f_T(94,80)\approx\dfrac{f(96,80)-f(94,80)}{2}=\dfrac{135-127}{2}=4$,

$f_T(94,80)\approx\dfrac{f(92,80)-f(94,80)}{-2}=\dfrac{119-127}{-2}=4$.

Averaging these values gives $f_T(94,80)\approx 4$. Similarly, $f_H(94,80)=\lim_{h\to 0}\dfrac{f(94,80+h)-f(94,80)}{h}$,

so we use $h=\pm 5$: $f_H(94,80)\approx\dfrac{f(94,85)-f(94,80)}{5}=\dfrac{132-127}{5}=1$,

$f_H(94,80)\approx\dfrac{f(94,75)-f(94,80)}{-5}=\dfrac{122-127}{-5}=1$. Averaging these values gives $f_H(94,80)\approx 1$. The linear approximation, then, is

$$\begin{aligned} f(\text{T,H}) &\approx f(94,80)+f_T(94,80)(T-94)+f_H(94,80)(H-80) \\ &\approx 127+4(T-94)+1(H-80) \end{aligned}$$

Thus when $T=95$ and $H=78$, $f(95,78)\approx 127+4(95-94)+1(78-80)=129$, so we estimate the heat index to be approximately $129°$ F .

22. From the table, $f(-15,50)=-29$. To estimate $f_T(-15,50)$ and $f_v(-15,50)$ we follow the procedure used in Section 15.3 . Since $f_T(-15,50)=\lim_{h\to 0}\dfrac{f(-15+h,50)-f(-15,50)}{h}$, we approximate this quantity with $h=\pm 5$ and use the values given in the table:

$f_T(-15,50)\approx\dfrac{f(-10,50)-f(-15,50)}{5}=\dfrac{-22-(-29)}{5}=1.4$,

$f_T(-15,50)\approx\dfrac{f(-20,50)-f(-15,50)}{-5}=\dfrac{-35-(-29)}{-5}=1.2$.

Averaging these values gives $f_T(-15,50)\approx 1.3$. Similarly $f_v(-15,50)=\lim_{h\to 0}\dfrac{f(-15,50+h)-f(-15,50)}{h}$

so we use $h=\pm 10$: $f_v(-15,50)\approx\dfrac{f(-15,60)-f(-15,50)}{10}=\dfrac{-30-(-29)}{10}=-0.1$,

$f_v(-15,50)\approx\dfrac{f(-15,40)-f(-15,50)}{-10}=\dfrac{-27-(-29)}{-10}=-0.2.$

Averaging these values gives $f_v(-15,50)\approx -0.15$. The linear approximation to the wind–chill index function, then, is

$$f(\text{T,v}) \approx f(-15,50)+f_T(-15,50)(T-(-15))+f_v(-15,50)(v-50)$$

$\approx\ -29+(1.3)(T+15)-(0.15)(v-50)$

Thus when $T=-17°$ C and $v=55$ km / h, $f(-17,55)\approx-29+(1.3)(-17+15)-(0.15)(55-50)=-32.35$, so we estimate the wind–chill index to be approximately $-32.35°$ C .

23. $z=x^3\ln(y^2)\Rightarrow$

$$dz=\frac{\partial z}{\partial x}\,dx+\frac{\partial z}{\partial y}\,dy=3x^2\ln\left(y^2\right)dx+x^3\cdot\frac{1}{y^2}\,(2y)\,dy=3x^2\ln\left(y^2\right)dx+\frac{2x^3}{y}\,dy\ .$$

24. $v=y\cos xy\Rightarrow$

$$dv=\frac{\partial v}{\partial x}\,dx+\frac{\partial v}{\partial y}\,dy=y(-\sin xy)y\,dx+[y(-\sin xy)x+\cos xy]\,dy=-y^2\sin xy\,dx+(\cos xy-xy\sin xy)\,dy$$

25. $u=e^t\sin\theta\Rightarrow\ du=\dfrac{\partial u}{\partial t}\,dt+\dfrac{\partial u}{\partial\theta}\,d\theta=e^t\sin\theta\,dt+e^t\cos\theta\,d\theta$

26. $u=\dfrac{r}{s+2t}\ \Rightarrow$

$$\begin{aligned}du &= \frac{\partial u}{\partial r}\,dr+\frac{\partial u}{\partial s}\,ds+\frac{\partial u}{\partial t}\,dt=\frac{1}{s+2t}\,dr+r(-1)(s+2t)^{-2}ds+r(-1)(s+2t)^{-2}(2)\,dt\\ &= \frac{1}{s+2t}\,dr-\frac{r}{(s+2t)^2}\,ds-\frac{2r}{(s+2t)^2}\,dt\end{aligned}$$

27. $w=\ln\sqrt{x^2+y^2+z^2}\ \Rightarrow$

$$\begin{aligned}dw &= \frac{\partial w}{\partial x}\,dx+\frac{\partial w}{\partial y}\,dy+\frac{\partial w}{\partial z}\,dz\\ &= \left(\frac{1}{2}\right)\frac{2x\left(x^2+y^2+z^2\right)^{-1/2}dx+2y\left(x^2+y^2+z^2\right)^{-1/2}dy+2z\left(x^2+y^2+z^2\right)^{-1/2}dz}{\left(x^2+y^2+z^2\right)^{1/2}}\\ &= \frac{x\,dx+y\,dy+z\,dz}{x^2+y^2+z^2}\end{aligned}$$

28. $w=xye^{xz}\Rightarrow$

$$\begin{aligned}dw &= \frac{\partial w}{\partial x}\,dx+\frac{\partial w}{\partial y}\,dy+\frac{\partial w}{\partial z}\,dz=(xyze^{xz}+ye^{xz})\,dx+xe^{xz}\,dy+x^2ye^{xz}\,dz\\ &=(xz+1)ye^{xz}\,dx+xe^{xz}\,dy+x^2ye^{xz}\,dz\ .\end{aligned}$$

29. $dx=\Delta x=0.05$, $dy=\Delta y=0.1$, $z=5x^2+y^2$, $z_x=10x$, $z_y=2y$. Thus when $x=1$ and $y=2$,
$dz=z_x(1,2)dx+z_y(1,2)dy=(10)(0.05)+(4)(0.1)=0.9$ while
$\Delta z=f(1.05,2.1)-f(1,2)=5(1.05)^2+(2.1)^2-5-4=0.9225$.

30. $dx=\Delta x=-0.04$, $dy=\Delta y=0.05$, $z=x^2-xy+3y^2$, $z_x=2x-y$, $z_y=6y-x$. Thus when $x=3$ and $y=-1$,
$dz=(7)(-0.04)+(-9)(0.05)=-0.73$ while $\Delta z=(2.96)^2-(2.96)(-0.95)+3(-0.95)^2-(9+3+3)=-0.7189$.

31. $dA=\frac{\partial A}{\partial x}dx+\frac{\partial A}{\partial y}dy=y\,dx+x\,dy$ and $|\Delta x|\leq 0.1$, $|\Delta y|\leq 0.1$. We use $dx=0.1$, $dy=0.1$ with $x=30$, $y=24$; then the maximum error in the area is about $dA=24(0.1)+30(0.1)=5.4\text{ cm}^2$.

32. Let S be surface area. Then $S=2(xy+xz+yz)$ and $dS=2(y+z)dx+2(x+z)dy+2(x+y)dz$. The maximum error occurs with $\Delta x=\Delta y=\Delta z=0.2$. Using $dx=\Delta x$, $dy=\Delta y$, $dz=\Delta z$ we find the maximum error in calculated surface area to be about $dS=(220)(0.2)+(260)(0.2)+(280)(0.2)=152\text{ cm}^2$.

33. The volume of a can is $V=\pi r^2h$ and $\Delta V\approx dV$ is an estimate of the amount of tin. Here $dV=2\pi rh\,dr+\pi r^2dh$, so put $dr=0.04$, $dh=0.08$ (0.04 on top, 0.04 on bottom) and then $\Delta V\approx dV=2\pi(48)(0.04)+\pi(16)(0.08)\approx 16.08\text{ cm}^3$. Thus the amount of tin is about 16 cm^3 .

34. Let V be the volume. Then $V=\pi r^2h$ and $\Delta V\approx dV=2\pi rh\,dr+\pi r^2dh$ is an estimate of the amount of metal. With $dr=0.05$ and $dh=0.2$ we get $dV=2\pi(2)(10)(0.05)+\pi(2)^2(0.2)=2.80\pi\approx 8.8\text{ cm}^3$.

35. The area of the rectangle is $A=xy$, and $\Delta A\approx dA$ is an estimate of the area of paint in the stripe. Here $dA=y\,dx+x\,dy$, so with $dx=dy=\frac{3+3}{12}=\frac{1}{2}$, $\Delta A\approx dA=(100)\left(\frac{1}{2}\right)+(200)\left(\frac{1}{2}\right)=150\text{ ft}^2$. Thus there are approximately 150 ft^2 of paint in the stripe.

36. Here $dV=\Delta V=0.3$, $dT=\Delta T=-5$, $P=8.31\frac{T}{V}$, so
$$dP=\left(\frac{8.31}{V}\right)dT-\frac{8.31\cdot T}{V^2}dV=8.31\left[-\frac{5}{12}-\frac{310}{144}\cdot\frac{3}{10}\right]\approx -8.83 .$$
Thus the pressure will drop by about 8.83 kPa .

37. First we find $\frac{\partial R}{\partial R_1}$ implicitly by taking partial derivatives of both sides with respect to R_1 :

$\dfrac{\partial}{\partial R_1}\left[\dfrac{1}{R}\right]=\dfrac{\partial\left[\left(1/R_1\right)+\left(1/R_2\right)+\left(1/R_3\right)\right]}{\partial R_1}\Rightarrow -R^{-2}\dfrac{\partial R}{\partial R_1}=-R_1^{-2}\Rightarrow\dfrac{\partial R}{\partial R_1}=\dfrac{R^2}{R_1^2}$. Then by symmetry,

$\dfrac{\partial R}{\partial R_2}=\dfrac{R^2}{R_2^2}$, $\dfrac{\partial R}{\partial R_3}=\dfrac{R^2}{R_3^2}$. When $R_1=25$, $R_2=40$ and $R_3=50$, $\dfrac{1}{R}=\dfrac{17}{200}\Leftrightarrow R=\dfrac{200}{17}$ ohms. Since the possible error for each R_i is 0.5% , the maximum error of R is attained by setting $\Delta R_i=0.005R_i$. So

$$\Delta R\approx\ dR=\frac{\partial R}{\partial R_1}\ \Delta R_1+\frac{\partial R}{\partial R_2}\ \Delta R_2+\frac{\partial R}{\partial R_3}\ \Delta R_3=(0.005)R^2\left[\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right]$$

$$=\ (0.005)R=\frac{1}{17}\approx 0.059\text{ohms}$$

38. Let x,y,z and w be the four numbers with $p(x,y,z,w)=xyzw$. Since the largest error due to rounding for each number is 0.05 , the maximum error in the calculated product is approximated by $dp=(yzw)(0.05)+(xzw)(0.05)+(xyw)(0.05)+(xyz)(0.05)$. Furthermore, each of the numbers is positive but less than 50 , so the product of any three is between 0 and $(50)^3$. Thus $dp\leq 4(50)^3(0.05)=25{,}000$.

39.

$$\begin{aligned}\Delta z&=f(a+\Delta x,b+\Delta y)-f(a,b)=(a+\Delta x)^2+(b+\Delta y)^2-\left(a^2+b^2\right)\\&=a^2+2a\Delta x+(\Delta x)^2+b^2+2b\Delta y+(\Delta y)^2-a^2-b^2=2a\Delta x+(\Delta x)^2+2b\Delta y+(\Delta y)^2\end{aligned}$$

But $f_x(a,b)=2a$ and $f_y(a,b)=2b$ and so $\Delta z=f_x(a,b)\Delta x+f_y(a,b)\Delta y+\Delta x\Delta x+\Delta y\Delta y$, which is Definition 7 with $\varepsilon_1=\Delta x$ and $\varepsilon_2=\Delta y$. Hence f is differentiable.

40.

$$\begin{aligned}\Delta z&=f(a+\Delta x,b+\Delta y)-f(a,b)=(a+\Delta x)(b+\Delta y)-5(b+\Delta y)^2-\left(ab-5b^2\right)\\&=ab+a\Delta y+b\Delta x+\Delta x\Delta y-5b^2-10b\Delta y-5(\Delta y)^2-ab+5b^2\\&=(a-10b)\Delta y+b\Delta x+\Delta x\Delta y-5\Delta y\Delta y\ ,\end{aligned}$$

but $f_x(a,b)=b$ and $f_y(a,b)=a-10b$ and so $\Delta z=f_x(a,b)\Delta x+f_y(a,b)\Delta y+\Delta x\Delta y-5\Delta y\Delta y$, which is Definition 7 with $\varepsilon_1=\Delta y$ and $\varepsilon_2=-5\Delta y$. Hence f is differentiable.

41. To show that f is continuous at (a,b) we need to show that

$\lim_{(x,y)\to(a,b)} f(x,y)=f(a,b)$ or equivalently $\lim_{(\Delta x,\Delta y)\to(0,0)} f(a+\Delta x,b+\Delta y)=f(a,b)$. Since f is differentiable at (a,b) , $f(a+\Delta x,b+\Delta y)-f(a,b)=\Delta z=f_x(a,b)\Delta x+f_y(a,b)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y$, where ε_1 and $\varepsilon_2\to 0$ as $(\Delta x,\Delta y)\to(0,0)$. Thus $f(a+\Delta x,b+\Delta y)=f(a,b)+f_x(a,b)\Delta x+f_y(a,b)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y$. Taking the limit of both sides as $(\Delta x,\Delta y)\to(0,0)$ gives $\lim_{(\Delta x,\Delta y)\to(0,0)} f(a+\Delta x,b+\Delta y)=f(a,b)$. Thus f is continuous at (a,b) .

42. **(a)** $\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$ and $\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$. Thus $f_x(0,0)=f_y(0,0)=0$. To show that f isn't differentiable at (0,0) we need only show that f is not continuous at (0,0) and apply Exercise 41 . As $(x,y)\to(0,0)$ along the x –axis $f(x,y)=0/x^2=0$ for $x\neq 0$ so $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the x –axis. But as $(x,y)\to(0,0)$ along the line $y=x$, $f(x,x)=x^2/\left(2x^2\right)=\frac{1}{2}$ for $x\neq 0$ so $f(x,y)\to\frac{1}{2}$ as $(x,y)\to(0,0)$ along this line. Thus $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so f is discontinuous at (0,0) and thus not differentiable there.

(b) For $(x,y)\neq(0,0)$, $f_x(x,y)=\frac{\left(x^2+y^2\right)y-xy(2x)}{\left(x^2+y^2\right)^2}=\frac{y\left(y^2+x^2\right)}{\left(x^2+y^2\right)^2}$. If we approach (0,0) along the y –axis, then $f_x(x,y)=f_x(0,y)=\frac{y^3}{y^4}=\frac{1}{y}$, so $f_x(x,y)\to\pm\infty$ as $(x,y)\to(0,0)$. Thus $\lim_{(x,y)\to(0,0)} f_x(x,y)$ does not exist and $f_x(x,y)$ is not continuous at (0,0). Similarly, $f_y(x,y)=\frac{\left(x^2+y^2\right)x-xy(2y)}{\left(x^2+y^2\right)^2}=\frac{x(x^2-y^2)}{\left(x^2+y^2\right)^2}$ for $(x,y)\neq(0,0)$, and if we approach (0,0) along the x –axis, then $f_y(x,y)=f_x(x,0)=\frac{x^3}{x^4}=\frac{1}{x}$. Thus $\lim_{(x,y)\to(0,0)} f_y(x,y)$ does not exist and $f_y(x,y)$ is not continuous at (0,0).

1. $z=x^2y+xy^2$, $x=2+t^4$, $y=1-t^3 \Rightarrow$

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=\left(2xy+y^2\right)\left(4t^3\right)+\left(x^2+2xy\right)\left(-3t^2\right)=4\left(2xy+y^2\right)^3-3\left(x^2+2xy\right)t^2$$

2. $z=\sqrt{x^2+y^2}$, $x=e^{2t}$, $y=e^{-2t} \Rightarrow$

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=\frac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x)\cdot e^{2t}(2)+\frac{1}{2}\left(x^2+y^2\right)^{-1/2}(2y)\cdot e^{-2t}(-2)=\frac{2xe^{2t}-2ye^{2t}}{\sqrt{x^2+y^2}}$$

3. $z=\sin x\cos y$, $x=\pi t$, $y=\sqrt{t} \Rightarrow$

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=\cos x\cos y\cdot\pi+\sin x(-\sin y)\cdot\frac{1}{2}t^{-1/2}=\pi\cos x\cos y-\frac{1}{2\sqrt{t}}\sin x\sin y$$

4. $z=x\ln(x+2y)$, $x=\sin t$, $y=\cos t \Rightarrow$

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=\left[x\cdot\frac{1}{x+2y}+1\cdot\ln(x+2y)\right]\cos t+x\cdot\frac{1}{x+2y}(2)\cdot(-\sin t)$$

$$=\left[\frac{x}{x+2y}+\ln(x+2y)\right]\cos t-\frac{2x}{x+2y}(\sin t)$$

5. $w=xe^{y/z}$, $x=t^2$, $y=1-t$, $z=1+2t \Rightarrow$

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}=e^{y/z}\cdot 2t+xe^{y/z}\left(\frac{1}{z}\right)\cdot(-1)+xe^{y/z}\left(-\frac{y}{z^2}\right)\cdot 2=e^{y/z}\left(2t-\frac{x}{z}-\frac{2xy}{z^2}\right)$$

6. $w=xy+yz^2$, $x=e^t$, $y=e^t\sin t$, $z=e^t\cos t \Rightarrow$

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}=y\cdot e^t+\left(x+z^2\right)\cdot\left(e^t\cos t+e^t\sin t\right)+2yz\cdot(-e^t\sin t+e^t\cos t)$$

$$=e^t\left[y+\left(x+z^2\right)(\cos t+\sin t)+2yz(\cos t-\sin t)\right]$$

7. $z=x^2+xy+y^2$, $x=s+t$, $y=st \Rightarrow$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}=(2x+y)(1)+(x+2y)(t)=2x+y+xt+2yt$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}=(2x+y)(1)+(x+2y)(s)=2x+y+xs+2ys$$

8. $z=x/y$, $x=se^t$, $y=1+se^{-t} \Rightarrow$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}=\frac{1}{y}\left(e^t\right)+\left(-\frac{x}{y^2}\right)\left(e^{-t}\right)=\frac{1}{y}e^t-\frac{x}{y^2}e^{-t}$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}=\frac{1}{y}\left(se^{t}\right)+\left(-\frac{x}{y^2}\right)\left(-se^{-t}\right)=\frac{s}{y}e^{t}+\frac{xs}{y^2}e^{-t}$$

9. $z=\arctan(2x+y)$, $x=s^2t$, $y=s\ln t\Rightarrow$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}=\frac{2}{1+(2x+y)^2}\cdot 2st+\frac{1}{1+(2x+y)^2}\cdot\ln t=\frac{4st+\ln t}{1+(2x+y)^2}$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}=\frac{2}{1+(2x+y)^2}\cdot s^2+\frac{1}{1+(2x+y)^2}\cdot\frac{s}{t}=\frac{2s^2+s/t}{1+(2x+y)^2}$$

10. $z=e^{xy}\tan y$, $x=s+2t$, $y=s/t\Rightarrow$

$$\frac{\partial z}{\partial s}=ye^{xy}\tan y\cdot 1+\left(e^{xy}\sec^2 y+xe^{xy}\tan y\right)\cdot\frac{1}{t}=ye^{xy}\tan y+\frac{e^{xy}}{t}\left(\sec^2 y+x\tan y\right)$$

$$\frac{\partial z}{\partial t}=ye^{xy}\tan y\cdot 2+\left(e^{xy}\sec^2 y+xe^{xy}\tan y\right)\left(\frac{-s}{t^2}\right)=2ye^{xy}\tan y-\frac{se^{xy}}{t^2}\left(\sec^2 y+x\tan y\right)$$

11. $z=e^{r}\cos\theta$, $r=st$, $\theta=\sqrt{s^2+t^2}\Rightarrow$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial r}\frac{\partial r}{\partial s}+\frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}=e^{r}\cos\theta\cdot t+e^{r}(-\sin\theta)\cdot\frac{1}{2}\left(s^2+t^2\right)^{-1/2}(2s)$$

$$=te^{r}\cos\theta-e^{r}\sin\theta\cdot\frac{s}{\sqrt{s^2+t^2}}=e^{r}\left(t\cos\theta-\frac{s}{\sqrt{s^2+t^2}}\sin\theta\right)$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial r}\frac{\partial r}{\partial t}+\frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t}=e^{r}\cos\theta\cdot s+e^{r}(-\sin\theta)\cdot\frac{1}{2}\left(s^2+t^2\right)^{-1/2}(2t)$$

$$=se^{r}\cos\theta-e^{r}\sin\theta\cdot\frac{t}{\sqrt{s^2+t^2}}=e^{r}\left(s\cos\theta-\frac{t}{\sqrt{s^2+t^2}}\sin\theta\right)$$

12. $z=\sin\alpha\tan\beta$, $\alpha=3s+t$, $\beta=s-t\Rightarrow$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial s}+\frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial s}=\cos\alpha\tan\beta\cdot 3+\sin\alpha\sec^2\beta\cdot 1=3\cos\alpha\tan\beta+\sin\alpha\sec^2\beta$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial t}+\frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial t}=\cos\alpha\tan\beta\cdot 1+\sin\alpha\sec^2\beta\cdot(-1)=\cos\alpha\tan\beta-\sin\alpha\sec^2\beta$$

13. When $t=3$, $x=g(3)=2$ and $y=h(3)=7$. By the Chain Rule (2),

$$\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=f_x(2,7)g'(3)+f_y(2,7)h'(3)=(6)(5)+(-8)(-4)=62\ .$$

14. By the Chain Rule (3), $\frac{\partial W}{\partial s}=\frac{\partial W}{\partial u}\frac{\partial u}{\partial s}+\frac{\partial W}{\partial v}\frac{\partial v}{\partial s}$. Then

$$W_s(1,0) = F_u(u(1,0),\mathrm{v}(1,0))\,u_s(1,0)+F_v(u(1,0),\mathrm{v}(1,0))\,v_s(1,0)$$
$$= F_u(2,3)u_s(1,0)+F_v(2,3)v_s(1,0)=(-1)(-2)+(10)(5)=52$$

Similarly, $\frac{\partial W}{\partial t}=\frac{\partial W}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial W}{\partial v}\frac{\partial v}{\partial t}\Rightarrow$

$$W_t(1,0) = F_u(u(1,0),\mathrm{v}(1,0))\,u_t(1,0)+F_v(u(1,0),\mathrm{v}(1,0))\,v_t(1,0)$$
$$= F_u(2,3)u_t(1,0)+F_v(2,3)v_t(1,0)=(-1)(6)+(10)(4)=34$$

15. $g(\mathrm{u,v})=f(x(\mathrm{u,v}),\mathrm{y}(\mathrm{u,v}))$ where $x=e^u+\sin v$, $y=e^u+\cos v\Rightarrow$ $\frac{\partial x}{\partial u}=e^u$, $\frac{\partial x}{\partial v}=\cos v$, $\frac{\partial y}{\partial u}=e^u$, $\frac{\partial y}{\partial v}=-\sin v$. By the Chain Rule (3), $\frac{\partial g}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$. Then

$$g_u(0,0) = f_x(x(0,0),\mathrm{y}(0,0))\,x_u(0,0)+f_y(x(0,0),\mathrm{y}(0,0))\,y_u(0,0)$$
$$= f_x(1,2)\left(e^0\right)+f_y(1,2)\left(e^0\right)=2(1)+5(1)=7$$

Similarly $\frac{\partial g}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$. Then

$$g_v(0,0) = f_x(x(0,0),\mathrm{y}(0,0))\,x_v(0,0)+f_y(x(0,0),\mathrm{y}(0,0))\,y_v(0,0)$$
$$= f_x(1,2)(\cos 0)+f_y(1,2)(-\sin 0)=2(1)+5(0)=2$$

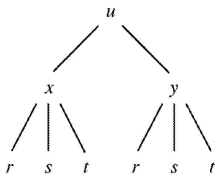
16. $g(\mathrm{r,s})=f(x(\mathrm{r,s}),\mathrm{y}(\mathrm{r,s}))$ where $x=2r-s$, $y=s^2-4r\Rightarrow$ $\frac{\partial x}{\partial r}=2$, $\frac{\partial x}{\partial s}=-1$, $\frac{\partial y}{\partial r}=-4$, $\frac{\partial y}{\partial s}=2s$. By the Chain Rule (3) $\frac{\partial g}{\partial r}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$. Then

$$g_r(1,2) = f_x(x(1,2),\mathrm{y}(1,2))\,x_r(1,2)+f_y(x(1,2),\mathrm{y}(1,2))\,y_r(1,2)$$
$$= f_x(0,0)(2)+f_y(0,0)(-4)=4(2)+8(-4)=-24$$

Similarly $\frac{\partial g}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$. Then

$$g_s(1,2) = f_x(x(1,2),\mathrm{y}(1,2))\,x_s(1,2)+f_y(x(1,2),\mathrm{y}(1,2))\,y_s(1,2)$$
$$= f_x(0,0)(-1)+f_y(0,0)(4)=4(-1)+8(4)=28$$

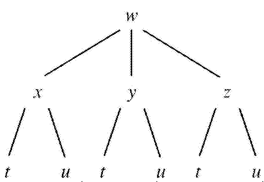
17.



$u=f\,(\mathrm{x,y})$, $x=x\,(\mathrm{r,s,t})$, $y=y\,(\mathrm{r,s,t}) \Rightarrow$

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\ \frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\ \frac{\partial y}{\partial r}\ ,\ \frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\ \frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\ \frac{\partial y}{\partial s}\ ,\ \frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\ \frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\ \frac{\partial y}{\partial t}$$

18.

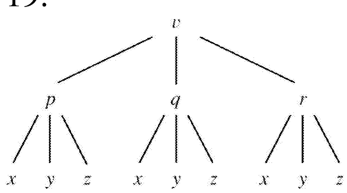


$w=f\,(\mathrm{x,y,z})$, $x=x\,(\mathrm{t,u})$, $y=y\,(\mathrm{t,u})$, $z=z\,(\mathrm{t,u}) \Rightarrow$

$$\frac{\partial w}{\partial t}=\frac{\partial w}{\partial x}\ \frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\ \frac{\partial y}{\partial t}+\frac{\partial w}{\partial z}\ \frac{\partial z}{\partial t}\ ,$$

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\ \frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\ \frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}\ \frac{\partial z}{\partial u}$$

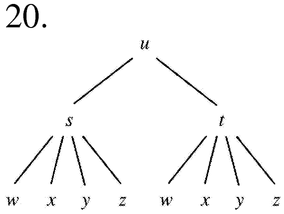
19.



$v=f(\mathrm{p,q,r})$, $p=p(\mathrm{x,y,z})$, $q=q(\mathrm{x,y,z})$, $r=r(\mathrm{x,y,z})\Rightarrow$

$$\frac{\partial v}{\partial x}=\frac{\partial v}{\partial p}\ \frac{\partial p}{\partial x}+\frac{\partial v}{\partial q}\ \frac{\partial q}{\partial x}+\frac{\partial v}{\partial r}\ \frac{\partial r}{\partial x}\ ,\ \frac{\partial v}{\partial y}=\frac{\partial v}{\partial p}\ \frac{\partial p}{\partial y}+\frac{\partial v}{\partial q}\ \frac{\partial q}{\partial y}+\frac{\partial v}{\partial r}\ \frac{\partial r}{\partial y}\ ,\ \frac{\partial v}{\partial z}=\frac{\partial v}{\partial p}\ \frac{\partial p}{\partial z}+\frac{\partial v}{\partial q}\ \frac{\partial q}{\partial z}+\frac{\partial v}{\partial r}\ \frac{\partial r}{\partial z}$$

20.



$u=f(\mathrm{s,t})$, $s=s(\mathrm{w,x,y,z})$, $t=t(\mathrm{w,x,y,z})\Rightarrow$

$$\frac{\partial u}{\partial w}=\frac{\partial u}{\partial s}\ \frac{\partial s}{\partial w}+\frac{\partial u}{\partial t}\ \frac{\partial t}{\partial w}\ ,\ \frac{\partial u}{\partial x}=\frac{\partial u}{\partial s}\ \frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\ \frac{\partial t}{\partial x}\ ,\ \frac{\partial u}{\partial y}=\frac{\partial u}{\partial s}\ \frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\ \frac{\partial t}{\partial y}\ ,\ \frac{\partial u}{\partial z}=\frac{\partial u}{\partial s}\ \frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\ \frac{\partial t}{\partial z}$$

21. $z=x^2+xy^3$, $x=uv^2+w^3$, $y=u+ve^w \Rightarrow \dfrac{\partial z}{\partial u}=\dfrac{\partial z}{\partial x}\ \dfrac{\partial x}{\partial u}+\dfrac{\partial z}{\partial y}\ \dfrac{\partial y}{\partial u}=\left(2x+y^3\right)\left(v^2\right)+\left(3xy^2\right)(1)$,

$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=\left(2x+y^3\right)(2uv)+\left(3xy^2\right)\left(e^w\right)$,
$\frac{\partial z}{\partial w}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial w}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial w}=\left(2x+y^3\right)\left(3w^2\right)+\left(3xy^2\right)\left(ve^w\right)$. When $u=2$, $v=1$, and $w=0$, we have $x=2$, $y=3$, so $\frac{\partial z}{\partial u}=(31)(1)+(54)(1)=85$, $\frac{\partial z}{\partial v}=(31)(4)+(54)(1)=178$, $\frac{\partial z}{\partial w}=(31)(0)+(54)(1)=54$.

22. $u=\left(r^2+s^2\right)^{1/2}$, $r=y+x\cos t$, $s=x+y\sin t\Rightarrow$
$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x}=\frac{1}{2}\left(r^2+s^2\right)^{-1/2}(2r)(\cos t)+\frac{1}{2}\left(r^2+s^2\right)^{-1/2}(2s)(1)=(r\cos t+s)/\sqrt{r^2+s^2}$,
$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y}=\frac{1}{2}\left(r^2+s^2\right)^{-1/2}(2r)(1)+\frac{1}{2}\left(r^2+s^2\right)^{-1/2}(2s)(\sin t)=(r+s\sin t)/\sqrt{r^2+s^2}$,
$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial t}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial t}=\frac{1}{2}\left(r^2+s^2\right)^{-1/2}(2r)(-x\sin t)+\frac{1}{2}\left(r^2+s^2\right)^{-1/2}(2s)(y\cos t)=\frac{-rx\sin t+sy\cos t}{\sqrt{r^2+s^2}}$.
When $x=1$, $y=2$, and $t=0$ we have $r=3$ and $s=1$, so $\frac{\partial u}{\partial x}=\frac{4}{\sqrt{10}}$, $\frac{\partial u}{\partial y}=\frac{3}{\sqrt{10}}$, and $\frac{\partial u}{\partial t}=\frac{2}{\sqrt{10}}$.

23. $R=\ln\left(u^2+v^2+w^2\right)$, $u=x+2y$, $v=2x-y$, $w=2xy\Rightarrow$

$$\frac{\partial R}{\partial x}=\frac{\partial R}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial R}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial R}{\partial w}\frac{\partial w}{\partial x}=\frac{2u}{u^2+v^2+w^2}(1)+\frac{2v}{u^2+v^2+w^2}(2)+\frac{2w}{u^2+v^2+w^2}(2y)$$
$$=\frac{2u+4v+4wy}{u^2+v^2+w^2},$$

$$\frac{\partial R}{\partial y}=\frac{\partial R}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial R}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial R}{\partial w}\frac{\partial w}{\partial y}=\frac{2u}{u^2+v^2+w^2}(2)+\frac{2v}{u^2+v^2+w^2}(-1)+\frac{2w}{u^2+v^2+w^2}(2x)$$
$$=\frac{4u-2v+4wx}{u^2+v^2+w^2}.$$

When $x=y=1$ we have $u=3$, $v=1$, and $w=2$, so $\frac{\partial R}{\partial x}=\frac{9}{7}$ and $\frac{\partial R}{\partial y}=\frac{9}{7}$.

24. $M=xe^{y-z^2}$, $x=2uv$, $y=u-v$, $z=u+v\Rightarrow$

$$\frac{\partial M}{\partial u}=\frac{\partial M}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial M}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial M}{\partial z}\frac{\partial z}{\partial u}=e^{y-z^2}(2v)+xe^{y-z^2}(1)+x(-2z)e^{y-z^2}(1)$$

$$=e^{y-z^2}(2v+x-2xz)\ ,$$

$$\frac{\partial M}{\partial v}=\frac{\partial M}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial M}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial M}{\partial z}\frac{\partial z}{\partial v}=e^{y-z^2}(2u)+xe^{y-z^2}(-1)+x(-2z)e^{y-z^2}(1)$$

$$=e^{y-z^2}(2u-x-2xz)\ .$$

When $u=3$, $v=-1$ we have $x=-6$, $y=4$, and $z=2$, so $\frac{\partial M}{\partial u}=16$ and $\frac{\partial M}{\partial v}=36$.

25. $u=x^2+yz$, $x=pr\cos\theta$, $y=pr\sin\theta$, $z=p+r\Rightarrow$

$$\frac{\partial u}{\partial p}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial p}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial p}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial p}=(2x)(r\cos\theta)+(z)(r\sin\theta)+(y)(1)=2xr\cos\theta+zr\sin\theta+y\ ,$$

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial r}=(2x)(p\cos\theta)+(z)(p\sin\theta)+(y)(1)=2xp\cos\theta+zp\sin\theta+y\ ,$$

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta}=(2x)(-pr\sin\theta)+(z)(pr\cos\theta)+(y)(0)=-2xpr\sin\theta+zpr\cos\theta\ .$$

When $p=2$, $r=3$, and $\theta=0$ we have $x=6$, $y=0$, and $z=5$, so $\frac{\partial u}{\partial p}=36$, $\frac{\partial u}{\partial r}=24$, and $\frac{\partial u}{\partial \theta}=30$.

26. $Y=w\tan^{-1}(uv)$, $u=r+s$, $v=s+t$, $w=t+r\Rightarrow$

$$\frac{\partial Y}{\partial r}=\frac{\partial Y}{\partial u}\frac{\partial u}{\partial r}+\frac{\partial Y}{\partial v}\frac{\partial v}{\partial r}+\frac{\partial Y}{\partial w}\frac{\partial w}{\partial r}=\frac{w}{1+(uv)^2}(v)(1)+\frac{w}{1+(uv)^2}(u)(0)+\tan^{-1}(uv)(1)$$

$$=\frac{vw}{1+u^2v^2}+\tan^{-1}(uv)$$

$$\frac{\partial Y}{\partial s}=\frac{\partial Y}{\partial u}\frac{\partial u}{\partial s}+\frac{\partial Y}{\partial v}\frac{\partial v}{\partial s}+\frac{\partial Y}{\partial w}\frac{\partial w}{\partial s}=\frac{wv}{1+u^2v^2}(1)+\frac{wu}{1+u^2v^2}(1)+\tan^{-1}(uv)(0)$$

$$=\frac{w(v+u)}{1+u^2v^2}$$

$$\frac{\partial Y}{\partial t}=\frac{\partial Y}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial Y}{\partial v}\frac{\partial v}{\partial t}+\frac{\partial Y}{\partial w}\frac{\partial w}{\partial t}=\frac{wv}{1+u^2v^2}(0)+\frac{wu}{1+u^2v^2}(1)+\tan^{-1}(uv)(1)$$

$$=\frac{wu}{1+u^2v^2}+\tan^{-1}(uv)$$

When $r=1$, $s=0$, and $t=1$, we have $u=1$, $v=1$, and $w=2$, so $\dfrac{\partial Y}{\partial r}=1+\dfrac{\pi}{4}$, $\dfrac{\partial Y}{\partial s}=2$, and $\dfrac{\partial Y}{\partial t}=1+\dfrac{\pi}{4}$.

27. $\sqrt{xy}=1+x^2y$, so let $F(\mathrm{x,y})=(xy)^{1/2}-1-x^2y=0$. Then by Equation 6

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{\frac{1}{2}(xy)^{-1/2}(y)-2xy}{\frac{1}{2}(xy)^{-1/2}(x)-x^2}=-\frac{y-4xy\sqrt{xy}}{x-2x^2\sqrt{xy}}=\frac{4(xy)^{3/2}-y}{x-2x^2\sqrt{xy}} .$$

28. $y^5+x^2y^3=1+ye^{x^2}$, so let $F(\mathrm{x,y})=y^5+x^2y^3-1-ye^{x^2}=0$. Then

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{2xy^3-2xye^{x^2}}{5y^4+3x^2y^2-e^{x^2}}=\frac{2xye^{x^2}-2xy^3}{5y^4+3x^2y^2-e^{x^2}} .$$

29. $\cos(x-y)=xe^y$, so let $F(\mathrm{x,y})=\cos(x-y)-xe^y=0$.

Then $\dfrac{dy}{dx}=-\dfrac{F_x}{F_y}=-\dfrac{-\sin(x-y)-e^y}{-\sin(x-y)(-1)-xe^y}=\dfrac{\sin(x-y)+e^y}{\sin(x-y)-xe^y}$.

30. $\sin x+\cos y=\sin x\cos y$, so let $F(\mathrm{x,y})=\sin x+\cos y-\sin x\cos y=0$. Then

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{\cos x-\cos x\cos y}{-\sin y+\sin x\sin y}=\frac{\cos x(\cos y-1)}{\sin y(\sin x-1)} .$$

31. $x^2+y^2+z^2=3xyz$, so let $F(\mathrm{x,y,z})=x^2+y^2+z^2-3xyz=0$. Then by Equations 7

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{2x-3yz}{2z-3xy}=\frac{3yz-2x}{2z-3xy} \text{ and } \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{2y-3xz}{2z-3xy}=\frac{3xz-2y}{2z-3xy} .$$

32. $xyz=\cos(x+y+z)$. Let $F(\mathrm{x,y,z})=xyz-\cos(x+y+z)=0$, so

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{yz+\sin(x+y+z)}{xy+\sin(x+y+z)} , \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{xz+\sin(x+y+z)}{xy+\sin(x+y+z)} .$$

33. $x-z=\arctan(yz)$, so let $F(\mathrm{x,y,z})=x-z-\arctan(yz)=0$. Then $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}=-\dfrac{1}{-1-\dfrac{1}{1+(yz)^2}(y)}=\dfrac{1+y^2z^2}{1+y+y^2z^2}$

and $\dfrac{\partial z}{\partial y}=-\dfrac{F_y}{F_z}=-\dfrac{-\dfrac{1}{1+(yz)^2}(z)}{-1-\dfrac{1}{1+(yz)^2}(y)}=-\dfrac{\dfrac{z}{1+y^2z^2}}{\dfrac{1+y^2z^2+y}{1+y^2z^2}}=-\dfrac{z}{1+y+y^2z^2}$.

34. $yz=\ln(x+z)$, so let $F(x,y,z)=yz-\ln(x+z)=0$. Then $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}=-\dfrac{-\dfrac{1}{x+z}(1)}{y-\dfrac{1}{x+z}(1)}=\dfrac{1}{y(x+z)-1}$,

$\dfrac{\partial z}{\partial y}=-\dfrac{F_y}{F_z}=-\dfrac{z}{y-\dfrac{1}{x+z}}=-\dfrac{z(x+z)}{y(x+z)-1}$.

35. Since x and y are each functions of t , $T(x,y)$ is a function of t , so by the Chain Rule, $\dfrac{dT}{dt}=\dfrac{\partial T}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial T}{\partial y}\dfrac{dy}{dt}$. After 3 seconds, $x=\sqrt{1+t}=\sqrt{1+3}=2$, $y=2+\frac{1}{3}t=2+\frac{1}{3}(3)=3$, $\dfrac{dx}{dt}=\dfrac{1}{2\sqrt{1+t}}=\dfrac{1}{2\sqrt{1+3}}=\dfrac{1}{4}$, and $\dfrac{dy}{dt}=\dfrac{1}{3}$. Then $\dfrac{dT}{dt}=T_x(2,3)\dfrac{dx}{dt}+T_y(2,3)\dfrac{dy}{dt}=4\left(\dfrac{1}{4}\right)+3\left(\dfrac{1}{3}\right)=2$. Thus the temperature is rising at a rate of $2°$ C/s .

36. **(a)** Since $\partial W/\partial T$ is negative, a rise in average temperature (while annual rainfall remains constant) causes a decrease in wheat production at the current production levels. Since $\partial W/\partial R$ is positive, an increase in annual rainfall (while the average temperature remains constant) causes an increase in wheat production.
(b) Since the average temperature is rising at a rate of $0.15°$ C/year , we know that $dT/dt=0.15$. Since rainfall is decreasing at a rate of 0.1 cm / year, we know $dR/dt=-0.1$. Then, by the Chain Rule, $\dfrac{dW}{dt}=\dfrac{\partial W}{\partial T}\dfrac{dT}{dt}+\dfrac{\partial W}{\partial R}\dfrac{dR}{dt}=(-2)(0.15)+(8)(-0.1)=-1.1$. Thus we estimate that wheat production will decrease at a rate of 1.1 units / year.

37. $C=1449.2+4.6T-0.055T^2+0.00029T^3+0.016D$, so $\dfrac{\partial C}{\partial T}=4.6-0.11T+0.00087T^2$ and $\dfrac{\partial C}{\partial D}=0.016$. According to the graph, the diver is experiencing a temperature of approximately $12.5°$ C at $t=20$ minutes, so $\dfrac{\partial C}{\partial T}=4.6-0.11(12.5)+0.00087(12.5)^2\approx 3.36$. By sketching tangent lines at $t=20$ to the graphs given, we estimate $\dfrac{dD}{dt}\approx\dfrac{1}{2}$ and $\dfrac{dT}{dt}\approx-\dfrac{1}{10}$. Then, by the Chain Rule,

$\frac{dC}{dt}=\frac{\partial C}{\partial T}\frac{dT}{dt}+\frac{\partial C}{\partial D}\frac{dD}{dt}\approx(3.36)\left(-\frac{1}{10}\right)+(0.016)\left(\frac{1}{2}\right)\approx-0.33$. Thus the speed of sound experienced by the diver is decreasing at a rate of approximately 0.33 m / s per minute.

38. $V=\pi r^2 h/3$, so $\frac{dV}{dt}=\frac{\partial V}{\partial r}\frac{dr}{dt}+\frac{\partial V}{\partial h}\frac{dh}{dt}=\frac{2\pi rh}{3}1.8+\frac{\pi r^2}{3}(-2.5)=20,160\pi-12,000\pi=8160\pi$ in^3/ s.

39. **(a)** $V=\ell wh$, so by the Chain Rule,

$$\frac{dV}{dt}=\frac{\partial V}{\partial \ell}\frac{d\ell}{dt}+\frac{\partial V}{\partial w}\frac{dw}{dt}+\frac{\partial V}{\partial h}\frac{dh}{dt}=wh\frac{d\ell}{dt}+\ell h\frac{dw}{dt}+\ell w\frac{dh}{dt}$$

$$=2\cdot 2\cdot 2+1\cdot 2\cdot 2+1\cdot 2\cdot(-3)=6\text{m}^3/\text{s}$$

(b) $S=2(\ell w+\ell h+wh)$, so by the Chain Rule,

$$\frac{dS}{dt}=\frac{\partial S}{\partial \ell}\frac{d\ell}{dt}+\frac{\partial S}{\partial w}\frac{dw}{dt}+\frac{\partial S}{\partial h}\frac{dh}{dt}=2(w+h)\frac{d\ell}{dt}+2(\ell+h)\frac{dw}{dt}+2(\ell+w)\frac{dh}{dt}$$

$$=2(2+2)2+2(1+2)2+2(1+2)(-3)=10\text{m}^2/\text{s}$$

(c) $L^2=\ell^2+w^2+h^2\Rightarrow 2L\frac{dL}{dt}=2\ell\frac{d\ell}{dt}+2w\frac{dw}{dt}+2h\frac{dh}{dt}=2(1)(2)+2(2)(2)+2(2)(-3)=0\Rightarrow dL/dt=0$ m / s.

40. $I=\frac{V}{R}\Rightarrow$

$$\frac{dI}{dt}=\frac{\partial I}{\partial V}\frac{dV}{dt}+\frac{\partial I}{\partial R}\frac{dR}{dt}=\frac{1}{R}\frac{dV}{dt}-\frac{V}{R^2}\frac{dR}{dt}=\frac{1}{R}\frac{dV}{dt}-\frac{I}{R}\frac{dR}{dt}$$

$$=\frac{1}{400}(-0.01)-\frac{0.08}{400}(0.03)=-0.000031\text{ A / s}$$

41. $\frac{dP}{dt}=0.05$, $\frac{dT}{dt}=0.15$, $V=8.31\frac{T}{P}$ and $\frac{dV}{dt}=\frac{8.31}{P}\frac{dT}{dt}-8.31\frac{T}{P^2}\frac{dP}{dt}$. Thus when $P=20$ and $T=320$, $\frac{dV}{dt}=8.31\left[\frac{0.15}{20}-\frac{(0.05)(320)}{400}\right]\approx-0.27$ L / s.

42. Let x and y be the respective distances of car A and car B from the intersection and let z be the distance between the two cars. Then $dx/dt=-90$, $dy/dt=-80$ and $z^2=x^2+y^2$. When $x=0.3$ and $y=0.4$, $z=\sqrt{0.25}=0.5$ and $2z(dz/dt)=2x(dx/dt)+2y(dy/dt)$ or $dz/dt=0.6(-90)+0.8(-80)=-118$ km / h.

43. **(a)** By the Chain Rule, $\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta$, $\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}(-r\sin\theta)+\frac{\partial z}{\partial y}r\cos\theta$.

(b) $\left(\frac{\partial z}{\partial r}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta+2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta+\left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta$,

$\left(\frac{\partial z}{\partial \theta}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta-2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}r^2\cos\theta\sin\theta+\left(\frac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta$. Thus

$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right](\cos^2\theta+\sin^2\theta)=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$.

44. By the Chain Rule, $\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}e^s\cos t+\frac{\partial u}{\partial y}e^s\sin t$, $\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\left(-e^s\sin t\right)+\frac{\partial u}{\partial y}e^s\cos t$. Then

$\left(\frac{\partial u}{\partial s}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2 e^{2s}\cos^2 t+2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}e^{2s}\cos t\sin t+\left(\frac{\partial u}{\partial y}\right)^2 e^{2s}\sin^2 t$ and

$\left(\frac{\partial u}{\partial t}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2 e^{2s}\sin^2 t-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}e^{2s}\cos t\sin t+\left(\frac{\partial u}{\partial y}\right)^2 e^{2s}\sin^2 t$. Thus

$\left[\left(\frac{\partial u}{\partial s}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right]e^{-2s}=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$.

45. Let $u=x-y$. Then $\frac{\partial z}{\partial x}=\frac{dz}{du}\frac{\partial u}{\partial x}=\frac{dz}{du}$ and $\frac{\partial z}{\partial y}=\frac{dz}{du}(-1)$. Thus $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=0$.

46. $\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}$ and $\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}$. Thus $\frac{\partial z}{\partial s}\frac{\partial z}{\partial t}=\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2$.

47. Let $u=x+at$, $v=x-at$. Then $z=f(u)+g(v)$, so $\partial z/\partial u=f'(u)$ and $\partial z/\partial v=g'(v)$.

Thus $\frac{\partial z}{\partial t}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial t}=af'(u)-ag'(v)$ and

$\frac{\partial^2 z}{\partial t^2}=a\frac{\partial}{\partial t}\left[f'(u)-g'(v)\right]=a\left(\frac{df'(u)}{du}\frac{\partial u}{\partial t}-\frac{dg'(v)}{dv}\frac{\partial v}{\partial t}\right)=a^2f''(u)+a^2g''(v)$. Similarly

$\frac{\partial z}{\partial x}=f'(u)+g'(v)$ and $\frac{\partial^2 z}{\partial x^2}=f''(u)+g''(v)$. Thus $\frac{\partial^2 z}{\partial t^2}=a^2\frac{\partial^2 z}{\partial x^2}$.

48. By the Chain Rule, $\frac{\partial u}{\partial s}=e^s\cos t\frac{\partial u}{\partial x}+e^s\sin t\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial t}=-e^s\sin t\frac{\partial u}{\partial x}+e^s\cos t\frac{\partial u}{\partial y}$. Then

$\frac{\partial^2 u}{\partial s^2}=e^s\cos t\frac{\partial u}{\partial x}+e^s\cos t\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)+e^s\sin t\frac{\partial u}{\partial y}+e^s\sin t\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right)$. But

$\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial s}+\frac{\partial^2 u}{\partial y\partial x}\frac{\partial y}{\partial s}=e^s\cos t\frac{\partial^2 u}{\partial x^2}+e^s\sin t\frac{\partial^2 u}{\partial y\partial x}$ and

$\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial s}+\frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial s}=e^s\sin t\frac{\partial^2 u}{\partial y^2}+e^s\cos t\frac{\partial^2 u}{\partial x\partial y}$. Also, by continuity of the partials,

$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$. Thus

$$\begin{aligned}\frac{\partial^2 u}{\partial s^2} &= e^s\cos t\frac{\partial u}{\partial x}+e^s\cos t\left(e^s\cos t\frac{\partial^2 u}{\partial x^2}+e^s\sin t\frac{\partial^2 u}{\partial x\partial y}\right)+e^s\sin t\frac{\partial u}{\partial y}\\ &\quad + e^s\sin t\left(e^s\sin t\frac{\partial^2 u}{\partial y^2}+e^s\cos t\frac{\partial^2 u}{\partial x\partial y}\right)\\ &= e^s\cos t\frac{\partial u}{\partial x}+e^s\sin t\frac{\partial u}{\partial y}+e^{2s}\cos^2 t\frac{\partial^2 u}{\partial x^2}+2e^{2s}\cos t\sin t\frac{\partial^2 u}{\partial x\partial y}+e^{2s}\sin^2 t\frac{\partial^2 u}{\partial y^2}\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= -e^s\cos t\frac{\partial u}{\partial x}-e^s\sin t\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)-e^s\sin t\frac{\partial u}{\partial y}+e^s\cos t\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)\\ &= -e^s\cos t\frac{\partial u}{\partial x}-e^s\sin t\left(-e^s\sin t\frac{\partial^2 u}{\partial x^2}+e^s\cos t\frac{\partial^2 u}{\partial x\partial y}\right)\\ &\quad - e^s\sin t\frac{\partial u}{\partial y}+e^s\cos t\left(e^s\cos t\frac{\partial^2 u}{\partial y^2}-e^s\sin t\frac{\partial^2 u}{\partial x\partial y}\right)\\ &= -e^s\cos t\frac{\partial u}{\partial x}-e^s\sin t\frac{\partial u}{\partial y}+e^{2s}\sin^2 t\frac{\partial^2 u}{\partial x^2}-2e^{2s}\cos t\sin t\frac{\partial^2 u}{\partial x\partial y}+e^{2s}\cos^2 t\frac{\partial^2 u}{\partial y^2}\end{aligned}$$

Thus $e^{-2s}\left(\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}\right)=(\cos^2 t+\sin^2 t)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$, as desired.

49. $\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}2s+\frac{\partial z}{\partial y}2r$. Then

$$\frac{\partial^2 z}{\partial r\,\partial s} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}2s\right)+\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}2r\right)$$

$$= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r}2s+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r}2s+\frac{\partial z}{\partial x}\frac{\partial}{\partial r}2s+\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}2r+\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r}2r+\frac{\partial z}{\partial y}2$$

$$= 4rs\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y\partial x}4s^2+0+4rs\frac{\partial^2 z}{\partial y^2}+\frac{\partial^2 z}{\partial x\partial y}4r^2+2\frac{\partial z}{\partial y}$$

By the continuity of the partials, $\frac{\partial^2 z}{\partial r\partial s}=4rs\frac{\partial^2 z}{\partial x^2}+4rs\frac{\partial^2 z}{\partial y^2}+(4r^2+4s^2)\frac{\partial^2 z}{\partial x\partial y}+2\frac{\partial z}{\partial y}$.

50. By the Chain Rule,

(a) $\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta$

(b) $\frac{\partial z}{\partial \theta}=-\frac{\partial z}{\partial x}r\sin\theta+\frac{\partial z}{\partial y}r\cos\theta$

(c)

$$\frac{\partial^2 z}{\partial r\partial\theta}=\frac{\partial^2 z}{\partial\theta\,\partial r}=\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta\right)$$

$$=-\sin\theta\frac{\partial z}{\partial x}+\cos\theta\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\right)+\cos\theta\frac{\partial z}{\partial y}+\sin\theta\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial y}\right)$$

$$=-\sin\theta\frac{\partial z}{\partial x}+\cos\theta\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial\theta}+\frac{\partial^2 z}{\partial y\partial x}\frac{\partial y}{\partial\theta}\right)+\cos\theta\frac{\partial z}{\partial y}+\sin\theta\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial\theta}+\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial\theta}$$

$$=-\sin\theta\frac{\partial z}{\partial x}+\cos\theta\left(-r\sin\theta\frac{\partial^2 z}{\partial x^2}+r\cos\theta\frac{\partial^2 z}{\partial y\partial x}\right)+\cos\theta\frac{\partial z}{\partial y}$$

$$+\sin\theta\left(r\cos\theta\frac{\partial^2 z}{\partial y^2}-r\sin\theta\frac{\partial^2 z}{\partial x\partial y}\right)$$

$$=-\sin\theta\frac{\partial z}{\partial x}-r\cos\theta\sin\theta\frac{\partial^2 z}{\partial x^2}+r\cos^2\theta\frac{\partial^2 z}{\partial y\partial x}+\cos\theta\frac{\partial z}{\partial y}$$

$$+r\cos\theta\sin\theta\frac{\partial^2 z}{\partial y^2}-r\sin^2\theta\frac{\partial^2 z}{\partial y\partial x}$$

$$=\cos\theta\frac{\partial z}{\partial y}-\sin\theta\frac{\partial z}{\partial x}+r\cos\theta\sin\theta\left(\frac{\partial^2 z}{\partial y^2}-\frac{\partial^2 z}{\partial x^2}\right)+r\left(\cos^2\theta-\sin^2\theta\right)\frac{\partial^2 z}{\partial y\partial x}$$

51. $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cos\theta + \frac{\partial z}{\partial y} \sin\theta$ and $\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x}\, r\sin\theta + \frac{\partial z}{\partial y}\, r\cos\theta$. Then

$$\frac{\partial^2 z}{\partial r^2} = \cos\theta \left(\frac{\partial^2 z}{\partial x^2} \cos\theta + \frac{\partial^2 z}{\partial y \partial x} \sin\theta \right) + \sin\theta \left(\frac{\partial^2 z}{\partial y^2} \sin\theta + \frac{\partial^2 z}{\partial x \partial y} \cos\theta \right)$$

$$= \cos^2\theta\, \frac{\partial^2 z}{\partial x^2} + 2\cos\theta \sin\theta\, \frac{\partial^2 z}{\partial x \partial y} + \sin^2\theta\, \frac{\partial^2 z}{\partial y^2}$$

and

$$\frac{\partial^2 z}{\partial \theta^2} = -r\cos\theta\, \frac{\partial z}{\partial x} + (-r\sin\theta) \left(\frac{\partial^2 z}{\partial x^2} (-r\sin\theta) + \frac{\partial^2 z}{\partial y \partial x}\, r\cos\theta \right)$$

$$-r\sin\theta\, \frac{\partial z}{\partial y} + r\cos\theta \left(\frac{\partial^2 z}{\partial y^2}\, r\cos\theta + \frac{\partial^2 z}{\partial x \partial y} (-r\sin\theta) \right)$$

$$= -r\cos\theta\, \frac{\partial z}{\partial x} - r\sin\theta\, \frac{\partial z}{\partial y} + r^2 \sin^2\theta\, \frac{\partial^2 z}{\partial x^2} - 2r^2 \cos\theta \sin\theta\, \frac{\partial^2 z}{\partial x \partial y} + r^2 \cos^2\theta\, \frac{\partial^2 z}{\partial y^2}$$

Thus

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r} \frac{\partial z}{\partial r} = \left(\cos^2\theta + \sin^2\theta\right) \frac{\partial^2 z}{\partial x^2} + \left(\sin^2\theta + \cos^2\theta\right) \frac{\partial^2 z}{\partial y^2} - \frac{1}{r} \cos\theta\, \frac{\partial z}{\partial x}$$

$$- \frac{1}{r} \sin\theta\, \frac{\partial z}{\partial y} + \frac{1}{r} \left(\cos\theta\, \frac{\partial z}{\partial x} + \sin\theta\, \frac{\partial z}{\partial y} \right)$$

$$= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \quad \text{as desired.}$$

52. **(a)** $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$. Then

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t} \left(\frac{\partial z}{\partial x} \frac{\partial x}{\partial t} \right) + \frac{\partial}{\partial t} \left(\frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \right)$$

$$= \frac{\partial}{\partial t} \left(\frac{\partial z}{\partial x} \right) \frac{\partial x}{\partial t} + \frac{\partial^2 x}{\partial t^2} \frac{\partial z}{\partial x} + \frac{\partial}{\partial t} \left(\frac{\partial z}{\partial y} \right) \frac{\partial y}{\partial t} + \frac{\partial^2 y}{\partial t^2} \frac{\partial z}{\partial y}$$

$$= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial y}{\partial t}\frac{\partial x}{\partial t} + \frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}$$

$$= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x} + \frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}$$

(b)

$$\frac{\partial^2 z}{\partial s \partial t} = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)$$

$$= \left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial s}\right)\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s \partial t} + \left(\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s} + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial s}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s \partial t}$$

$$= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial x \partial y}\left(\frac{\partial y}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial y}{\partial t}\frac{\partial x}{\partial s}\right) + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s \partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s \partial t} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s}\frac{\partial y}{\partial t}$$

53. **(a)** Since f is a polynomial, it has continuous second–order partial derivatives, and

$$f(\text{tx},\text{ty}) = (tx)^2(ty)+2(tx)(ty)^2+5(ty)^3=t^3x^2y+2t^3xy^2+5t^3y^3$$
$$= t^3\left(x^2y+2xy^2+5y^3\right)=t^3 f(\text{x},\text{y})$$

Thus, f is homogeneous of degree 3.

(b) Differentiating both sides of $f(\text{tx},\text{ty})=t^n f(\text{x},\text{y})$ with respect to t using the Chain Rule, we get

$$\frac{\partial}{\partial t} f(\text{tx},\text{ty})= \frac{\partial}{\partial t}\left[t^n f(\text{x},\text{y})\right] \Leftrightarrow$$

$$\frac{\partial}{\partial(tx)} f(\text{tx},\text{ty})\cdot\frac{\partial(tx)}{\partial t} + \frac{\partial}{\partial(ty)} f(\text{tx},\text{ty})\cdot\frac{\partial(ty)}{\partial t} = x\frac{\partial}{\partial(tx)} f(\text{tx},\text{ty})+y\frac{\partial}{\partial(ty)} f(\text{tx},\text{ty})=nt^{n-1}f(\text{x},\text{y})\ .$$

Setting $t=1$: $x\frac{\partial}{\partial x} f(\text{x},\text{y})+y\frac{\partial}{\partial y} f(\text{x},\text{y})=nf(\text{x},\text{y})$.

54. Differentiating both sides of $f(\text{tx},\text{ty})=t^n f(\text{x},\text{y})$ with respect to t using the Chain Rule, we get $\frac{\partial}{\partial(tx)} f(\text{tx},\text{ty})\cdot\frac{\partial(tx)}{\partial t} + \frac{\partial}{\partial(ty)} f(\text{tx},\text{ty})\cdot\frac{\partial(ty)}{\partial t} = x\frac{\partial}{\partial(tx)} f(\text{tx},\text{ty})+y\frac{\partial}{\partial(ty)} f(\text{tx},\text{ty})=nt^{n-1}f(\text{x},\text{y})$ and differentiating again with respect to t gives

$$x\left[\frac{\partial^2}{\partial(tx)^2} f(\text{tx},\text{ty})\cdot\frac{\partial(tx)}{\partial t} + \frac{\partial^2}{\partial(ty)\partial(tx)} f(\text{tx},\text{ty})\cdot\frac{\partial(ty)}{\partial t}\right]$$

$$+\ y\left[\frac{\partial^2}{\partial(tx)\partial(ty)}f(\text{tx,ty})\cdot\frac{\partial(tx)}{\partial t}+\frac{\partial^2}{\partial(ty)^2}f(\text{tx,ty})\cdot\frac{\partial(ty)}{\partial t}\right]=n(n-1)t^{n-1}f(\text{x,y})\ .$$

Setting $t=1$ and using the fact that $f_{yx}=f_{xy}$, we have $x^2f_{xx}+2xyf_{xy}+y^2f_{yy}=n(n-1)f(\text{x,y})$.

55. Differentiating both sides of $f(\text{tx,ty})=t^nf(\text{x,y})$ with respect to x using the Chain Rule, we get

$$\frac{\partial}{\partial x}f(\text{tx,ty})=\frac{\partial}{\partial x}\left[t^nf(\text{x,y})\right]\Leftrightarrow$$

$$\frac{\partial}{\partial(tx)}f(\text{tx,ty})\cdot\frac{\partial(tx)}{\partial x}+\frac{\partial}{\partial(ty)}f(\text{tx,ty})\cdot\frac{\partial(ty)}{\partial x}=t^n\frac{\partial}{\partial x}f(\text{x,y})\Leftrightarrow tf_x(\text{tx,ty})=t^nf_x(\text{x,y})\ .$$

Thus $f_x(\text{tx,ty})=t^{n-1}f_x(\text{x,y})$.

56. $F(\text{x,y,z})=0$ is assumed to define z as a function of x and y , that is, $z=f(\text{x,y})$. So by (7), $\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}$ since $F_z\neq 0$. Similarly, it is assumed that $F(\text{x,y,z})=0$ defines x as a function of y and z , that is $x=h(\text{x,z})$. Then $F(h(\text{y,z}),\text{y,z})=0$ and by the Chain Rule, $F_x\frac{\partial x}{\partial y}+F_y\frac{\partial y}{\partial y}+F_z\frac{\partial z}{\partial y}=0$. But $\frac{\partial z}{\partial y}=0$ and $\frac{\partial y}{\partial y}=1$, so $F_x\frac{\partial x}{\partial y}+F_y=0\Rightarrow\frac{\partial x}{\partial y}=-\frac{F_y}{F_x}$. A similar calculation shows that $\frac{\partial y}{\partial z}=-\frac{F_z}{F_y}$. Thus

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}=\left(-\frac{F_x}{F_z}\right)\left(-\frac{F_y}{F_x}\right)\left(-\frac{F_z}{F_y}\right)=-1\ .$$

1. First we draw a line passing through Raleigh and the eye of the hurricane. We can approximate the directional derivative at Raleigh in the direction of the eye of the hurricane by the average rate of change of pressure between the points where this line intersects the contour lines closest to Raleigh. In the direction of the eye of the hurricane, the pressure changes from 996 millibars to 992 millibars. We estimate the distance between these two points to be approximately 40 miles, so the rate of change of pressure in the direction given is approximately $\frac{992-996}{40}=-0.1$ millibar / mi.

2. First we draw a line passing through Muskegon and Ludington. We approximate the directional derivative at Muskegon in the direction of Ludington by the average rate of change of snowfall between the points where the line
intersects the contour lines closest to Muskegon. In the direction of Ludington, the snowfall changes from 60 to 70 inches. We estimate the distance between these two points to be approximately 28 miles, so the rate of change of annual snowfall in the direction given is approximately $\frac{70-60}{28}\approx 0.36$ in / mi.

3. $D_{\boldsymbol{u}}f(-20,30)=\nabla f(-20,30)\cdot \boldsymbol{u}=f_T(-20,30)\left(\frac{1}{\sqrt{2}}\right)+f_v(-20,30)\left(\frac{1}{\sqrt{2}}\right)$.

$f_T(-20,30)=\lim_{h\to 0}\frac{f(-20+h,30)-f(-20,30)}{h}$, so we can approximate $f_T(-20,30)$ by considering $h=\pm 5$ and using the values given in the table: $f_T(-20,30)\approx\frac{f(-15,30)-f(-20,30)}{5}=\frac{-26-(-33)}{5}=1.4$,

$f_T(-20,30)\approx\frac{f(-25,30)-f(-20,30)}{-5}=\frac{-39-(-33)}{-5}=1.2$. Averaging these values gives $f_T(-20,30)\approx 1.3$.

Similarly, $f_v(-20,30)=\lim_{h\to 0}\frac{f(-20,30+h)-f(-20,30)}{h}$, so we can approximate $f_v(-20,30)$ with $h=\pm 10$:

$f_v(-20,30)\approx\frac{f(-20,40)-f(-20,30)}{10}=\frac{-34-(-33)}{10}=-0.1$,

$f_v(-20,30)\approx\frac{f(-20,20)-f(-20,30)}{-10}=\frac{-30-(-33)}{-10}=-0.3$. Averaging these values gives $f_v(-20,30)\approx -0.2$. Then $D_{\boldsymbol{u}}f(-20,30)\approx 1.3\left(\frac{1}{\sqrt{2}}\right)+(-0.2)\left(\frac{1}{\sqrt{2}}\right)\approx 0.778$.

4. $f(x,y)=x^2y^3-y^4\Rightarrow f_x(x,y)=2xy^3$ and $f_y(x,y)=3x^2y^2-4y^3$. If $\boldsymbol{u}$ is a unit vector in the direction of $\theta=\frac{\pi}{4}$, then from Equation 6,

$D_{\boldsymbol{u}}f(2,1)=f_x(2,1)\cos\left(\frac{\pi}{4}\right)+f_y(2,1)\sin\left(\frac{\pi}{4}\right)=4\cdot\frac{\sqrt{2}}{2}+8\cdot\frac{\sqrt{2}}{2}=6\sqrt{2}$.

5. $f(x,y)=\sqrt{5x-4y} \Rightarrow f_x(x,y)=\frac{1}{2}(5x-4y)^{-1/2}(5)=\frac{5}{2\sqrt{5x-4y}}$ and $f_y(x,y)=\frac{1}{2}(5x-4y)^{-1/2}(-4)=-\frac{2}{\sqrt{5x-4y}}$. If $\boldsymbol{u}$ is a unit vector in the direction of $\theta=-\frac{\pi}{6}$, then from Equation 6,

$D_{\boldsymbol{u}}f(4,1)=f_x(4,1)\cos\left(-\frac{\pi}{6}\right)+f_y(4,1)\sin\left(-\frac{\pi}{6}\right)=\frac{5}{8}\cdot\frac{\sqrt{3}}{2}+\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)=\frac{5\sqrt{3}}{16}+\frac{1}{4}$.

6. $f(x,y)=x\sin(xy) \Rightarrow f_x(x,y)=x\cos(xy)\cdot y+\sin(xy)=xy\cos(xy)+\sin(xy)$ and

$f_y(x,y)=x\cos(xy)\cdot x=x^2\cos(xy)$. If $\boldsymbol{u}$ is a unit vector in the direction of $\theta=\frac{\pi}{3}$, then from Equation 6,

$D_{\boldsymbol{u}}f(2,0)=f_x(2,0)\cos\frac{\pi}{3}+f_y(2,0)\sin\frac{\pi}{3}=0+4\left(\frac{\sqrt{3}}{2}\right)=2\sqrt{3}$.

7. $f(x,y)=5xy^2-4x^3y$

(a) $\nabla f(x,y)=\left\langle f_x(x,y),f_y(x,y)\right\rangle=\left\langle 5y^2-12x^2y,10xy-4x^3\right\rangle$

(b) $\nabla f(1,2)=\left\langle 5(2)^2-12(1)^2(2),10(1)(2)-4(1)^3\right\rangle=\langle -4,16\rangle$

(c) By Equation 9, $D_{\boldsymbol{u}}f(1,2)=\nabla f(1,2)\cdot \boldsymbol{u}=\langle -4,16\rangle\cdot\left\langle\frac{5}{13},\frac{12}{13}\right\rangle=(-4)\left(\frac{5}{13}\right)+(16)\left(\frac{12}{13}\right)=\frac{172}{13}$

.

8. **(a)** $\nabla f(x,y)=\left\langle f_x(x,y),f_y(x,y)\right\rangle=\langle y/x,\ln x\rangle$

(b) $\nabla f(1,-3)=\left\langle\frac{-3}{1},\ln 1\right\rangle=\langle -3,0\rangle$

(c) By Equation 9, $D_{\boldsymbol{u}}f(1,-3)=\nabla f(1,-3)\cdot \boldsymbol{u}=\langle -3,0\rangle\cdot\left\langle-\frac{4}{5},\frac{3}{5}\right\rangle=\frac{12}{5}$.

9. $f(x,y,z)=xe^{2yz}$

(a) $\nabla f(x,y,z)=\left\langle f_x(x,y,z),f_y(x,y,z),f_z(x,y,z)\right\rangle=\left\langle e^{2yz},2xze^{2yz},2xye^{2yz}\right\rangle$

(b) $\nabla f(3,0,2)=\langle 1,12,0\rangle$

(c) By Equation 14, $D_{\boldsymbol{u}}f(3,0,2)=\nabla f(3,0,2)\cdot \boldsymbol{u}=\langle 1,12,0\rangle\cdot\left\langle\frac{2}{3},-\frac{2}{3},\frac{1}{3}\right\rangle=\frac{2}{3}-\frac{24}{3}+0=-\frac{22}{3}$.

10.

$f(\mathrm{x,y,z})=\sqrt{x+yz}=(x+yz)^{1/2}$

(a)

$$\nabla f(\mathrm{x,y,z})=\left\langle \frac{1}{2}(x+yz)^{-1/2}(1), \frac{1}{2}(x+yz)^{-1/2}(z), \frac{1}{2}(x+yz)^{-1/2}(y)\right\rangle$$

$$=\left\langle 1/\left(2\sqrt{x+yz}\right), \mathrm{z}/\left(2\sqrt{x+yz}\right), \mathrm{y}/\left(2\sqrt{x+yz}\right)\right\rangle$$

(b) $\nabla f(1,3,1)=\left\langle \frac{1}{4},\frac{1}{4},\frac{3}{4}\right\rangle$

(c) $D_{\boldsymbol{u}}f(1,3,1)=\nabla f(1,3,1)\cdot \boldsymbol{u}=\left\langle \frac{1}{4},\frac{1}{4},\frac{3}{4}\right\rangle\cdot\left\langle \frac{2}{7},\frac{3}{7},\frac{6}{7}\right\rangle=\frac{2}{28}+\frac{3}{28}+\frac{18}{28}=\frac{23}{28}$.

11. $f(\mathrm{x,y})=1+2x\sqrt{y}\Rightarrow\nabla f(\mathrm{x,y})=\left\langle 2\sqrt{y},2x\cdot\frac{1}{2}y^{-1/2}\right\rangle=\left\langle 2\sqrt{y},\mathrm{x}/\sqrt{y}\right\rangle$, $\nabla f(3,4)=\left\langle 4,\frac{3}{2}\right\rangle$, and a unit vector in the direction of $\boldsymbol{v}$ is $\boldsymbol{u}=\frac{1}{\sqrt{4^2+(-3)^2}}\langle 4,-3\rangle=\left\langle \frac{4}{5},-\frac{3}{5}\right\rangle$, so

$D_{\boldsymbol{u}}f(3,4)=\nabla f(3,4)\cdot\boldsymbol{u}=\left\langle 4,\frac{3}{2}\right\rangle\cdot\left\langle \frac{4}{5},-\frac{3}{5}\right\rangle=\frac{23}{10}$.

12. $f(\mathrm{x,y})=\ln\left(x^2+y^2\right)\Rightarrow\nabla f(\mathrm{x,y})=\left\langle \frac{2x}{x^2+y^2},\frac{2y}{x^2+y^2}\right\rangle$, $\nabla f(2,1)=\left\langle \frac{4}{5},\frac{2}{5}\right\rangle$, and a unit vector in the direction of $\boldsymbol{v}=\langle -1,2\rangle$ is $\boldsymbol{u}=\frac{1}{\sqrt{1+4}}\langle -1,2\rangle=\left\langle -\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right\rangle$, so

$D_{\boldsymbol{u}}f(2,1)=\nabla f(2,1)\cdot\boldsymbol{u}=\left\langle \frac{4}{5},\frac{2}{5}\right\rangle\cdot\left\langle -\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right\rangle=-\frac{4}{5\sqrt{5}}+\frac{4}{5\sqrt{5}}=0$.

13. $g(\mathrm{s,t})=s^2e^t\Rightarrow\nabla g(\mathrm{s,t})=2se^t\boldsymbol{i}+s^2e^t\boldsymbol{j}$, $\nabla g(2,0)=4\boldsymbol{i}+4\boldsymbol{j}$, and a unit vector in the direction of $\boldsymbol{v}$ is $\boldsymbol{u}=\frac{1}{\sqrt{2}}(\boldsymbol{i}+\boldsymbol{j})$, so $D_{\boldsymbol{u}}g(2,0)=\nabla g(2,0)\cdot\boldsymbol{u}=(4\boldsymbol{i}+4\boldsymbol{j})\cdot\frac{1}{\sqrt{2}}(\boldsymbol{i}+\boldsymbol{j})=\frac{8}{\sqrt{2}}=4\sqrt{2}$.

14. $g(\mathrm{r},\theta)=e^{-r}\sin\theta\Rightarrow\nabla g(\mathrm{r},\theta)=\left(-e^{-r}\sin\theta\right)\boldsymbol{i}+\left(e^{-r}\cos\theta\right)\boldsymbol{j}$, $\nabla g\left(0,\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}\boldsymbol{i}+\frac{1}{2}\boldsymbol{j}$, and a unit vector in the direction of $\boldsymbol{v}$ is

$\boldsymbol{u}=\frac{1}{\sqrt{13}}\ (3\boldsymbol{i}-2\boldsymbol{j})$, so

$D_{\boldsymbol{u}}g\left(0,\frac{\pi}{3}\right)=\nabla g\left(0,\frac{\pi}{3}\right)\cdot\boldsymbol{u}=\left(-\frac{\sqrt{3}}{2}\ \boldsymbol{i}+\frac{1}{2}\ \boldsymbol{j}\right)\cdot\frac{1}{\sqrt{13}}\ (3\boldsymbol{i}-2\boldsymbol{j})=-\frac{3\sqrt{3}}{2\sqrt{13}}-\frac{1}{\sqrt{13}}=-\frac{3\sqrt{3}+2}{2\sqrt{13}}$.

15. $f(x,y,z)=\sqrt{x^2+y^2+z^2}\Rightarrow\nabla f(x,y,z)=\left\langle\frac{x}{\sqrt{x^2+y^2+z^2}},\frac{y}{\sqrt{x^2+y^2+z^2}},\frac{z}{\sqrt{x^2+y^2+z^2}}\right\rangle$,

$\nabla f(1,2,-2)=\left\langle\frac{1}{3},\frac{2}{3},-\frac{2}{3}\right\rangle$, and a unit vector in the direction of $\boldsymbol{v}$ is

$\boldsymbol{u}=\frac{1}{9}\ \langle-6,6,-3\rangle=\left\langle-\frac{2}{3},\frac{2}{3},-\frac{1}{3}\right\rangle$, so

$D_{\boldsymbol{u}}f(1,2,-2)=\nabla f(1,2,-2)\cdot\boldsymbol{u}=\left\langle\frac{1}{3},\frac{2}{3},-\frac{2}{3}\right\rangle\cdot\left\langle-\frac{2}{3},\frac{2}{3},-\frac{1}{3}\right\rangle=\frac{4}{9}$.

16. $f(x,y,z)=\frac{x}{y+z}\Rightarrow\nabla f(x,y,z)=\left\langle\frac{1}{y+z},-\frac{x}{(y+z)^2},-\frac{x}{(y+z)^2}\right\rangle$, $\nabla f(4,1,1)=\left\langle\frac{1}{2},-1,-1\right\rangle$, and a unit vector in the direction of $\boldsymbol{v}$ is $\boldsymbol{u}=\frac{1}{\sqrt{14}}\ \langle 1,2,3\rangle$, so

$D_{\boldsymbol{u}}f(4,1,1)=\nabla f(4,1,1)\cdot\boldsymbol{u}=\left\langle\frac{1}{2},-1,-1\right\rangle\cdot\frac{1}{\sqrt{14}}\ \langle 1,2,3\rangle=-\frac{9}{2\sqrt{14}}$.

17. $g(x,y,z)=(x+2y+3z)^{3/2}\Rightarrow$

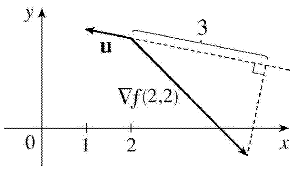
$\nabla g(x,y,z)=\left\langle\frac{3}{2}\,(x+2y+3z)^{1/2}(1),\frac{3}{2}\,(x+2y+3z)^{1/2}(2),\frac{3}{2}\,(x+2y+3z)^{1/2}(3)\right\rangle$

$=\left\langle\frac{3}{2}\sqrt{x+2y+3z},3\sqrt{x+2y+3z},\frac{9}{2}\sqrt{x+2y+3z}\right\rangle$, $\nabla g(1,1,2)=\left\langle\frac{9}{2},9,\frac{27}{2}\right\rangle$,

and a unit vector in the direction of $\boldsymbol{v}=2\boldsymbol{j}-\boldsymbol{k}$ is $\boldsymbol{u}=\frac{2}{\sqrt{5}}\ \boldsymbol{j}-\frac{1}{\sqrt{5}}\ \boldsymbol{k}$, so

$D_{\boldsymbol{u}}g(1,1,2)=\left\langle\frac{9}{2},9,\frac{27}{2}\right\rangle\cdot\left\langle 0,\frac{2}{\sqrt{5}},-\frac{1}{\sqrt{5}}\right\rangle=\frac{18}{\sqrt{5}}-\frac{27}{2\sqrt{5}}=\frac{9}{2\sqrt{5}}$.

18. $D_{\boldsymbol{u}} f(2{,}2)=\nabla f(2{,}2)\cdot \boldsymbol{u}$, the scalar projection of $\nabla f(2{,}2)$ onto $\boldsymbol{u}$, so we draw a perpendicular from the tip of $\nabla f(2{,}2)$ to the line containing $\boldsymbol{u}$. We can use the point (2,2) to determine the scale of the axes, and we estimate the length of



the projection to be approximately 3.0 units. Since the angle between $\nabla f(2{,}2)$ and $\boldsymbol{u}$ is greater than $90°$, the scalar projection is negative. Thus $D_{\boldsymbol{u}} f(2{,}2)\approx -3$.

19. $f(x,y)=\sqrt{xy} \Rightarrow \nabla f(x,y)=\left\langle \frac{1}{2}(xy)^{-1/2}(y), \frac{1}{2}(xy)^{-1/2}(x) \right\rangle = \left\langle \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}} \right\rangle$, so $\nabla f(2{,}8)=\left\langle 1, \frac{1}{4} \right\rangle$. The unit vector in the direction of $\overrightarrow{PQ}=\langle 5-2,4-8\rangle=\langle 3,-4\rangle$ is $\boldsymbol{u}=\left\langle \frac{3}{5},-\frac{4}{5} \right\rangle$, so $D_{\boldsymbol{u}} f(2{,}8)=\nabla f(2{,}8)\cdot \boldsymbol{u}=\left\langle 1,\frac{1}{4} \right\rangle \cdot \left\langle \frac{3}{5},-\frac{4}{5} \right\rangle = \frac{2}{5}$.

20. $f(x,y,z)=x^2+y^2+z^2 \Rightarrow \nabla f(x,y,z)=\langle 2x,2y,2z\rangle$, so $\nabla f(2{,}1{,}3)=\langle 4,2,6\rangle$. The unit vector in the direction of $\overrightarrow{PO}=\langle -2,-1,-3\rangle$ is $\boldsymbol{u}=\frac{1}{\sqrt{14}}\langle -2,-1,-3\rangle$, so $D_{\boldsymbol{u}} f(2{,}1{,}3)=\nabla f(2{,}1{,}3)\cdot \boldsymbol{u}=\langle 4,2,6\rangle \cdot \frac{1}{\sqrt{14}}\langle -2,-1,-3\rangle=-\frac{28}{\sqrt{14}}=-2\sqrt{14}$.

21. $f(x,y)=y^2/x=y^2x^{-1} \Rightarrow \nabla f(x,y)=\langle -y^2x^{-2},2yx^{-1}\rangle=\langle -y^2/x^2,2y/x\rangle$.
$\nabla f(2{,}4)=\langle -4,4\rangle$, or equivalently $\langle -1,1\rangle$, is the direction of maximum rate of change, and the maximum rate is $|\nabla f(2{,}4)|=\sqrt{16+16}=4\sqrt{2}$.

22. $f(p,q)=qe^{-p}+pe^{-q} \Rightarrow \nabla f(p,q)=\langle -qe^{-p}+e^{-q},e^{-p}-pe^{-q}\rangle$.
$\nabla f(0{,}0)=\langle 1,1\rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(0{,}0)|=\sqrt{2}$.

23. $f(x,y)=\sin(xy) \Rightarrow \nabla f(x,y)=\langle y\cos(xy),x\cos(xy)\rangle$, $\nabla f(1{,}0)=\langle 0,1\rangle$. Thus the maximum rate of change is $|\nabla f(1{,}0)|=1$ in the direction $\langle 0,1\rangle$.

24. $f(x,y,z)=x^2y^3z^4 \Rightarrow \nabla f(x,y,z)=\langle 2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3\rangle$, $\nabla f(1,1,1)=\langle 2,3,4\rangle$. Thus the maximum rate of change is $|\nabla f(1,1,1)|=\sqrt{29}$ in the direction $\langle 2,3,4\rangle$.

25. $f(x,y,z)=\ln(xy^2z^3) \Rightarrow \nabla f(x,y,z)=\left\langle \frac{y^2z^3}{xy^2z^3}, \frac{2xyz^3}{xy^2z^3}, \frac{3xy^2z^2}{xy^2z^3} \right\rangle = \left\langle \frac{1}{x}, \frac{2}{y}, \frac{3}{z} \right\rangle$.

$\nabla f(1,-2,-3)=\langle 1,-1,-1\rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(1,-2,-3)|=\sqrt{3}$.

26. $f(x,y,z)=\tan(x+2y+3z) \Rightarrow \nabla f(x,y,z)=\langle \sec^2(x+2y+3z)(1), \sec^2(x+2y+3z)(2), \sec^2(x+2y+3z)(3)\rangle$.

$\nabla f(-5,1,1)=\langle \sec^2(0), 2\sec^2(0), 3\sec^2(0)\rangle=\langle 1,2,3\rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(-5,1,1)|=\sqrt{14}$.

27. **(a)** As in the proof of Theorem 15, $D_{\boldsymbol{u}}f=|\nabla f|\cos\theta$. Since the minimum value of $\cos\theta$ is -1 occurring when $\theta=\pi$, the minimum value of $D_{\boldsymbol{u}}f$ is $-|\nabla f|$ occurring when $\theta=\pi$, that is when $\boldsymbol{u}$ is in the opposite direction of ∇f (assuming $\nabla f\neq\mathbf{0}$).

(b) $f(x,y)=x^4y-x^2y^3 \Rightarrow \nabla f(x,y)=\langle 4x^3y-2xy^3, x^4-3x^2y^2\rangle$, so f decreases fastest at the point (2,–3) in the direction $-\nabla f(2,-3)=-\langle 12,-92\rangle=\langle -12,92\rangle$.

28. $f(x,y)=x^2+\sin xy \Rightarrow f_x(x,y)=2x+y\cos xy$, $f_y(x,y)=x\cos xy$ and $f_x(1,0)=2(1)+(0)\cos 0=2$, $f_y(1,0)=(1)\cos 0=1$. If $\boldsymbol{u}$ is a unit vector which makes an angle θ with the positive x –axis, then $D_{\boldsymbol{u}}f(1,0)=f_x(1,0)\cos\theta+f_y(1,0)\sin\theta=2\cos\theta+\sin\theta$. We want $D_{\boldsymbol{u}}f(1,0)=1$, so $2\cos\theta+\sin\theta=1 \Rightarrow$
$\sin\theta=1-2\cos\theta \Rightarrow \sin^2\theta=(1-2\cos\theta)^2 \Rightarrow 1-\cos^2\theta=1-4\cos\theta+4\cos^2\theta \Rightarrow 5\cos^2\theta-4\cos\theta=0 \Rightarrow$
$\cos\theta(5\cos\theta-4)=0 \Rightarrow$
$\cos\theta=0$ or $\cos\theta=\frac{4}{5} \Rightarrow \theta=\frac{\pi}{2}$ or $\theta=2\pi-\cos^{-1}\left(\frac{4}{5}\right)\approx 5.64$.

29. The direction of fastest change is $\nabla f(x,y)=(2x-2)\boldsymbol{i}+(2y-4)\boldsymbol{j}$, so we need to find all points (x,y) where $\nabla f(x,y)$ is parallel to $\boldsymbol{i}+\boldsymbol{j} \Leftrightarrow (2x-2)\boldsymbol{i}+(2y-4)\boldsymbol{j}= k(\boldsymbol{i}+\boldsymbol{j}) \Leftrightarrow k=2x-2$ and $k=2y-4$. Then $2x-2=2y-4 \Rightarrow y=x+1$, so the direction of fastest change is $\boldsymbol{i}+\boldsymbol{j}$ at all points on the line $y=x+1$.

30. The fisherman is traveling in the direction $\langle -80,-60\rangle$. A unit vector in this direction is

$\boldsymbol{u}=\frac{1}{100}\langle -80,-60\rangle=\left\langle -\frac{4}{5},-\frac{3}{5}\right\rangle$, and if the depth of the lake is given by $f(x,y)=200+0.02x^2-0.001y^3$, then $\nabla f(x,y)=\langle 0.04x,-0.003y^2\rangle$.
$D_{\boldsymbol{u}}f(80,60)=\nabla f(80,60)\cdot \boldsymbol{u}=\langle 3.2,-10.8\rangle\cdot \left\langle -\frac{4}{5},-\frac{3}{5}\right\rangle=3.92$. Since $D_{\boldsymbol{u}}f(80,60)$ is positive, the depth of the lake is increasing near (80,60) in the direction toward the buoy.

31. $T=\frac{k}{\sqrt{x^2+y^2+z^2}}$ and $120=T(1,2,2)=\frac{k}{3}$ so $k=360$.

(a) $\boldsymbol{u}=\frac{\langle 1,-1,1\rangle}{\sqrt{3}}$,

$$D_{\boldsymbol{u}}T(1,2,2) = \nabla T(1,2,2)\cdot \boldsymbol{u}=\left[-360\left(x^2+y^2+z^2\right)^{-3/2}\langle x,y,z\rangle\right]_{(1,2,2)}\cdot \boldsymbol{u}$$
$$= -\frac{40}{3}\langle 1,2,2\rangle\cdot \frac{1}{\sqrt{3}}\langle 1,-1,1\rangle=-\frac{40}{3\sqrt{3}}$$

(b) From (a), $\nabla T=-360\left(x^2+y^2+z^2\right)^{-3/2}\langle x,y,z\rangle$, and since $\langle x,y,z\rangle$ is the position vector of the point (x,y,z) , the vector $-\langle x,y,z\rangle$, and thus ∇T , always points toward the origin.

32. $\nabla T=-400e^{-x^2-3y^2-9z^2}\langle x,3y,9z\rangle$
(a) $\boldsymbol{u}=\frac{1}{\sqrt{6}}\langle 1,-2,1\rangle$, $\nabla T(2,-1,2)=-400e^{-43}\langle 2,-3,18\rangle$ and
$$D_{\boldsymbol{u}}T(2,-1,2)=\left(-\frac{400e^{-43}}{\sqrt{6}}\right)(26)=-\frac{5200\sqrt{6}}{3e^{43}}\ \square/\text{ m.}$$
(b) $\nabla T(2,-1,2)=400e^{-43}\langle -2,3,-18\rangle$ or equivalently $\langle -2,3,-18\rangle$.
(c) $|\nabla T|=400e^{-x^2-3y^2-9z^2}\sqrt{x^2+9y^2+81z^2}$ °C/m is the maximum rate of increase. At $(2,-1,2)$ the maximum rate of increase is $400e^{-43}\sqrt{337}$ °C/m.
33. $\nabla V(x,y,z)=\langle 10x-3y+yz,xz-3x,xy\rangle$, $\nabla V(3,4,5)=\langle 38,6,12\rangle$
(a) $D_{\boldsymbol{u}}V(3,4,5)=\langle 38,6,12\rangle\cdot \frac{1}{\sqrt{3}}\langle 1,1,-1\rangle=\frac{32}{\sqrt{3}}$
(b) $\nabla V(3,4,5)=\langle 38,6,12\rangle$ or equivalently $\langle 19,3,6\rangle$.

(c) $|\nabla V(3,4,5)|=\sqrt{38^2+6^2+12^2}=\sqrt{1624}=2\sqrt{406}$

34. $z=f(x,y)=1000-0.01x^2-0.02y^2 \Rightarrow \nabla f(x,y)=\langle -0.02x,-0.04y\rangle$ and $\nabla f(50,80)=\langle -1,-3.2\rangle$

(a) Due south is in the direction of the unit vector $\boldsymbol{u}=-\boldsymbol{j}$ and $D_{\boldsymbol{u}}f(50,80)=\nabla f(50,80)\cdot\langle 0,-1\rangle=\langle -1,-3.2\rangle\cdot\langle 0,-1\rangle=0+3.2=3.2$. Thus, if you walk due south from $(50,80,847)$ you will ascend at a rate of 3.2 vertical meters per horizontal meter.

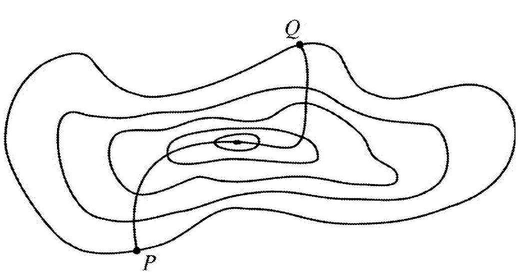
(b) Northwest is in the direction of the unit vector $\boldsymbol{u}=\frac{1}{\sqrt{2}}\langle -1,1\rangle$ and

$D_{\boldsymbol{u}}f(50,80)=\nabla f(50,80)\cdot\frac{1}{\sqrt{2}}\langle -1,1\rangle=\langle -1,-3.2\rangle\cdot\frac{1}{\sqrt{2}}\langle -1,1\rangle=-\frac{2.2}{\sqrt{2}}\approx -1.56$. Thus, if you walk northwest from $(50,80,847)$ you will descend at a rate of approximately 1.56 vertical meters per horizontal meter.

(c) $\nabla f(50,80)=\langle -1,-3.2\rangle$ is the direction of largest slope with a rate of ascent $|\nabla f(50,80)|=\sqrt{11.24}\approx 3.35$. The angle above the horizontal in which the path begins is given by $\tan\theta\approx 3.35\Rightarrow\theta\approx\tan^{-1}(3.35)\approx 73.4°$.

35. A unit vector in the direction of $\overrightarrow{AB}$ is $\boldsymbol{i}$ and a unit vector in the direction of $\overrightarrow{AC}$ is $\boldsymbol{j}$. Thus $D_{\overrightarrow{AB}}f(1,3)=f_x(1,3)=3$ and $D_{\overrightarrow{AC}}f(1,3)=f_y(1,3)=26$. Therefore $\nabla f(1,3)=\left\langle f_x(1,3),f_y(1,3)\right\rangle=\langle 3,26\rangle$, and by definition, $D_{\overrightarrow{AD}}f(1,3)=\nabla f\cdot\boldsymbol{u}$ where $\boldsymbol{u}$ is a unit vector in the direction of $\overrightarrow{AD}$, which is $\left\langle \frac{5}{13},\frac{12}{13}\right\rangle$. Therefore, $D_{\overrightarrow{AD}}f(1,3)=\langle 3,26\rangle\cdot\left\langle \frac{5}{13},\frac{12}{13}\right\rangle=3\cdot\frac{5}{13}+26\cdot\frac{12}{13}=\frac{327}{13}$.

36. The curve of steepest ascent is perpendicular to all of the contour lines.



37. **(a)**

$$\nabla(au+bv)=\left\langle \frac{\partial(au+bv)}{\partial x},\frac{\partial(au+bv)}{\partial y}\right\rangle=\left\langle a\frac{\partial u}{\partial x}+b\frac{\partial v}{\partial x},a\frac{\partial u}{\partial y}+b\frac{\partial v}{\partial y}\right\rangle$$

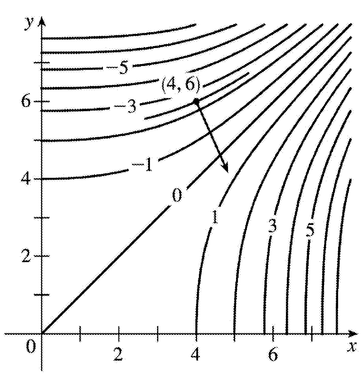
$$=a\left\langle \frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right\rangle+b\left\langle \frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right\rangle=a\nabla u+b\nabla v$$

(b) $\nabla(uv)=\left\langle v\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial x},v\frac{\partial u}{\partial y}+u\frac{\partial v}{\partial y}\right\rangle=v\left\langle \frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right\rangle+u\left\langle \frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right\rangle=v\nabla u+u\nabla v$

(c) $\nabla\left(\frac{u}{v}\right)=\left\langle \frac{v\frac{\partial u}{\partial x}-u\frac{\partial v}{\partial x}}{v^2},\frac{v\frac{\partial u}{\partial y}-u\frac{\partial v}{\partial y}}{v^2}\right\rangle=\frac{v\left\langle \frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right\rangle-u\left\langle \frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right\rangle}{v^2}=\frac{v\nabla u-u\nabla v}{v^2}$

(d) $\nabla u^n=\left\langle \frac{\partial\left(u^n\right)}{\partial x},\frac{\partial\left(u^n\right)}{\partial y}\right\rangle=\left\langle nu^{n-1}\frac{\partial u}{\partial x},nu^{n-1}\frac{\partial u}{\partial y}\right\rangle=nu^{n-1}\nabla u$.

38. If we place the initial point of the gradient vector $\nabla f(4,6)$ at $(4,6)$, the vector is perpendicular to the level curve of f that includes $(4,6)$, so we sketch a portion of the level curve through $(4,6)$ (using the nearby level curves as a
guideline) and draw a line perpendicular to the curve at $(4,6)$. The gradient vector is parallel to this line, pointing in the direction of increasing function values, and with length equal to the maximum value of the directional derivative of f at $(4,6)$. We can estimate this length by finding the average rate of change in the direction of the gradient. The line intersects the contour lines corresponding to -2 and -3 with an estimated distance of 0.5 units. Thus the rate of change is approxi–
mately $\frac{-2-(-3)}{0.5}=2$, and we sketch the gradient vector with length 2 .



39. Let $F(x,y,z)=x^2+2y^2+3z^2$. Then $x^2+2y^2+3z^2=21$ is a level surface of F . $F_x(x,y,z)=2x\Rightarrow$ $F_x(4,-1,1)=8$, $F_y(x,y,z)=4y\Rightarrow F_y(4,-1,1)=-4$, and $F_z(x,y,z)=6z\Rightarrow F_z(4,-1,1)=6$.

(a) Equation 19 gives an equation of the tangent plane at $(4,-1,1)$ as $8(x-4)-4[y-(-1)]+6(z-1)=0$ or $4x-2y+3z=21$.

(b) By Equation 20, the normal line has symmetric equations
$\frac{x-4}{8}=\frac{y+1}{-4}=\frac{z-1}{6}$ or $\frac{x-4}{4}=\frac{y+1}{-2}=\frac{z-1}{3}$.

40. Let $F(x,y,z)=y^2+z^2-x$. Then $x=y^2+z^2-2$ is the level surface $F(x,y,z)=2$.
$F_x(x,y,z)=-1 \Rightarrow F_x(-1,1,0)=-1$, $F_y(x,y,z)=2y \Rightarrow F_y(-1,1,0)=2$,
and $F_z(x,y,z)=2z \Rightarrow F_z(-1,1,0)=0$.
(a) An equation of the tangent plane is $-1(x+1)+2(y-1)+0(z-0)=0$ or $-x+2y=3$.
(b) The normal line has symmetric equations $\frac{x+1}{-1}=\frac{y-1}{2}$, $z=0$.

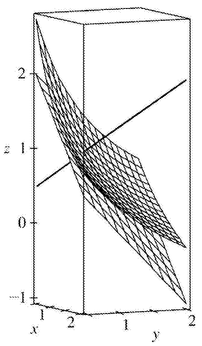
41. Let $F(x,y,z)=x^2-2y^2+z^2+yz$. Then $x^2-2y^2+z^2+yz=2$ is a level surface of F and
$\nabla F(x,y,z)=\langle 2x,-4y+z,2z+y\rangle$.
(a) $\nabla F(2,1,-1)=\langle 4,-5,-1\rangle$ is a normal vector for the tangent plane at $(2,1,-1)$, so an equation of the tangent plane is $4(x-2)-5(y-1)-1(z+1)=0$ or $4x-5y-z=4$.
(b) The normal line has direction $\langle 4,-5,-1\rangle$, so parametric equations are $x=2+4t$, $y=1-5t$, $z=-1-t$, and symmetric equations are $\frac{x-2}{4}=\frac{y-1}{-5}=\frac{z+1}{-1}$.

42. Let $F(x,y,z)=x-z-4\arctan(yz)$. Then $x-z=4\arctan(yz)$ is the level surface $F(x,y,z)=0$, and
$\nabla F(x,y,z)=\left\langle 1,-\frac{4z}{1+y^2z^2},-1-\frac{4y}{1+y^2z^2}\right\rangle$.
(a) $\nabla F(1+\pi,1,1)=\langle 1,-2,-3\rangle$ and an equation of the tangent plane is $1(x-(1+\pi))-2(y-1)-3(z-1)=0$ or $x-2y-3z=-4+\pi$.
(b) The normal line has direction $\langle 1,-2,-3\rangle$, so parametric equations are $x=1+\pi+t$, $y=1-2t$, $z=1-3t$, and symmetric equations are $x-1-\pi=\frac{y-1}{-2}=\frac{z-1}{-3}$.

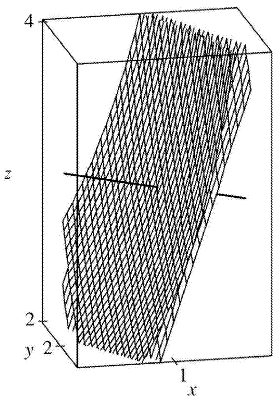
43. $F(x,y,z)=-z+xe^y\cos z \Rightarrow \nabla F(x,y,z)=\langle e^y\cos z, xe^y\cos z,-1-xe^y\sin z\rangle$, $\nabla F(1,0,0)=\langle 1,1,-1\rangle$
(a) $1(x-1)+1(y-0)-1(z-0)=0$ or $x+y-z=1$
(b) $x-1=y=-z$

44. $F(x,y,z)=yz-\ln(x+z) \Rightarrow \nabla F(x,y,z)=\left\langle -\frac{1}{x+z},z,y-\frac{1}{x+z}\right\rangle$, $\nabla F(0,0,1)=\langle -1,1,-1\rangle$.
(a) $(-1)(x-0)+(1)(y-0)-1(z-1)=0$ or $x-y+z=1$.
(b) Parametric equations are $x=-t$, $y=t$, $z=1-t$ and symmetric equations are $\frac{x}{-1}=\frac{y}{1}=\frac{z-1}{-1}$ or $-x=y=1-z$.

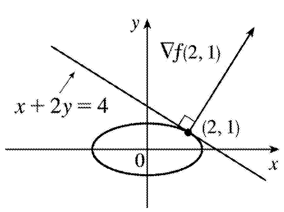
45. $F(x,y,z)=xy+yz+zx$, $\nabla F(x,y,z)=\langle y+z,x+z,y+x\rangle$, $\nabla F(1,1,1)=\langle 2,2,2\rangle$, so an equation of the tangent plane is $2x+2y+2z=6$ or $x+y+z=3$, and the normal line is given by $x-1=y-1=z-1$ or $x=y=z$.



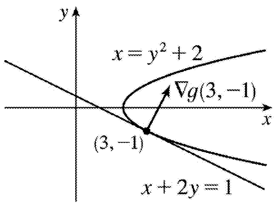
46. $F(x,y,z)=xyz$, $\nabla F(x,y,z)=\langle yz,xz,yx\rangle$, $\nabla F(1,2,3)=\langle 6,3,2\rangle$, so an equation of the tangent plane is $6x+3y+2z=18$, and the normal line is given by $\dfrac{x-1}{6}=\dfrac{y-2}{3}=\dfrac{z-3}{2}$ or $x=1+6t$, $y=2+3t$, $z=3+2t$.



47. $\nabla f(x,y)=\langle 2x,8y\rangle$, $\nabla f(2,1)=\langle 4,8\rangle$. The tangent line has equation $\nabla f(2,1)\cdot\langle x-2,y-1\rangle=0\Rightarrow$ $4(x-2)+8(y-1)=0$, which simplifies to $x+2y=4$.



48. $\nabla g(x,y)=\langle 1,-2y\rangle$, $\nabla g(3,-1)=\langle 1,2\rangle$. The tangent line has equation $\nabla g(3,-1)\cdot\langle x-3,y+1\rangle=0\Rightarrow$ $1(x-3)+2(y+1)=0$, which simplifies to $x+2y=1$.



49. $\nabla F\left(x_0,y_0,z_0\right)=\left\langle \frac{2x_0}{a^2},\frac{2y_0}{b^2},\frac{2z_0}{c^2}\right\rangle$. Thus an equation of the tangent plane at $\left(x_0,y_0,z_0\right)$ is $\frac{2x_0}{a^2}x+\frac{2y_0}{b^2}y+\frac{2z_0}{c^2}z=2\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}\right)=2(1)=2$ since $\left(x_0,y_0,z_0\right)$ is a point on the ellipsoid. Hence $\frac{x_0}{a^2}x+\frac{y_0}{b^2}y+\frac{z_0}{c^2}z=1$ is an equation of the tangent plane.

50. $\nabla F\left(x_0,y_0,z_0\right)=\left\langle \frac{2x_0}{a^2},\frac{2y_0}{b^2},\frac{-2z_0}{c^2}\right\rangle$, so an equation of the tangent plane at $\left(x_0,y_0,z_0\right)$ is $\frac{2x_0}{a^2}x+\frac{2y_0}{b^2}y-\frac{2z_0}{c^2}z=2\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}-\frac{z_0^2}{c^2}\right)=2$ or $\frac{x_0}{a^2}x+\frac{y_0}{b^2}y-\frac{z_0}{c^2}z=1$.

51. $\nabla F\left(x_0,y_0,z_0\right)=\left\langle \frac{2x_0}{a^2},\frac{2y_0}{b^2},\frac{-1}{c}\right\rangle$, so an equation of the tangent plane is $\frac{2x_0}{a^2}x+\frac{2y_0}{b^2}y-\frac{1}{c}z=\frac{2x_0^2}{a^2}+\frac{2y_0^2}{b^2}-\frac{z_0}{c}$ or $\frac{2x_0}{a^2}x+\frac{2y_0}{b^2}y=\frac{z}{c}+2\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}\right)-\frac{z_0}{c}$. But $\frac{z_0}{c}=\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}$, so the equation can be written as $\frac{2x_0}{a^2}x+\frac{2y_0}{b^2}y=\frac{z+z_0}{c}$.

52. Since $\nabla f\left(x_0,y_0,z_0\right)=\left\langle 2x_0,4y_0,6z_0\right\rangle$ and $\left\langle 3,-1,3\right\rangle$ are both normal vectors to the surface at $\left(x_0,y_0,z_0\right)$, we need $\left\langle 2x_0,4y_0,6z_0\right\rangle=c\left\langle 3,-1,3\right\rangle$ or $\left\langle x_0,2y_0,3z_0\right\rangle=k\left\langle 3,-1,3\right\rangle$. Thus $x_0=3k$, $y_0=-\frac{1}{2}k$ and

$z_0=k$. But $x_0^2+2y_0^2+3z_0^2=1$ or $\left(9+\frac{1}{2}+3\right)k^2=1$, so $k=\pm\frac{\sqrt{2}}{5}$ and there are two such points: $\left(\pm\frac{3\sqrt{2}}{5},\mp\frac{1}{5\sqrt{2}},\pm\frac{\sqrt{2}}{5}\right)$.

53. $\nabla f\left(x_0,y_0,z_0\right)=\left\langle 2x_0,-2y_0,4z_0\right\rangle$ and the given line has direction numbers 2,4,6 , so $\left\langle 2x_0,-2y_0,4z_0\right\rangle=k\langle 2,4,6\rangle$ or $x_0=k$, $y_0=-2k$ and $z_0=\frac{3}{2}k$. But $x_0^2-y_0^2+2z_0^2=1$ or $\left(1-4+\frac{9}{2}\right)k^2=1$, so $k=\pm\sqrt{\frac{2}{3}}=\pm\frac{\sqrt{6}}{3}$ and there are two such points: $\left(\pm\frac{\sqrt{6}}{3},\mp\frac{2\sqrt{6}}{3},\pm\frac{\sqrt{6}}{2}\right)$.

54. First note that the point (1,1,2) is on both surfaces. For the ellipsoid, an equation of the tangent plane at (1,1,2) is $6x+4y+4z=18$ or $3x+2y+2z=9$, and for the sphere, an equation of the tangent plane at (1,1,2) is $(2-8)x+(2-6)y+(4-8)z=-18$ or $-6x-4y-4z=-18$ or $3x+2y+2z=9$. Since these tangent planes are the same, the surfaces are tangent to each other at the point (1,1,2) .

55. Let $\left(x_0,y_0,z_0\right)$ be a point on the cone . Then an equation of the tangent plane to the cone at this point is $2x_0x+2y_0y-2z_0z=2\left(x_0^2+y_0^2-z_0^2\right)$. But $x_0^2+y_0^2=z_0^2$ so the tangent plane is given by $x_0x+y_0y-z_0z=0$, a plane which always contains the origin.

56. Let $\left(x_0,y_0,z_0\right)$ be a point on the sphere. Then the normal line is given by $\frac{x-x_0}{2x_0}=\frac{y-y_0}{2y_0}=\frac{z-z_0}{2z_0}$. For the center $(0,0,0)$ to be on the line, we need $-\frac{x_0}{2x_0}=-\frac{y_0}{2y_0}=-\frac{z_0}{2z_0}$ or equivalently $1=1=1$, which is true.

57. Let $\left(x_0,y_0,z_0\right)$ be a point on the surface. Then an equation of the tangent plane at the point is

$$\frac{x}{2\sqrt{x_0}}+\frac{y}{2\sqrt{y_0}}+\frac{z}{2\sqrt{z_0}}=\frac{\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}}{2}$$

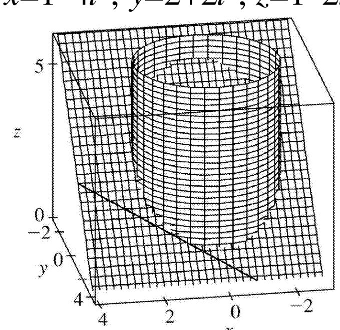
. But $\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}=\sqrt{c}$, so the equation is $\frac{x}{\sqrt{x_0}}+\frac{y}{\sqrt{y_0}}+\frac{z}{\sqrt{z_0}}=\sqrt{c}$. The x –, y –, and z –intercepts are $\sqrt{cx_0}$, $\sqrt{cy_0}$ and $\sqrt{cz_0}$ respectively. (The x –intercept is found by setting $y=z=0$ and solving the resulting equation for x , and the y – and z –intercepts are found similarly.) So the sum of the intercepts is

$\sqrt{c}\left(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}\right)=c$, a constant.

58. Here the equation of the tangent plane to the point $\left(x_0,y_0,z_0\right)$ is $y_0z_0x+x_0z_0y+x_0y_0z=3x_0y_0z_0$ or $\frac{x}{3x_0}+\frac{y}{3y_0}+\frac{z}{3z_0}=1$. Then the x –, y – , and z –intercepts are $3x_0$, $3y_0$ and $3z_0$ respectively, and their product is $27x_0y_0z_0=27c^3$, a constant.

59. If $f(x,y,z)=z-x^2-y^2$ and $g(x,y,z)=4x^2+y^2+z^2$, then the tangent line is perpendicular to both ∇f and ∇g at $(-1,1,2)$. The vector $\boldsymbol{v}=\nabla f\times\nabla g$ will therefore be parallel to the tangent line. We have: $\nabla f(x,y,z)=\langle -2x,-2y,1\rangle \Rightarrow \nabla f(-1,1,2)=\langle 2,-2,1\rangle$, and $\nabla g(x,y,z)=\langle 8x,2y,2z\rangle \Rightarrow \nabla g(-1,1,2)=\langle -8,2,4\rangle$. Hence $\boldsymbol{v}=\nabla f\times\nabla g=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -2 & 1 \\ -8 & 2 & 4 \end{vmatrix}=-10\boldsymbol{i}-16\boldsymbol{j}-12\boldsymbol{k}$. Parametric equations are: $x=-1-10t$, $y=1-16t$, $z=2-12t$.

60. **(a)** Let $f(x,y,z)=y+z$ and $g(x,y,z)=x^2+y^2$. Then the required tangent line is perpendicular to both ∇f and ∇g at $(1,2,1)$ and the vector $\boldsymbol{v}=\nabla f\times\nabla g$ is parallel to the tangent line. We have $\nabla f(x,y,z)=\langle 0,1,1\rangle \Rightarrow \nabla f(1,2,1)=\langle 0,1,1\rangle$, and $\nabla g(x,y,z)=\langle 2x,2y,0\rangle \Rightarrow \nabla g(1,2,1)=\langle 2,4,0\rangle$. Hence $\boldsymbol{v}=\nabla f\times\nabla g=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 1 & 1 \\ 2 & 4 & 0 \end{vmatrix}=-4\boldsymbol{i}+2\boldsymbol{j}-2\boldsymbol{k}$. So parametric equations of the desired tangent line are $x=1-4t$, $y=2+2t$, $z=1-2t$.

(b) 

61. **(a)** The direction of the normal line of F is given by ∇F , and that of G by ∇G . Assuming that $\nabla F\neq 0\neq \nabla G$, the two normal lines are perpendicular at P if $\nabla F\cdot \nabla G=0$ at $P\Leftrightarrow$ $\langle \partial F/\partial x,\partial F/\partial y,\partial F/\partial z\rangle\cdot \langle \partial G/\partial x,\partial G/\partial y,\partial G/\partial z\rangle=0$ at $P\Leftrightarrow F_xG_x+F_yG_y+F_zG_z=0$ at P .

(b) Here $F=x^2+y^2-z^2$ and $G=x^2+y^2+z^2-r^2$, so $\nabla F\cdot \nabla G=\langle 2x,2y,-2z\rangle\cdot \langle 2x,2y,2z\rangle=4x^2+4y^2-4z^2=4F=0$

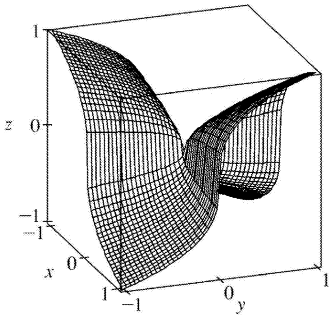
, since the point $\langle x,y,z\rangle$ lies on the graph of $F=0$. To see that this is true without using calculus, note that $G=0$ is the equation of a sphere centered at the origin and $F=0$ is the equation of a right circular cone with vertex at the origin (which is generated by lines through the origin). At any point of intersection, the sphere's normal line (which passes through the origin) lies on the cone, and thus is perpendicular to the cone's normal line. So the surfaces with equations $F=0$ and $G=0$ are everywhere orthogonal.

62. **(a)** The function $f(x,y)=(xy)^{1/3}$ is continuous on R^2 since it is a composition of a polynomial and the cube root function, both of which are continuous. (See the text just after Example 15.2.8 .)

$f_x(0,0)=\lim_{h\to 0}\dfrac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\dfrac{(h\cdot 0)^{1/3}-0}{h}=0$,

$f_y(0,0)=\lim_{h\to 0}\dfrac{f(0,0+h)-f(0,0)}{h}=\lim_{h\to 0}\dfrac{(0\cdot h)^{1/3}-0}{h}=0$. Therefore, $f_x(0,0)$ and $f_y(0,0)$ do exist and are equal to 0 . Now let $\boldsymbol{u}$ be any unit vector other than $\boldsymbol{i}$ and $\boldsymbol{j}$ (these correspond to f_x and f_y respectively.) Then $\boldsymbol{u}=a\boldsymbol{i}+b\boldsymbol{j}$ where $a\neq 0$ and $b\neq 0$. Thus

$D_{\boldsymbol{u}}f(0,0)=\lim_{h\to 0}\dfrac{f(0+ha,0+hb)-f(0,0)}{h}=\lim_{h\to 0}\dfrac{\sqrt[3]{(ha)(hb)}}{h}=\lim_{h\to 0}\dfrac{\sqrt[3]{ab}}{h^{1/3}}$ and this limit does not exist, so $D_{\boldsymbol{u}}f(0,0)$ does not exist.

(b) 

Notice that if we start at the origin and proceed in the direction of the x – or y –axis, then the graph is flat. But if we proceed in any other direction, then the graph is extremely steep.

63. Let $\boldsymbol{u}=\langle a,b\rangle$ and $\boldsymbol{v}=\langle c,d\rangle$. Then we know that at the given point, $D_{\boldsymbol{u}}f=\nabla f\cdot \boldsymbol{u}=af_x+bf_y$ and $D_{\boldsymbol{v}}f=\nabla f\cdot \boldsymbol{v}=cf_x+df_y$. But these are just two linear equations in the two unknowns f_x and f_y , and since $\boldsymbol{u}$ and $\boldsymbol{v}$ are not parallel, we can solve the equations to find $\nabla f=\langle f_x,f_y\rangle$ at the given point. In fact, $\nabla f=\left\langle \dfrac{dD_{\boldsymbol{u}}f-bD_{\boldsymbol{v}}f}{ad-bc},\dfrac{aD_{\boldsymbol{v}}f-cD_{\boldsymbol{u}}f}{ad-bc}\right\rangle$.

64. Since $z=f(x,y)$ is differentiable at $\boldsymbol{x}_0=(x_0,y_0)$, by Definition 15.4.7 we have $\Delta z=f_x(x_0,y_0)\Delta x+f_y(x_0,y_0)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y$ where $\varepsilon_1,\varepsilon_2\to 0$ as $(\Delta x,\Delta y)\to(0,0)$. Now $\Delta z=f(\boldsymbol{x})-f(\boldsymbol{x}_0)$, $\langle\Delta x,\Delta y\rangle=\boldsymbol{x}-\boldsymbol{x}_0$ so $(\Delta x,\Delta y)\to(0,0)$ is equivalent to $\boldsymbol{x}\to\boldsymbol{x}_0$ and $\left\langle f_x(x_0,y_0),f_y(x_0,y_0)\right\rangle=\nabla f(\boldsymbol{x}_0)$. Substituting into (15.4.7) gives $f(\boldsymbol{x})-f(\boldsymbol{x}_0)=\nabla f(\boldsymbol{x}_0)\cdot(\boldsymbol{x}-\boldsymbol{x}_0)+\left\langle\varepsilon_1,\varepsilon_2\right\rangle\cdot\langle\Delta x,\Delta y\rangle$ or $\left\langle\varepsilon_1,\varepsilon_2\right\rangle\cdot(\boldsymbol{x}-\boldsymbol{x}_0)=f(\boldsymbol{x})-f(\boldsymbol{x}_0)-\nabla f(\boldsymbol{x}_0)\cdot(\boldsymbol{x}-\boldsymbol{x}_0)$, and so $\dfrac{f(\boldsymbol{x})-f(\boldsymbol{x}_0)-\nabla f(\boldsymbol{x}_0)\cdot(\boldsymbol{x}-\boldsymbol{x}_0)}{\left|\boldsymbol{x}-\boldsymbol{x}_0\right|}=\dfrac{\left\langle\varepsilon_1,\varepsilon_2\right\rangle\cdot(\boldsymbol{x}-\boldsymbol{x}_0)}{\left|\boldsymbol{x}-\boldsymbol{x}_0\right|}$. But $\dfrac{\boldsymbol{x}-\boldsymbol{x}_0}{\left|\boldsymbol{x}-\boldsymbol{x}_0\right|}$ is a unit vector so $\lim\limits_{\boldsymbol{x}\to\boldsymbol{x}_0}\dfrac{\left\langle\varepsilon_1,\varepsilon_2\right\rangle\cdot(\boldsymbol{x}-\boldsymbol{x}_0)}{\left|\boldsymbol{x}-\boldsymbol{x}_0\right|}=0$ since $\varepsilon_1,\varepsilon_2\to 0$ as $\boldsymbol{x}\to\boldsymbol{x}_0$. Hence $\lim\limits_{\boldsymbol{x}\to\boldsymbol{x}_0}\dfrac{f(\boldsymbol{x})-f(\boldsymbol{x}_0)-\nabla f(\boldsymbol{x}_0)\cdot(\boldsymbol{x}-\boldsymbol{x}_0)}{\left|\boldsymbol{x}-\boldsymbol{x}_0\right|}=0$.

1. **(a)** First we compute $D(1,1)=f_{xx}(1,1)\,f_{yy}(1,1)-[f_{xy}(1,1)]^2=(4)(2)-(1)^2=7$. Since $D(1,1)>0$ and $f_{xx}(1,1)>0$, f has a local minimum at (1,1) by the Second Derivatives Test.

(b) $D(1,1)=f_{xx}(1,1)\,f_{yy}(1,1)-[f_{xy}(1,1)]^2=(4)(2)-(3)^2=-1$. Since $D(1,1)<0$, f has a saddle point at (1,1) by the Second Derivatives Test.

2. **(a)** $D=g_{xx}(0,2)\,g_{yy}(0,2)-[g_{xy}(0,2)]^2=(-1)(1)-(6)^2=-37$. Since $D<0$, g has a saddle point at (0,2) by the Second Derivatives Test.

(b) $D=g_{xx}(0,2)\,g_{yy}(0,2)-[g_{xy}(0,2)]^2=(-1)(-8)-(2)^2=4$. Since $D>0$ and $g_{xx}(0,2)<0$, g has a local maximum at (0,2) by the Second Derivatives Test.

(c) $D=g_{xx}(0,2)\,g_{yy}(0,2)-[g_{xy}(0,2)]^2=(4)(9)-(6)^2=0$. In this case the Second Derivatives Test gives no information about g at the point (0,2) .

3. In the figure, a point at approximately $(1,1)$ is enclosed by level curves which are oval in shape and indicate that as we move away from the point in any direction the values of f are increasing. Hence we would expect a local minimum at or near $(1,1)$. The level curves near $(0,0)$ resemble hyperbolas, and as we move away from the origin, the values of f increase in some directions and decrease in others, so we would expect to find a saddle point there.

To verify our predictions, we have $f(x,y)=4+x^3+y^3-3xy \Rightarrow f_x(x,y)=3x^2-3y$, $f_y(x,y)=3y^2-3x$. We have critical points where these partial derivatives are equal to 0 : $3x^2-3y=0$, $3y^2-3x=0$. Substituting $y=x^2$ from the first equation into the second equation gives $3\left(x^2\right)^2-3x=0 \Rightarrow 3x(x^3-1)=0 \Rightarrow x=0$ or $x=1$. Then we have two critical points, $(0,0)$ and $(1,1)$. The second partial derivatives are $f_{xx}(x,y)=6x$, $f_{xy}(x,y)=-3$, and $f_{yy}(x,y)=6y$, so

$D(x,y)=f_{xx}(x,y)\,f_{yy}(x,y)-[f_{xy}(x,y)]^2=(6x)(6y)-(-3)^2=36xy-9$. Then $D(0,0)=36(0)(0)-9=-9$, and $D(1,1)=36(1)(1)-9=27$. Since $D(0,0)<0$, f has a saddle point at (0,0) by the Second Derivatives Test. Since $D(1,1)>0$ and $f_{xx}(1,1)>0$, f has a local minimum at (1,1) .

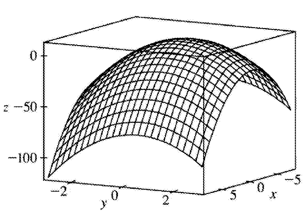
4. In the figure, points at approximately (−1,1) and (−1,−1) are enclosed by oval–shaped level curves which indicate that as we move away from either point in any direction, the values of f are increasing. Hence we would expect local minima at or near $(-1,\pm 1)$. Similarly, the point (1,0) appears to be enclosed by oval–shaped level curves which indicate that as we move away from the point in any direction the values of f are decreasing, so we should have a local maximum there. We also show hyperbola–shaped level curves near the points $(-1,0)$, $(1,1)$, and $(1,-1)$. The values of f increase along some paths leaving these points and decrease in others, so we should have a saddle

point at each of these points.

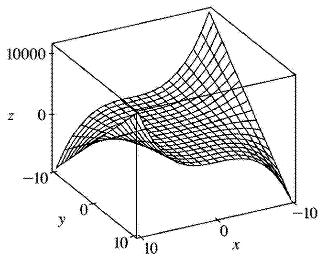
To confirm our predictions, we have $f(x,y)=3x-x^3-2y^2+y^4 \Rightarrow f_x(x,y)=3-3x^2$, $f_y(x,y)=-4y+4y^3$. Setting these partial derivatives equal to 0 , we have $3-3x^2=0 \Rightarrow x=\pm 1$ and $-4y+4y^3=0 \Rightarrow y\left(y^2-1\right)=0 \Rightarrow y=0,\pm 1$. So our critical points are $(\pm 1,0)$, $(\pm 1,\pm 1)$. The second partial derivatives are $f_{xx}(x,y)=-6x$, $f_{xy}(x,y)=0$, and $f_{yy}(x,y)=12y^2-4$, so $D(x,y)=f_{xx}(x,y)\,f_{yy}(x,y)-[f_{xy}(x,y)]^2=(-6x)\left(12y^2-4\right)-(0)^2=-72xy^2+24x$. We use the Second Derivatives Test to classify the 6 critical points:

Critical Point	D	f_{xx}	Conclusion
$(1,0)$	24	-6	$D>0, f_{xx}<0 \Rightarrow f$ has a local maximum at $(1,0)$
$(1,1)$	-48		$D<0 \Rightarrow f$ has a saddle point at $(1,1)$
$(1,-1)$	-48		$D<0 \Rightarrow f$ has a saddle point at $(1,-1)$
$(-1,0)$	-24		$D<0 \Rightarrow f$ has a saddle point at $(-1,0)$
$(-1,1)$	48	6	$D>0, f_{xx}>0 \Rightarrow f$ has a local minimum at $(-1,1)$
$(-1,-1)$	48	6	$D>0, f_{xx}>0 \Rightarrow f$ has a local minimum at $(-1,-1)$

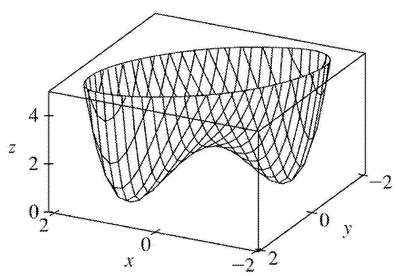
5. $f(x,y)=9-2x+4y-x^2-4y^2 \Rightarrow f_x=-2-2x$, $f_y=4-8y$, $f_{xx}=-2$, $f_{xy}=0$, $f_{yy}=-8$. Then $f_x=0$ and $f_y=0$ imply $x=-1$ and $y=\frac{1}{2}$, and the only critical point is $\left(-1,\frac{1}{2}\right)$.
$D(x,y)=f_{xx}f_{yy}-\left(f_{xy}\right)^2=(-2)\,(-8)-0^2=16$, and since $D\left(-1,\frac{1}{2}\right)=16>0$ and $f_{xx}\left(-1,\frac{1}{2}\right)=-2<0$, $f\left(-1,\frac{1}{2}\right)=11$ is a local maximum by the Second Derivatives Test.



6. $f(x,y)=x^3y+12x^2-8y \Rightarrow f_x=3x^2y+24x$,
$f_y=x^3-8$, $f_{xx}=6xy+24$, $f_{xy}=3x^2$, $f_{yy}=0$. Then $f_y=0$ implies $x=2$, and substitution into $f_x=0$ gives $12y+48=0 \Rightarrow y=-4$. Thus, the only critical point is $(2,-4)$. $D(2,-4)=(-24)(0)-12^2=-144<0$, so $(2,-4)$ is a saddle point.

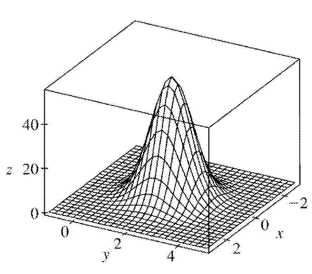


7. $f(\mathrm{x,y})=x^4+y^4-4xy+2 \Rightarrow f_x=4x^3-4y$, $f_y=4y^3-4x$, $f_{xx}=12x^2$, $f_{xy}=-4$, $f_{yy}=12y^2$. Then $f_x=0$ implies $y=x^3$, and substitution into $f_y=0 \Rightarrow x=y^3$ gives $x^9-x=0 \Rightarrow x(x^8-1)=0 \Rightarrow x=0$ or $x=\pm 1$. Thus the critical points are $(0,0)$, $(1,1)$, and $(-1,-1)$. Now $D(0,0)=0\cdot 0-(-4)^2=-16<0$, so $(0,0)$ is a saddle point. $D(1,1)=(12)(12)-(-4)^2>0$ and $f_{xx}(1,1)=12>0$, so

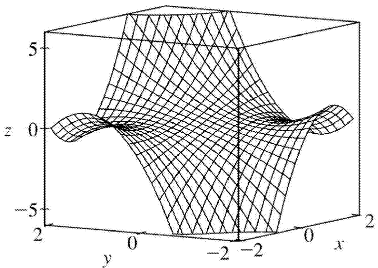


$f(1,1)=0$ is a local minimum. $D(-1,-1)=(12)(12)-(-4)^2>0$ and $f_{xx}=(-1,-1)=12>0$, so $f(-1,-1)=0$ is also a local minimum.

8. $f(\mathrm{x,y})=e^{4y-x^2-y^2} \Rightarrow f_x=-2xe^{4y-x^2-y^2}$, $f_y=(4-2y)e^{4y-x^2-y^2}$, $f_{xx}=(4x^2-2)e^{4y-x^2-y^2}$, $f_{xy}=-2x(4-2y)e^{4y-x^2-y^2}$, $f_{yy}=(4y^2-16y+14)e^{4y-x^2-y^2}$. Then $f_x=0$ and $f_y=0$ implies $x=0$ and $y=2$, so the only critical point is $(0,2)$. $D(0,2)=(-2e^4)(-2e^4)-0^2=4e^8>0$ and $f_{xx}(0,2)=-2e^4<0$, so $f(0,2)=e^4$ is a local maximum.

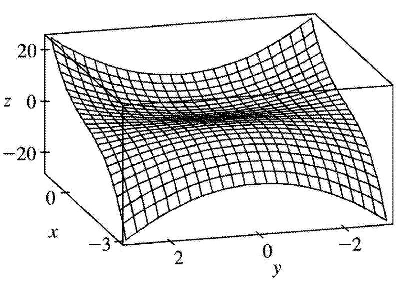


9. $f(\mathrm{x,y})=(1+xy)(x+y)=x+y+x^2y+xy^2 \Rightarrow f_x=1+2xy+y^2$, $f_y=1+x^2+2xy$, $f_{xx}=2y$, $f_{xy}=2x+2y$, $f_{yy}=2x$. Then $f_x=0$ implies $1+2xy+y^2=0$ and $f_y=0$ implies $1+x^2+2xy=0$. Subtracting the second equation from the first gives $y^2-x^2=0 \Rightarrow y=\pm x$, but if $y=x$ then $1+2xy+y^2=0 \Rightarrow 1+3x^2=0$ which has no real solution. If $y=-x$ then $1+2xy+y^2=0 \Rightarrow 1-x^2=0 \Rightarrow x=\pm 1$,

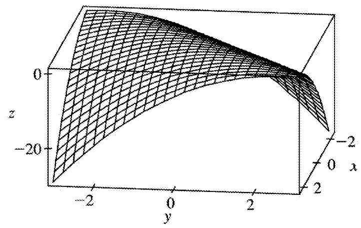


so critical points are $(1,-1)$ and $(-1,1)$. $D(1,-1)=(-2)(2)-0<0$ and $D(-1,1)=(2)(-2)-0<0$, so $(-1,1)$ and $(1,-1)$ are saddle points.

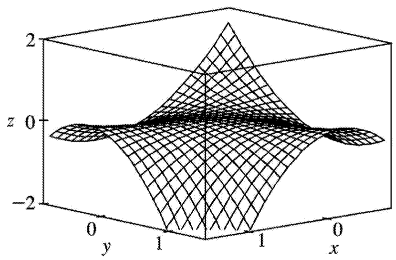
10. $f(\mathrm{x,y})=2x^3+xy^2+5x^2+y^2 \Rightarrow f_x=6x^2+y^2+10x$, $f_y=2xy+2y$, $f_{xx}=12x+10$, $f_{yy}=2x+2$, $f_{xy}=2y$. Then $f_y=0$ implies $y=0$ or $x=-1$. Substituting into $f_x=0$ gives the critical points $(0,0)$, $\left(-\frac{5}{3},0\right)$, $(-1,\pm 2)$. Now $D(0,0)=20>0$ and $f_{xx}(0,0)=10>0$, so $f(0,0)=0$ is a local minimum. Also $f_{xx}\left(-\frac{5}{3},0\right)<0$, $D\left(-\frac{5}{3},0\right)>0$, and $D(-1,\pm 2)<0$. Hence $f\left(-\frac{5}{3},0\right)=\frac{125}{27}$ is a local maximum while $(-1,\pm 2)$ are saddle points.



11. $f(\mathrm{x,y})=1+2xy-x^2-y^2 \Rightarrow f_x=2y-2x$, $f_y=2x-2y$, $f_{xx}=f_{yy}=-2$, $f_{xy}=2$. Then $f_x=0$ and $f_y=0$ implies $x=y$ so the critical points are all points of the form $\left(x_0,\mathrm{x}_0\right)$. But $D\left(x_0,\mathrm{x}_0\right)=4-4=0$ so the Second Derivatives Test gives no information. However $1+2xy-x^2-y^2=1-(x-y)^2$ and $1-(x-y)^2\leq 1$ for all $(\mathrm{x,y})$, with equality if and only if $x=y$. Thus $f\left(x_0,\mathrm{x}_0\right)=1$ are local maxima.

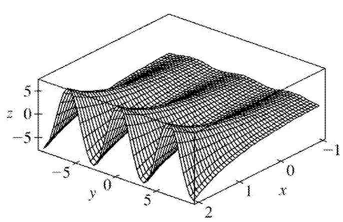


12. $f(x,y)=xy(1-x-y) \Rightarrow f_x = y-2xy-y^2$, $f_y = x-x^2-2xy$, $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 1-2x-2y$. Then $f_x = 0$ implies $y=0$ or $y=1-2x$. Substituting $y=0$ into $f_y=0$ gives $x=0$ or $x=1$ and substituting $y=1-2x$ into $f_y=0$ gives $3x^2-x=0$ so $x=0$ or $\frac{1}{3}$. Thus the critical points are $(0,0)$, $(1,0)$, $(0,1)$ and $\left(\frac{1}{3},\frac{1}{3}\right)$.



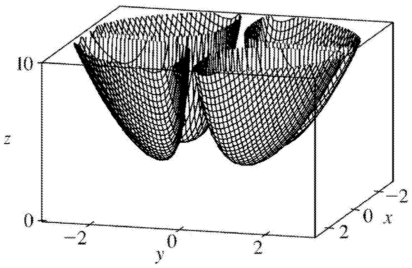
$D(0,0)=D(1,0)=D(0,1)=-1$ while $D\left(\frac{1}{3},\frac{1}{3}\right)=\frac{1}{3}$ and $f_{xx}\left(\frac{1}{3},\frac{1}{3}\right)=-\frac{2}{3}<0$. Thus $(0,0)$, $(1,0)$ and $(0,1)$ are saddle points, and $f\left(\frac{1}{3},\frac{1}{3}\right)=\frac{1}{27}$ is a local maximum.

13. $f(x,y)=e^x \cos y \Rightarrow f_x = e^x \cos y$, $f_y = -e^x \sin y$. Now $f_x=0$ implies $\cos y=0$ or $y=\frac{\pi}{2}+n\pi$ for n an integer. But $\sin\left(\frac{\pi}{2}+n\pi\right)\neq 0$, so there are no critical points.



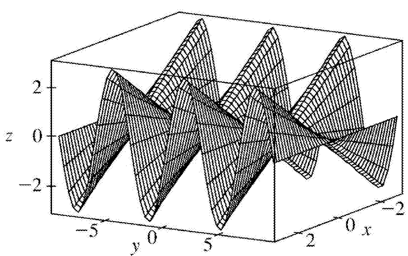
14. $f(x,y)=x^2+y^2+\frac{1}{x^2y^2} \Rightarrow f_x = 2x-2x^{-3}y^{-2}$, $f_y = 2y-2x^{-2}y^{-3}$, $f_{xx} = 2+6x^{-4}y^{-2}$, $f_{yy} = 2+6x^{-2}y^{-4}$, $f_{xy} = 4x^{-3}y^{-3}$. Then $f_x=0$ implies $2x^4y^2-2=0$ or $x^4y^2=1$ or $y^2=x^{-4}$. Note that neither x nor y can be zero. Now $f_y=0$ implies $2x^2y^4-2=0$, and with $y^2=x^{-4}$ this implies $2x^{-6}-2=0$ or $x^6=1$. Thus $x=\pm 1$ and

if $x=1$, $y=\pm 1$;

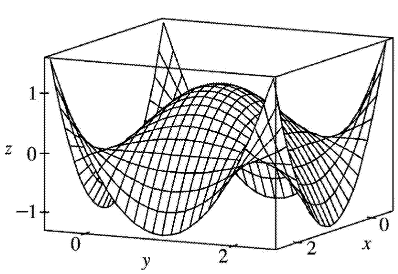


if $x=-1$, $y=\pm 1$. So the critical points are $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$. Now $D(\pm 1,\pm 1)=D(\pm 1,\mp 1)=64-16>0$ and $f_{xx}>0$ always, so $f(\pm 1,\pm 1)=f(\pm 1,\mp 1)=3$ are local minima.

15. $f(x,y)=x\sin y\Rightarrow f_x=\sin y$, $f_y=x\cos y$, $f_{xx}=0$, $f_{yy}=-x\sin y$ and $f_{xy}=\cos y$. Then $f_x=0$ if and only if $y=n\pi$, n an integer, and substituting into $f_y=0$ requires $x=0$ for each of these y –values. Thus the critical points are $(0,n\pi)$, n an integer. But $D(0,n\pi)=-\cos^2(n\pi)<0$ so each critical point is a saddle point.



16. $f(x,y)=\left(2x-x^2\right)\left(2y-y^2\right)\Rightarrow f_x=(2-2x)\left(2y-y^2\right)$, $f_y=\left(2x-x^2\right)(2-2y)$, $f_{xx}=-2\left(2y-y^2\right)$, $f_{yy}=-2\left(2x-x^2\right)$ and $f_{xy}=(2-2x)(2-2y)$. Then $f_x=0$ implies $x=1$ or $y=0$ or $y=2$ and when $x=1$, $f_y=0$ implies $y=1$, when $y=0$, $f_y=0$ implies $x=0$ or $x=2$ and when $y=2$, $f_y=0$ implies $x=0$ or $x=2$. Thus the critical points are $(1,1)$, $(0,0)$, $(2,0)$,



$(0,2)$ and $(2,2)$. Now $D(0,0)=D(2,0)=D(0,2)=D(2,2)=-16$ so these critical points are saddle points, and $D(1,1)=4$ with $f_{xx}(1,1)=-2$, so $f(1,1)=1$ is a local maximum.

17. $f(x,y)=\left(x^2+y^2\right)e^{y^2-x^2} \Rightarrow f_x=\left(x^2+y^2\right)e^{y^2-x^2}(-2x)+2xe^{y^2-x^2}=2xe^{y^2-x^2}\left(1-x^2-y^2\right)$,

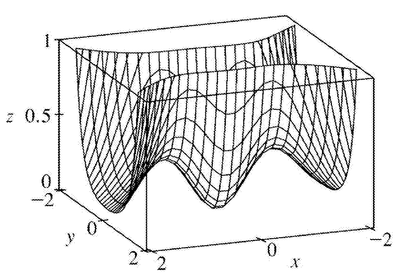
$f_y=\left(x^2+y^2\right)e^{y^2-x^2}(2y)+2ye^{y^2-x^2}=2ye^{y^2-x^2}\left(1+x^2+y^2\right)$,

$$f_{xx}=2xe^{y^2-x^2}(-2x)+\left(1-x^2-y^2\right)\left(2x\left(-2xe^{y^2-x^2}\right)+2e^{y^2-x^2}\right)$$
$$=2e^{y^2-x^2}\left(\left(1-x^2-y^2\right)\left(1-2x^2\right)-2x^2\right) ,$$

$f_{xy}=2xe^{y^2-x^2}(-2y)+2x(2y)e^{y^2-x^2}\left(1-x^2-y^2\right)=-4xye^{y^2-x^2}\left(x^2+y^2\right)$,

$$f_{yy}=2ye^{y^2-x^2}(2y)+\left(1+x^2+y^2\right)\left(2y\left(2ye^{y^2-x^2}\right)+2e^{y^2-x^2}\right)$$
$$=2e^{y^2-x^2}\left(\left(1+x^2+y^2\right)\left(1+2y^2\right)+2y^2\right) .$$

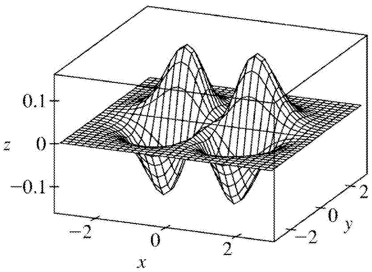
$f_y=0$ implies $y=0$, and substituting into $f_x=0$ gives $2xe^{-x^2}(1-x^2)=0 \Rightarrow x=0$ or $x=\pm 1$.
Thus the critical points are $(0,0)$ and $(\pm 1,0)$. $D(0,0)=(2)(2)-0>0$ and $f_{xx}(0,0)=2>0$, so $f(0,0)=0$ is a local minimum. $D(\pm 1,0)=\left(-4e^{-1}\right)\left(4e^{-1}\right)-0<0$ so $(\pm 1,0)$ are saddle points.



18. $f(x,y)=x^2ye^{-x^2-y^2} \Rightarrow f_x=x^2ye^{-x^2-y^2}(-2x)+2xye^{-x^2-y^2}=2xy\left(1-x^2\right)e^{-x^2-y^2}$,

$f_y=x^2ye^{-x^2-y^2}(-2y)+x^2e^{-x^2-y^2}=x^2\left(1-2y^2\right)e^{-x^2-y^2}$, $f_{xx}=2y\left(2x^4-5x^2+1\right)e^{-x^2-y^2}$,

$f_{xy}=2x\left(1-x^2\right)\left(1-2y^2\right)e^{-x^2-y^2}$, $f_{yy}=2x^2y\left(2y^2-3\right)e^{-x^2-y^2}$.

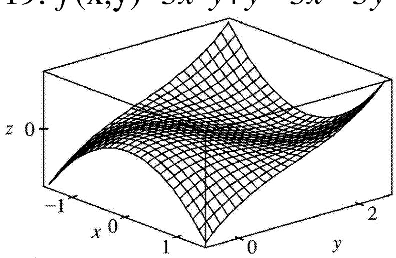


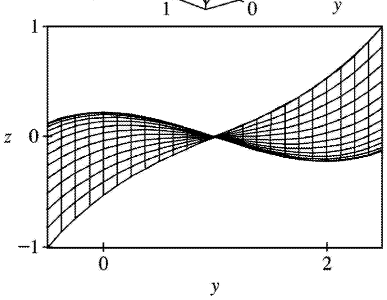
$f_x=0$ implies $x=0$, $y=0$, or $x=\pm 1$. If $x=0$ then $f_y=0$ for any y –value, so all points of the form $(0,y)$ are critical points. If $y=0$ then $f_y=0 \Rightarrow x^2 e^{-x^2}=0 \Rightarrow x=0$, so $(0,0)$ (already included above) is a critical point. If $x=\pm 1$ then $\left(1-2y^2\right)e^{-1-y^2}=0 \Rightarrow y=\pm \frac{1}{\sqrt{2}}$, so $\left(1,\pm \frac{1}{\sqrt{2}}\right)$ and $\left(-1,\pm \frac{1}{\sqrt{2}}\right)$ are critical points. $D(0,y)=0$, so the Second Derivatives Test gives no information. However, if $y>0$ then $x^2 ye^{-x^2-y^2} \geq 0$ with equality only when $x=0$, so we have local minimum values $f(0,y)=0$, $y>0$. Similarly, if $y<0$ then $x^2 ye^{-x^2-y^2} \leq 0$ with equality when $x=0$ so $f(0,y)=0$, $y<0$ are local maximum values, and $(0,0)$ is a saddle point.

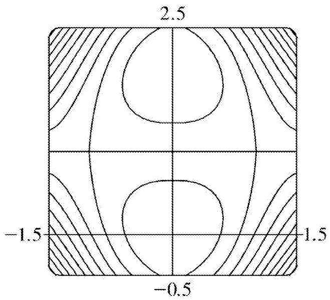
$D\left(\pm 1,\frac{1}{\sqrt{2}}\right)=8e^{-3}>0$, $f_{xx}\left(\pm 1,\frac{1}{\sqrt{2}}\right)=-2\sqrt{2}\,e^{-3/2}<0$ and

$D\left(\pm 1,-\frac{1}{\sqrt{2}}\right)=8e^{-3}>0$, $f_{xx}\left(\pm 1,-\frac{1}{\sqrt{2}}\right)=2\sqrt{2}\,e^{-3/2}>0$, so $f\left(\pm 1,\frac{1}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}\,e^{-3/2}$ are local maximum points while $f\left(\pm 1,-\frac{1}{\sqrt{2}}\right)=-\frac{1}{\sqrt{2}}\,e^{-3/2}$ are local minimum points.

19. $f(x,y)=3x^2 y+y^3-3x^2-3y^2+2$



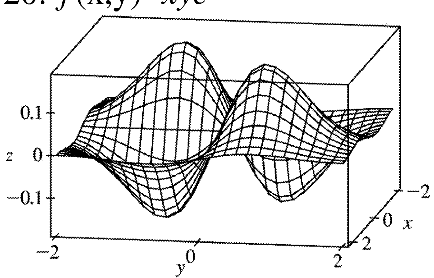


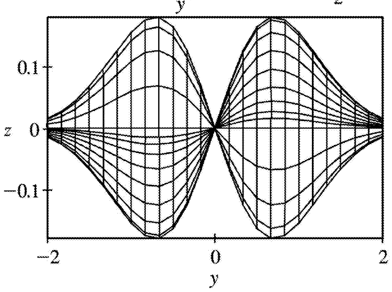


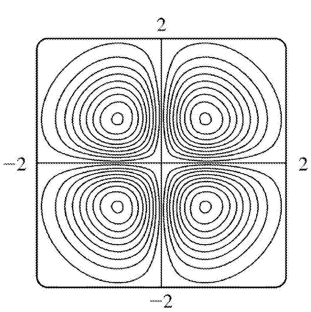
From the graphs, it appears that f has a local maximum $f(0,0)\approx 2$ and a local minimum $f(0,2)\approx -2$. There appear to be saddle points near $(\pm 1,1)$.

$f_x=6xy-6x$, $f_y=3x^2+3y^2-6y$. Then $f_x=0$ implies $x=0$ or $y=1$ and when $x=0$, $f_y=0$ implies $y=0$ or $y=2$; when $y=1$, $f_y=0$ implies $x^2=1$ or $x=\pm 1$. Thus the critical points are $(0,0)$, $(0,2)$, $(\pm 1,1)$. Now $f_{xx}=6y-6$, $f_{yy}=6y-6$ and $f_{xy}=6x$, so $D(0,0)=D(0,2)=36>0$ while $D(\pm 1,1)=-36<0$ and $f_{xx}(0,0)=-6$, $f_{xx}(0,2)=6$. Hence $(\pm 1,1)$ are saddle points while $f(0,0)=2$ is a local maximum and $f(0,2)=-2$ is a local minimum.

20. $f(\mathrm{x,y})=xye^{-x^2-y^2}$







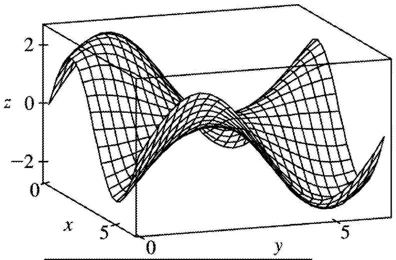
There appear to be local maxima of about $f(\pm 0.7,\pm 0.7)\approx 0.18$ and local minima of about $f(\pm 0.7,\mp 0.7)\approx -0.18$. Also, there seems to be a saddle point at the origin.

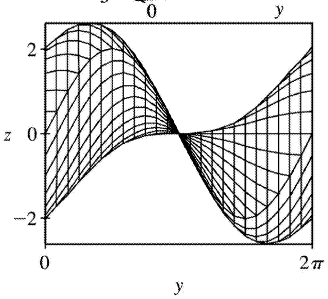
$f_x = ye^{-x^2-y^2}\left(1-2x^2\right)$, $f_y = xe^{-x^2-y^2}\left(1-2y^2\right)$, $f_{xx} = 2xye^{-x^2-y^2}\left(2x^2-3\right)$, $f_{yy} = 2xye^{-x^2-y^2}\left(2y^2-3\right)$, $f_{xy} = \left(1-2x^2\right)e^{-x^2-y^2}\left(1-2y^2\right)$. Then $f_x = 0$ implies $y = 0$ or $x = \pm\frac{1}{\sqrt{2}}$.

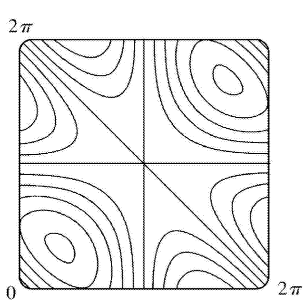
Substituting these values into $f_y = 0$ gives the critical points $(0,0)$, $\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$. Then

$D(\text{x,y}) = e^{2\left(-x^2-y^2\right)}\left[4x^2y^2\left(2x^2-3\right)\left(2y^2-3\right)-\left(1-2x^2\right)^2\left(1-2y^2\right)^2\right]$, so $D(0,0) = -1$, while $D\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) > 0$ and $D\left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) > 0$. But $f_{xx}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) < 0$, $f_{xx}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) > 0$, $f_{xx}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) > 0$ and $f_{xx}\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) < 0$. Hence $(0,0)$ is a saddle point; $f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2e}$ are local minima and $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2e}$ are local maxima.

21. $f(\text{x,y}) = \sin x + \sin y + \sin(x+y)$, $0 \le x \le 2\pi$, $0 \le y \le 2\pi$

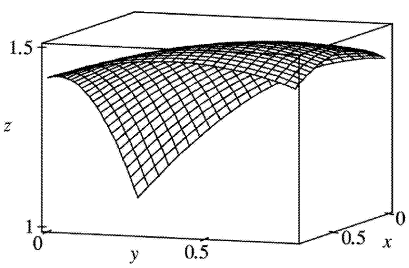


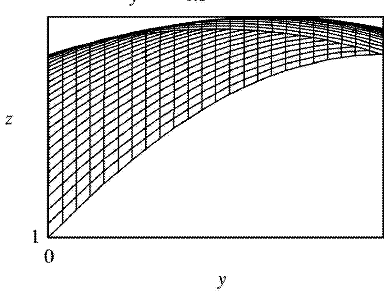


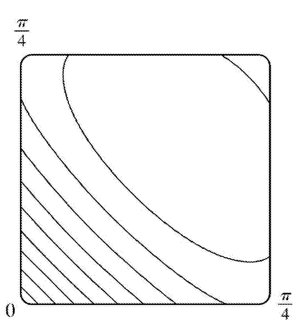


From the graphs it appears that f has a local maximum at about $(1,1)$ with value approximately 2.6 , a local minimum at about $(5,5)$ with value approximately -2.6 , and a saddle point at about $(3,3)$. $f_x=\cos x+\cos (x+y)$, $f_y=\cos y+\cos (x+y)$, $f_{xx}=-\sin x-\sin (x+y)$, $f_{yy}=-\sin y-\sin (x+y)$, $f_{xy}=-\sin (x+y)$. Setting $f_x=0$ and $f_y=0$ and subtracting gives $\cos x-\cos y=0$ or $\cos x=\cos y$. Thus $x=y$ or $x=2\pi-y$. If $x=y$, $f_x=0$ becomes $\cos x+\cos 2x=0$ or $2\cos^2 x+\cos x-1=0$, a quadratic in $\cos x$. Thus $\cos x=-1$ or $\frac{1}{2}$ and $x=\pi$, $\frac{\pi}{3}$, or $\frac{5\pi}{3}$, yielding the critical points (π,π) , $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)$. Similarly if $x=2\pi-y$, $f_x=0$ becomes $(\cos x)+1=0$ and the resulting critical point is (π,π) . Now $D(x,y)=\sin x\sin y+\sin x\sin (x+y)+\sin y\sin (x+y)$. So $D(\pi,\pi)=0$ and the Second Derivatives Test doesn't apply. $D\left(\frac{\pi}{3},\frac{\pi}{3}\right)=\frac{9}{4}>0$ and $f_{xx}\left(\frac{\pi}{3},\frac{\pi}{3}\right)<0$ so $f\left(\frac{\pi}{3},\frac{\pi}{3}\right)=\frac{3\sqrt{3}}{2}$ is a local maximum while $D\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)=\frac{9}{4}>0$ and $f_{xx}\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)>0$, so $f\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)=-\frac{3\sqrt{3}}{2}$ is a local minimum.

22. $f(x,y)=\sin x+\sin y+\cos (x+y)$, $0\leq x\leq \frac{\pi}{4}$, $0\leq y\leq \frac{\pi}{4}$



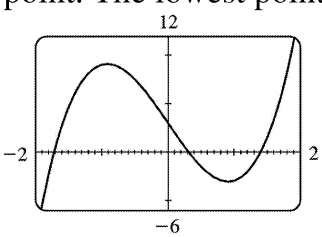


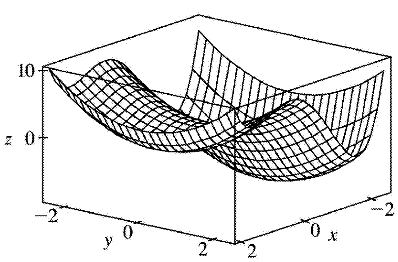


From the graphs, it seems that f has a local maximum at about $(0.5,0.5)$.

$f_x=\cos x-\sin(x+y)$, $f_y=\cos y-\sin(x+y)$, $f_{xx}=-\sin x-\cos(x+y)$, $f_{yy}=-\sin y-\cos(x+y)$, $f_{xy}=-\cos(x+y)$. Setting $f_x=0$ and $f_y=0$ and subtracting gives $\cos x=\cos y$. Thus $x=y$. Substituting $x=y$ into $f_x=0$ gives $\cos x-\sin 2x=0$ or $\cos x(1-2\sin x)=0$. But $\cos x\neq 0$ for $0\leq x\leq \frac{\pi}{4}$ and $1-2\sin x=0$ implies $x=\frac{\pi}{6}$, so the only critical point is $\left(\frac{\pi}{6},\frac{\pi}{6}\right)$. Here $f_{xx}\left(\frac{\pi}{6},\frac{\pi}{6}\right)=-1<0$ and $D\left(\frac{\pi}{6},\frac{\pi}{6}\right)=(-1)^2-\frac{1}{4}>0$. Thus $f\left(\frac{\pi}{6},\frac{\pi}{6}\right)=\frac{3}{2}$ is a local maximum.

23. $f(x,y)=x^4-5x^2+y^2+3x+2\Rightarrow f_x(x,y)=4x^3-10x+3$ and $f_y(x,y)=2y$. $f_y=0\Rightarrow y=0$, and the graph of f_x shows that the roots of $f_x=0$ are approximately $x=-1.714$, 0.312 and 1.402 . (Alternatively, we could have used a calculator or a CAS to find these roots.) So to three decimal places, the critical points are $(-1.714,0)$, $(1.402,0)$, and $(0.312,0)$. Now since $f_{xx}=12x^2-10$, $f_{xy}=0$, $f_{yy}=2$, and $D=24x^2-20$, we have $D(-1.714,0)>0$, $f_{xx}(-1.714,0)>0$, $D(1.402,0)>0$, $f_{xx}(1.402,0)>0$, and $D(0.312,0)<0$. Therefore $f(-1.714,0)\approx-9.200$ and $f(1.402,0)\approx 0.242$ are local minima, and (0.312,0) is a saddle point. The lowest point on the graph is approximately $(-1.714,0,-9.200)$.



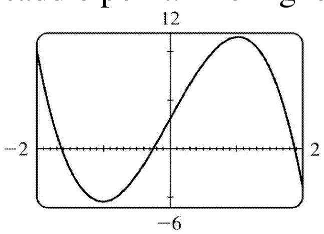


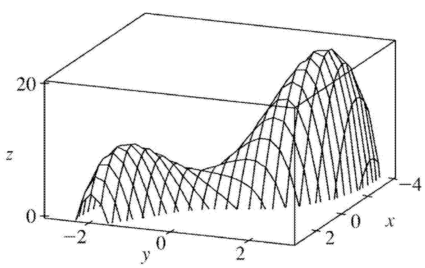
24. $f(x,y)=5-10xy-4x^2+3y-y^4\Rightarrow f_x(x,y)=-10y-8x$, $f_y(x,y)=-10x+3-4y^3$.

Now $f_x=0\Rightarrow x=-\frac{5}{4}y$, so using a graph, we find solutions to $0=f_y\left(-\frac{5}{4}y,y\right)=-10\left(-\frac{5}{4}y\right)+3-4y^3=-4y^3+\frac{25}{2}y+3$. (Alternatively, we could have found the roots of $f_x=f_y=0$ directly, using a calculator or a CAS.) To three decimal places, the solutions are $y\approx 1.877$, -0.245

and -1.633 , so f has critical points at approximately $(-2.347,1.877)$, $(0.306,-0.245)$, and $(2.041,-1.633)$.

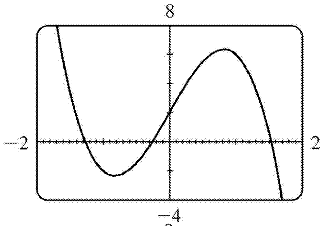
Now since $f_{xx}=-8$, $f_{xy}=-10$, $f_{yy}=-12y^2$, and $D=96y^2-100$, we have $D(-2.347,1.877)>0$, $D(0.306,-0.245)<0$, and $D(2.041,-1.633)>0$. Therefore, since $f_{xx}<0$ everywhere, $f(-2.347,1.877)\approx 20.238$ and $f(2.041,-1.633)\approx 9.657$ are local maxima, and $(0.306,-0.245)$ is a saddle point. The highest point on the graph is approximately $(-2.347,1.877,20.238)$.

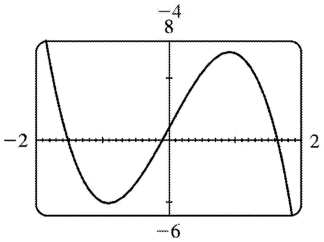


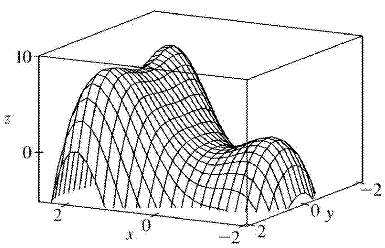


25. $f(x,y)=2x+4x^2-y^2+2xy^2-x^4-y^4 \Rightarrow f_x(x,y)=2+8x+2y^2-4x^3$, $f_y(x,y)=-2y+4xy-4y^3$. Now $f_y=0\Leftrightarrow$ $2y\left(2y^2-2x+1\right)=0\Leftrightarrow y=0$ or $y^2=x-\frac{1}{2}$.

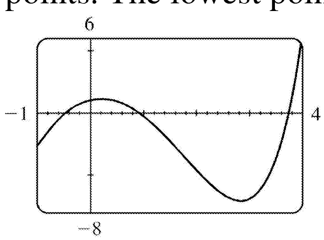
The first of these implies that $f_x=-4x^3+8x+2$, and the second implies that $f_x=2+8x+2\left(x-\frac{1}{2}\right)-4x^3=-4x^3+10x+1$. From the graphs, we see that the first possibility for f_x has roots at approximately -1.267 , -0.259 , and 1.526 , and the second has a root at approximately 1.629 (the negative roots do not give critical points, since $y^2=x-\frac{1}{2}$ must be positive). So to three decimal places, f has critical points at $(-1.267,0)$, $(-0.259,0)$, $(1.526,0)$, and $(1.629,\pm 1.063)$. Now since $f_{xx}=8-12x^2$, $f_{xy}=4y$, $f_{yy}=4x-12y^2$, and $D=(8-12x^2)(4x-12y^2)-16y^2$, we have $D(-1.267,0)>0$, $f_{xx}(-1.267,0)>0$, $D(-0.259,0)<0$, $D(1.526,0)<0$, $D(1.629,\pm 1.063)>0$, and $f_{xx}(1.629,\pm 1.063)<0$. Therefore, to three decimal places, $f(-1.267,0)\approx 1.310$ and $f(1.629,\pm 1.063)\approx 8.105$ are local maxima, and $(-0.259,0)$ and $(1.526,0)$ are saddle points. The highest points on the graph are approximately $(1.629,\pm 1.063,8.105)$.

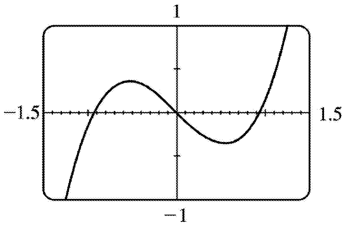


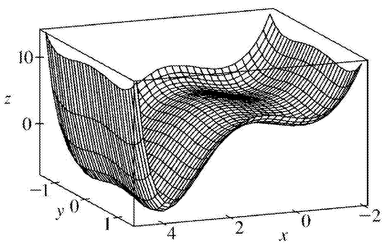




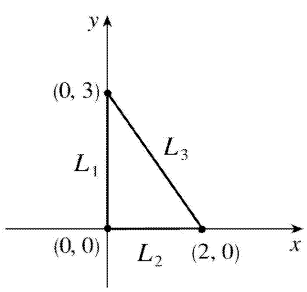
26. $f(x,y)=e^x+y^4-x^3+4\cos y \Rightarrow f_x(x,y)=e^x-3x^2$ and $f_y(x,y)=4y^3-4\sin y$. From the graphs, we see that to three decimal places, $f_x=0$ when $x\approx-0.459$, 0.910 , or 3.733 , and $f_y=0$ when $y\approx0$ or ±0.929 . (Alternatively, we could have used a calculator or a CAS to find the roots of $f_x=0$ and $f_y=0$.) So, to three decimal places, f has critical points at $(-0.459,0)$, $(-0.459,\pm0.929)$, $(0.910,0)$, $(0.910,\pm0.929)$, $(3.733,0)$, and $(3.733,\pm0.929)$. Now $f_{xx}=e^x-6x$, $f_{xy}=0$, $f_{yy}=12y^2-4\cos y$, and $D=\left(e^x-6x\right)\left(12y^2-4\cos y\right)$. Therefore $D(-0.459,0)<0$, $D(-0.459,\pm0.929)>0$, $f_{xx}(-0.459,\pm0.929)>0$, $D(0.910,0)>0$, $f_{xx}(0.910,0)<0$, $D(0.910,\pm0.929)<0$, $D(3.733,0)<0$, $D(3.733,\pm0.929)>0$, and $f_{xx}(3.733,\pm0.929)>0$. So $f(-0.459,\pm0.929)\approx3.868$ and $f(3.733,\pm0.929)\approx-7.077$ are local minima, $f(0.910,0)\approx5.731$ is a local maximum, and $(-0.459,0)$, $(0.910,\pm0.929)$, and $(3.733,0)$ are saddle points. The lowest points on the graph are approximately $(3.733,\pm0.929,-7.077)$.



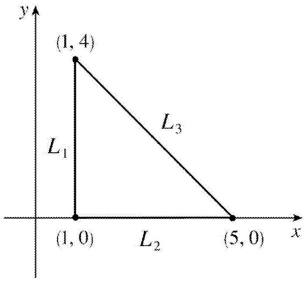




27. Since f is a polynomial it is continuous on D , so an absolute maximum and minimum exist. Here $f_x=4$, $f_y=-5$ so there are no critical points inside D . Thus the absolute extrema must both occur on the boundary. Along L_1 , $x=0$ and $f(0,y)=1-5y$ for $0\leq y\leq 3$, a decreasing function in y , so the maximum value is $f(0,0)=1$ and the minimum value is $f(0,3)=-14$. Along L_2 , $y=0$ and $f(x,0)=1+4x$ for $0\leq x\leq 2$, an increasing
function in x , so the minimum value is $f(0,0)=1$ and the maximum value is $f(2,0)=9$. Along L_3 , $y=-\frac{3}{2}x+3$ and $f\left(x,-\frac{3}{2}x+3\right)=\frac{23}{2}x-14$ for $0\leq x\leq 2$, an increasing function in x , so the minimum value is $f(0,3)=-14$ and the maximum value is $f(2,0)=9$. Thus the absolue maximum of f on D is $f(2,0)=9$ and the absolute minimum is $f(0,3)=-14$.



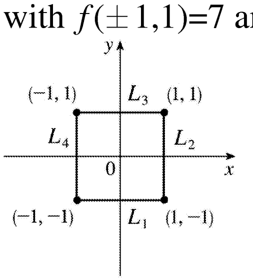
28. Since f is a polynomial it is continuous on D , so an absolute maximum and minimum exist. $f_x=y-1$, $f_y=x-2$, and setting $f_x=f_y=0$ gives $(2,1)$ as the only critical point, where $f(2,1)=1$. Along L_1 : $x=1$ and $f(1,y)=2-y$ for $0\leq y\leq 4$, a decreasing function in y , so the maximum value is $f(1,0)=2$ and the minimum value is $f(1,4)=-2$. Along L_2 : $y=0$ and $f(x,0)=3-x$ for $1\leq x\leq 5$, a decreasing function in x , so the maximum value is $f(1,0)=2$ and the minimum value is



$f(5,0)=-2$. Along L_3 , $y=5-x$ and $f(x,5-x)=-x^2+6x-7=-(x-3)^2+2$ for $1\le x\le 5$, which has a maximum at $x=3$ where $f(3,2)=2$ and a minimum at both $x=1$ and $x=5$, where $f(1,4)=f(5,0)=-2$. Thus the absolute maximum of f on D is $f(1,0)=f(3,2)=2$ and the absolute minimum is $f(1,4)=f(5,0)=-2$.

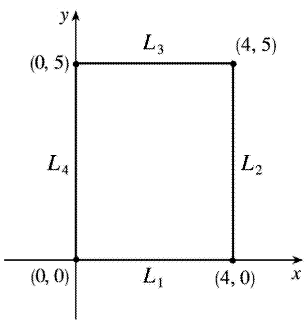
29. $f_x(x,y)=2x+2xy$, $f_y(x,y)=2y+x^2$, and setting $f_x=f_y=0$
gives $(0,0)$ as the only critical point in D , with $f(0,0)=4$.
On L_1 : $y=-1$, $f(x,-1)=5$, a constant.
On L_2 : $x=1$, $f(1,y)=y^2+y+5$, a quadratic in y which attains its maximum at $(1,1)$, $f(1,1)=7$ and its minimum at $\left(1,-\frac{1}{2}\right)$, $f\left(1,-\frac{1}{2}\right)=\frac{19}{4}$.
On L_3 : $f(x,1)=2x^2+5$ which attains its maximum at $(-1,1)$ and $(1,1)$ with $f(\pm 1,1)=7$ and its minimum at $(0,1)$, $f(0,1)=5$.
On L_4 : $f(-1,y)=y^2+y+5$ with maximum at $(-1,1)$, $f(-1,1)=7$ and

minimum at $\left(-1,-\frac{1}{2}\right)$, $f\left(-1,-\frac{1}{2}\right)=\frac{19}{4}$. Thus the absolute maximum is attained at both $(\pm 1,1)$ with $f(\pm 1,1)=7$ and the absolute minimum on D is attained at $(0,0)$ with $f(0,0)=4$.

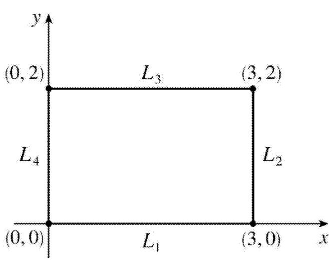


30. $f_x(x,y)=4-2x$ and $f_y(x,y)=6-2y$, so the only critical point is $(2,3)$ (which is in D) where $f(2,3)=13$. Along L_1 : $y=0$, so $f(x,0)=4x-x^2=-(x-2)^2+4$, $0\le x\le 4$, which has a maximum value when $x=2$ where $f(2,0)=4$ and a minimum value both when $x=0$ and $x=4$, where $f(0,0)=f(4,0)=0$.

Along L_2 : $x=4$, so $f(4,y)=6y-y^2=-(y-3)^2+9$, $0\le y\le 5$, which has a maximum value when $y=3$ where $f(4,3)=9$ and a minimum value when $y=0$ where $f(4,0)=0$. Along L_3 : $y=5$, so $f(x,5)=-x^2+4x+5=-(x-2)^2+9$, $0\le x\le 4$, which has a maximum value when $x=2$ where $f(2,5)=9$ and a minimum value both when $x=0$ and $x=4$, where $f(0,5)=f(4,5)=5$.
Along L_4 : $x=0$, so $f(0,y)=6y-y^2=-(y-3)^2+9$, $0\le y\le 5$, which has a maximum value when $y=3$ where $f(0,3)=9$ and a minimum value when $y=0$ where $f(0,0)=0$. Thus the absolute maximum is $f(2,3)=13$ and the absolute minimum is attained at both $(0,0)$ and $(4,0)$, where $f(0,0)=f(4,0)=0$.

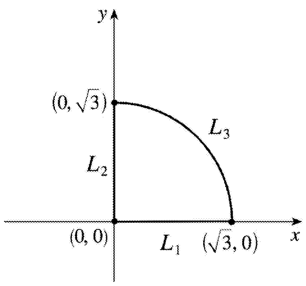


31. $f(x,y)=x^4+y^4-4xy+2$ is a polynomial and hence continuous on D , so it has an absolute maximum and minimum on D . In Exercise 7, we found the critical points of f ; only $(1,1)$ with $f(1,1)=0$ is inside D . On L_1 : $y=0$, $f(x,0)=x^4+2$, $0\le x\le 3$, a polynomial in x which attains its maximum at $x=3$, $f(3,0)=83$, and its minimum at $x=0$, $f(0,0)=2$. On L_2 : $x=3$, $f(3,y)=y^4-12y+83$, $0\le y\le 2$, a polynomial in y



which attains its minimum at $y=\sqrt[3]{3}$, $f\left(3,\sqrt[3]{3}\right)=83-9\sqrt[3]{3}\approx 70.0$, and its maximum at $y=0$, $f(3,0)=83$. On L_3 : $y=2$, $f(x,2)=x^4-8x+18$, $0\le x\le 3$, a polynomial in x which attains its minimum at $x=\sqrt[3]{2}$, $f\left(\sqrt[3]{2},2\right)=18-6\sqrt[3]{2}\approx 10.4$, and its maximum at $x=3, f(3,2)=75$. On L_4 : $x=0$, $f(0,y)=y^4+2$, $0\le y\le 2$, a polynomial in y which attains its maximum at $y=2$, $f(0,2)=18$, and its minimum at $y=0$, $f(0,0)=2$. Thus the absolute maximum of f on D is $f(3,0)=83$ and the absolute minimum is $f(1,1)=0$.

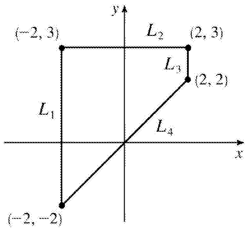
32. $f_x=y^2$ and $f_y=2xy$, and since $f_x=0 \Leftrightarrow y=0$, there are no critical points in the interior of D . Along L_1 , $y=0$ and $f(x,0)=0$. Along L_2, $x=0$ and $f(0,y)=0$. Along L_3 , $y=\sqrt{3-x^2}$, so let $g(x)=f\left(x,\sqrt{3-x^2}\right)=3x-x^3$ for $0\le x\le\sqrt{3}$. Then $g'(x)=3-3x^2=0 \Leftrightarrow x=1$. The maximum value is $f\left(1,\sqrt{2}\right)=2$ and the minimum occurs both at $x=0$ and $x=\sqrt{3}$ where $f\left(0,\sqrt{3}\right)=f\left(\sqrt{3},0\right)=0$. Thus the absolute maximum of f
on D is $f\left(1,\sqrt{2}\right)=2$, and the absolute minimum is 0 which occurs at all points along L_1 and L_2 .



33. $f_x(x,y)=6x^2$ and $f_y(x,y)=4y^3$. And so $f_x=0$ and $f_y=0$ only occur when $x=y=0$. Hence, the only critical point inside the disk is at $x=y=0$ where $f(0,0)=0$. Now on the circle $x^2+y^2=1$, $y^2=1-x^2$ so let $g(x)=f(x,y)=2x^3+\left(1-x^2\right)^2=x^4+2x^3-2x^2+1$, $-1\le x\le 1$. Then $g'(x)=4x^3+6x^2-4x=0 \Rightarrow x=0$, -2 , or $\frac{1}{2}$. $f(0,\pm 1)=g(0)=1$, $f\left(\frac{1}{2},\pm\frac{\sqrt{3}}{2}\right)=g\left(\frac{1}{2}\right)=\frac{13}{16}$, and $(-2,-3)$ is not in D . Checking the endpoints, we get $f(-1,0)=g(-1)=-2$ and $f(1,0)=g(1)=2$. Thus the absolute maximum and minimum of f on D are $f(1,0)=2$ and $f(-1,0)=-2$.
Another method: On the boundary $x^2+y^2=1$ we can write $x=\cos\theta$, $y=\sin\theta$, so $f(\cos\theta,\sin\theta)=2\cos^3\theta+\sin^4\theta$, $0\le\theta\le 2\pi$.

34. $f_x(x,y)=3x^2-3$ and $f_y(x,y)=-3y^2+12$ and the critical points are $(1,2)$, $(1,-2)$, $(-1,2)$, and $(-1,-2)$. But only $(1,2)$ and $(-1,2)$ are in D and $f(1,2)=14$, $f(-1,2)=18$. Along L_1 : $x=-2$ and $f(-2,y)=-2-y^3+12y$, $-2\le y\le 3$, which has a maximum at $y=2$ where $f(-2,2)=14$ and a minimum at $y=-2$ where $f(-2,-2)=-18$. Along L_2 : $x=2$ and $f(2,y)=2-y^3+12y$, $2\le y\le 3$, which has a maximum at
$y=2$ where $f(2,2)=18$ and a minimum at $y=3$ where $f(2,3)=11$. Along L_3 : $y=3$ and $f(x,3)=x^3-3x+9$, $-2\le x\le 2$, which has a maximum at $x=-1$ and $x=2$ where $f(-1,3)=f(2,3)=11$ and a minimum at $x=1$

and $x=-2$ where $f(1,3)=f(-2,3)=7$.
Along L_4 : $y=x$ and $f(x,x)=9x$, $-2\leq x\leq 2$, which has a maximum at $x=2$ where $f(2,2)=18$ and a minimum at $x=-2$ where $f(-2,-2)=-18$. So the absolute maximum value of f on D is $f(2,2)=18$ and the minimum is $f(-2,-2)=-18$.



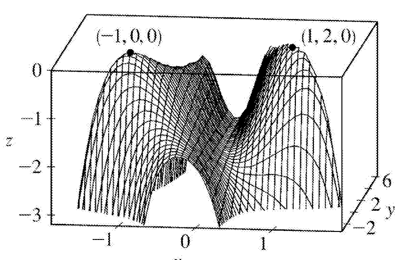
35. $f(x,y)=-\left(x^2-1\right)^2-\left(x^2y-x-1\right)^2\Rightarrow f_x(x,y)=-2\left(x^2-1\right)(2x)-2\left(x^2y-x-1\right)(2xy-1)$ and $f_y(x,y)=-2\left(x^2y-x-1\right)x^2$. Setting $f_y(x,y)=0$ gives either $x=0$ or $x^2y-x-1=0$. There are no critical points for $x=0$, since $f_x(0,y)=-2$, so we set $x^2y-x-1=0\Leftrightarrow y=\frac{x+1}{x^2}$ ($x\neq 0$), so

$$f_x\left(x,\frac{x+1}{x^2}\right)=-2\left(x^2-1\right)(2x)-2\left(x^2\frac{x+1}{x^2}-x-1\right)\left(2x\frac{x+1}{x^2}-1\right)=-4x\left(x^2-1\right)$$

. Therefore $f_x(x,y)=f_y(x,y)=0$ at the points $(1,2)$ and $(-1,0)$. To classify these critical points, we calculate $f_{xx}(x,y)=-12x^2-12x^2y^2+12xy+4y+2$, $f_{yy}(x,y)=-2x^4$, and $f_{xy}(x,y)=-8x^3y+6x^2+4x$.
In order to use the Second Derivatives Test we calculate

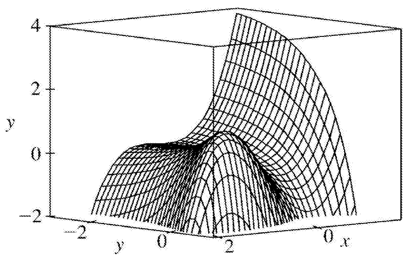
$$\begin{aligned} D(-1,0) &= f_{xx}(-1,0)f_{yy}(-1,0)-[f_{xy}(-1,0)]^2 \\ &= 16>0 , \end{aligned}$$

$f_{xx}(-1,0)=-10<0$, $D(1,2)=16>0$, and $f_{xx}(1,2)=-26<0$, so both $(-1,0)$ and $(1,2)$ give local maxima.



36. $f(x,y)=3xe^y-x^3-e^{3y}$ is differentiable everywhere, so the requirement for critical points is that (1) $f_x=3e^y-3x^2=0$ and (2) $f_y=3xe^y-3e^{3y}=0$. From (1) we obtain $e^y=x^2$, and then (2) gives $3x^3-3x^6=0\Rightarrow$

$x=1$ or 0 , but only $x=1$ is valid, since $x=0$ makes (1) impossible. So substituting $x=1$ into (1) gives $y=0$, and the only critical point is $(1,0)$.



The Second Derivatives Test shows that this gives a local maximum, since $D(1,0)=\left[-6x\left(3xe^{y}-9e^{3y}\right)-\left(3e^{y}\right)^{2}\right]_{(1,0)}=27>0$ and $f_{xx}(1,0)=[-6x]_{(1,0)}=-6<0$. But $f(1,0)=1$ is not an absolute maximum because, for instance, $f(-3,0)=17$. This can also be seen from the graph.

37. Let d be the distance from $(2,1,-1)$ to any point (x,y,z) on the plane $x+y-z=1$, so $d=\sqrt{(x-2)^{2}+(y-1)^{2}+(z+1)^{2}}$ where $z=x+y-1$, and we minimize $d^{2}=f(x,y)=(x-2)^{2}+(y-1)^{2}+(x+y)^{2}$. Then $f_{x}(x,y)=2(x-2)+2(x+y)=4x+2y-4$, $f_{y}(x,y)=2(y-1)+2(x+y)=2x+4y-2$. Solving $4x+2y-4=0$ and $2x+4y-2=0$ simultaneously gives $x=1$, $y=0$. An absolute minimum exists (since there is a minimum distance from the point to the plane) and it must occur at a critical point, so the shortest distance occurs for $x=1$, $y=0$ for which $d=\sqrt{(1-2)^{2}+(0-1)^{2}+(1+0)^{2}}=\sqrt{3}$.

38. Here the distance d from a point on the plane to the point $(1,2,3)$ is $d=\sqrt{(x-1)^{2}+(y-2)^{2}+(z-3)^{2}}$, where $z=4-x+y$. We can minimize $d^{2}=f(x,y)=(x-1)^{2}+(y-2)^{2}+(1-x+y)^{2}$, so $f_{x}(x,y)=2(x-1)+2(1-x+y)(-1)=4x-2y-4$ and $f_{y}(x,y)=2(y-2)+2(1-x+y)=4y-2x-2$. Solving $4x-2y-4=0$ and $4y-2x-2=0$ simultaneously gives $x=\frac{5}{3}$ and $y=\frac{4}{3}$, so the only critical point is $\left(\frac{5}{3},\frac{4}{3}\right)$. This point must correspond to the minimum distance, so the point on the plane closest to $(1,2,3)$ is $\left(\frac{5}{3},\frac{4}{3},\frac{11}{3}\right)$.

39. Minimize $d^{2}=x^{2}+y^{2}+z^{2}=x^{2}+y^{2}+xy+1$. Then $f_{x}=2x+y$, $f_{y}=2y+x$ so the critical point is $(0,0)$ and $D(0,0)=4-1>0$ with $f_{xx}(0,0)=2$ so this is a minimum. Thus $z^{2}=1$ or $z=\pm 1$ and the points on the surface are $(0,0,\pm 1)$.

40. Since $z=1/\left(x^2y^2\right)$ on the surface, we minimize $d^2=x^2+y^2+z^2=x^2+y^2+x^{-4}y^{-4}=f(\mathrm{x,y})$. $f_x=2x-\dfrac{4}{x^5y^4}$, $f_y=2y-\dfrac{4}{x^4y^5}$, so the critical points occur when $2x=\dfrac{4}{x^5y^4}$ and $2y=\dfrac{4}{x^4y^5}$ or $x^6y^4=2=x^4y^6$, so $x^2=y^2\Rightarrow x=\pm y$ and $x^{10}=2\Rightarrow x=\pm 2^{1/10}$, $y=\pm 2^{1/10}$. The four critical points are $(\pm 2^{1/10},\pm 2^{1/10})$. The absolute minimum must occur at these points (there is no maximum since the surface is infinite in extent). Thus the points on the surface closest to the origin are $(\pm 2^{1/10},\pm 2^{1/10},2^{-2/5})$.

41. $x+y+z=100$, so maximize $f(\mathrm{x,y})=xy(100-x-y)$. $f_x=100y-2xy-y^2$, $f_y=100x-x^2-2xy$, $f_{xx}=-2y$, $f_{yy}=-2x$, $f_{xy}=100-2x-2y$. Then $f_x=0$ implies $y=0$ or $y=100-2x$. Substituting $y=0$ into $f_y=0$ gives $x=0$ or $x=100$ and substituting $y=100-2x$ into $f_y=0$ gives $3x^2-100x=0$ so $x=0$ or $\dfrac{100}{3}$. Thus the critical points are $(0,0)$, $(100,0)$, $(0,100)$ and $\left(\dfrac{100}{3},\dfrac{100}{3}\right)$.
$D(0,0)=D(100,0)=D(0,100)=-10,000$ while $D\left(\dfrac{100}{3},\dfrac{100}{3}\right)=\dfrac{10,000}{3}$ and $f_{xx}\left(\dfrac{100}{3},\dfrac{100}{3}\right)=-\dfrac{200}{3}<0$. Thus $(0,0)$, $(100,0)$ and $(0,100)$ are saddle points whereas $f\left(\dfrac{100}{3},\dfrac{100}{3}\right)$ is a local maximum. Thus the numbers are $x=y=z=\dfrac{100}{3}$.

42. Maximize $f(\mathrm{x,y})=x^ay^b(100-x-y)^c$.
$f_x=ax^{a-1}y^b(100-x-y)^c-cx^ay^b(100-x-y)^{c-1}=x^{a-1}y^b(100-x-y)^{c-1}[a(100-x-y)-cx]$
and $f_y=x^ay^{b-1}(100-x-y)^{c-1}[b(100-x-y)-cy]$. Since x , y and z are all positive, the only critical point occurs when $x=a\dfrac{100-y}{a+c}$ and $y=\dfrac{100b}{a+b+c}$. Thus the point is $\left(\dfrac{100a}{a+b+c},\dfrac{100b}{a+b+c}\right)$ and the numbers are $x=\dfrac{100a}{a+b+c}$, $y=\dfrac{100b}{a+b+c}$, $z=\dfrac{100c}{a+b+c}$.

43. Maximize $f(\mathrm{x,y})=xy\left(36-9x^2-36y^2\right)^{1/2}/2$ with $(\mathrm{x,y,z})$ in first octant. Then
$$f_x=\frac{y\left(36-9x^2-36y^2\right)^{1/2}}{2}+\frac{-9x^2y\left(36-9x^2-36y^2\right)^{-1/2}}{2}=\frac{\left(36y-18x^2y-36y^3\right)}{2\left(36-9x^2-36y^2\right)^{1/2}}$$ and

$f_y = \dfrac{36x - 9x^3 - 72xy^2}{2\left(36 - 9x^2 - 36y^2\right)^{1/2}}$. Setting $f_x = 0$ gives $y = 0$ or $y^2 = \dfrac{2 - x^2}{2}$ but $y > 0$, so only the latter solution applies. Substituting this y into $f_y = 0$ gives $x^2 = \dfrac{4}{3}$ or $x = \dfrac{2}{\sqrt{3}}$, $y = \dfrac{1}{\sqrt{3}}$ and then $z^2 = (36 - 12 - 12)/4 = 3$. The fact that this gives a maximum volume follows from the geometry. This maximum volume is $V = (2x)(2y)(2z) = 8\left(\dfrac{2}{\sqrt{3}}\right)\left(\dfrac{1}{\sqrt{3}}\right)(\sqrt{3}) = \dfrac{16}{\sqrt{3}}$.

44. Here maximize $f(x,y) = xy\,\dfrac{\left(a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2\right)^{1/2}}{a^2b^2}$. Then

$f_x = yc^2\,\dfrac{a^2b^2 - 2b^2x^2 - a^2y^2}{a^2b^2\left(a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2\right)^{1/2}}$ and $f_y = xc^2\,\dfrac{a^2b^2 - 2a^2y^2 - b^2x^2}{a^2b^2\left(a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2\right)^{1/2}}$. Then $f_x = 0$ (with x , $y > 0$) implies $y^2 = \dfrac{a^2b^2 - 2b^2x^2}{a^2}$ and substituting into $f_y = 0$ implies $3b^2x^2 = a^2b^2$ or $x = \dfrac{1}{\sqrt{3}}\,a$, $y = \dfrac{1}{\sqrt{3}}\,b$ and then $z = \dfrac{1}{\sqrt{3}}\,c$. Thus the maximum volume of such a rectangle is $V = (2x)(2y)(2z) = \dfrac{8}{3\sqrt{3}}\,abc$.

45. Maximize $f(x,y) = \dfrac{xy}{3}(6 - x - 2y)$, then the maximum volume is $V = xyz$. $f_x = \dfrac{1}{3}\left(6y - 2xy - y^2\right) = \dfrac{1}{3}\,y(6 - 2x - 2y)$ and $f_y = \dfrac{1}{3}\,x(6 - x - 4y)$. Setting $f_x = 0$ and $f_y = 0$ gives the critical point $(2,1)$ which geometrically must yield a maximum. Thus the volume of the largest such box is $V = (2)(1)\left(\dfrac{2}{3}\right) = \dfrac{4}{3}$.

46. Surface area $= 2(xy + xz + yz) = 64\text{ cm}^2$, so $xy + xz + yz = 32$ or $z = \dfrac{32 - xy}{x + y}$. Maximize the volume $f(x,y) = xy\,\dfrac{32 - xy}{x + y}$. Then $f_x = \dfrac{32y^2 - 2xy^3 - x^2y^2}{(x+y)^2} = y^2\,\dfrac{32 - 2xy - x^2}{(x+y)^2}$ and $f_y = x^2\,\dfrac{32 - 2xy - y^2}{(x+y)^2}$. Setting $f_x = 0$ implies $y = \dfrac{32 - x^2}{2x}$ and substituting into $f_y = 0$ gives $32\left(4x^2\right) - \left(32 - x^2\right)\left(4x^2\right) - \left(32 - x^2\right)^2 = 0$ or $3x^4 + 64x^2 - (32)^2 = 0$. Thus $x^2 = \dfrac{64}{6}$ or $x = \dfrac{8}{\sqrt{6}}$, $y = \dfrac{64/3}{16/\sqrt{6}} = \dfrac{8}{\sqrt{6}}$ and $z = \dfrac{8}{\sqrt{6}}$. Thus the box is a cube

with edge length $\frac{8}{\sqrt{6}}$ cm.

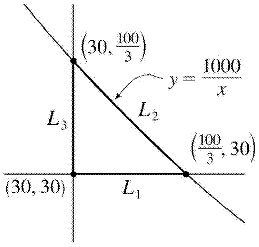
47. Let the dimensions be x , y , and z ; then $4x+4y+4z=c$ and the volume is $V=xyz=xy\left(\frac{1}{4}c-x-y\right)=\frac{1}{4}cxy-x^2y-xy^2$, $x>0$, $y>0$. Then $V_x=\frac{1}{4}cy-2xy-y^2$ and $V_y=\frac{1}{4}cx-x^2-2xy$, so $V_x=0=V_y$ when $2x+y=\frac{1}{4}c$ and $x+2y=\frac{1}{4}c$. Solving, we get $x=\frac{1}{12}c$, $y=\frac{1}{12}c$ and $z=\frac{1}{4}c-x-y=\frac{1}{12}c$. From the geometrical nature of the problem, this critical point must give an absolute maximum. Thus the box is a cube with edge length $\frac{1}{12}c$.

48. The cost equals $5xy+2(xz+yz)$ and $xyz=V$, so $C(\mathrm{x,y})=5xy+2V(x+y)/(xy)=5xy+2V\left(x^{-1}+y^{-1}\right)$. Then $C_x=5y-2Vx^{-2}$, $C_y=5x-2Vy^{-2}$, $f_x=0$ implies $y=2V/\left(5x^2\right)$, $f_y=0$ implies $x=\sqrt[3]{\frac{2}{5}V}=y$. Thus the dimensions of the aquarium which minimize the cost are $x=y=\sqrt[3]{\frac{2}{5}V}$ units, $z=V^{1/3}\left(\frac{5}{2}\right)^{2/3}$.

49. Let the dimensions be x , y and z , then minimize $xy+2(xz+yz)$ if $xyz=32\ ,000\ \mathrm{m}^3$. Then $f(\mathrm{x,y})=xy+[64]\ ,000(x+y)/xy]=xy+64\ ,000(x^{-1}+y^{-1})$, $f_x=y-64\ ,000x^{-2}$, $f_y=x-64\ ,000y^{-2}$. And $f_x=0$ implies $y=64\ ,000/x^2$; substituting into $f_y=0$ implies $x^3=64\ ,000$ or $x=40$ and then $y=40$. Now $D(\mathrm{x,y})=[(2)(64]\ ,000)]^2x^{-3}y^{-3}-1>0$ for $(40,40)$ and $f_{xx}(40,40)>0$ so this is indeed a minimum. Thus the dimensions of the box are $x=y=40$ cm, $z=20$ cm.

50. Let x be the length of the north and south walls, y the length of the east and west walls, and z the height of the building. The heat loss is given by $h=10(2yz)+8(2xz)+1(xy)+5(xy)=6xy+16xz+20\mathrm{yz}$. The volume is $4000\ \mathrm{m}^3$, so $xyz=4000$, and we substitute $z=\frac{4000}{xy}$ to obtain the heat loss function $h(\mathrm{x,y})=6xy+80\ ,000/x+64\ ,000/y$.

(a) Since $z=\frac{4000}{xy}\geq 4$, $xy\leq 1000\Rightarrow y\leq 1000/x$.
Also $x\geq 30$ and $y\geq 30$, so the domain of h is $D=\{(\mathrm{x,y})\mid x\geq 30,30\leq y\leq 1000/x\}$.



(b) $h(x,y)=6xy+80,000x^{-1}+64,000y^{-1} \Rightarrow h_x=6y-80,000x^{-2}$, $h_y=6x-64,000y^{-2}$.

$h_x=0$ implies $6x^2y=80,000 \Rightarrow y=\dfrac{80,000}{6x^2}$ and substituting into $h_y=0$ gives $6x=64,000\left(\dfrac{6x^2}{80,000}\right)^2$

$\Rightarrow x^3=\dfrac{80,000^2}{6\cdot 64,000}=\dfrac{50,000}{3}$, so $x=\sqrt[3]{\dfrac{50,000}{3}}=10\sqrt[3]{\dfrac{50}{3}} \Rightarrow y=\dfrac{80}{\sqrt[3]{60}}$,

and the only critical point of h is $\left(10\sqrt[3]{\dfrac{50}{3}}, \dfrac{80}{\sqrt[3]{60}}\right)\approx(25.54,20.43)$ which is not in D. Next we check the boundary of D. On L_1: $y=30$, $h(x,30)=180x+80,000/x+6400/3$, $30\le x\le \dfrac{100}{3}$. Since $h'(x,30)=180-80,000/x^2>0$ for $30\le x\le \dfrac{100}{3}$, $h(x,30)$ is an increasing function with minimum $h(30,30)=10,200$ and maximum $h\left(\dfrac{100}{3},30\right)\approx 10,533$. On L_2: $y=1000/x$, $h(x,1000/x)=6000+64x+80,000/x$, $30\le x\le \dfrac{100}{3}$. Since $h'(x,1000/x)=64-80,000/x^2<0$ for $30\le x\le \dfrac{100}{3}$, $h(x,1000/x)$ is a decreasing function with minimum $h\left(\dfrac{100}{3},30\right)\approx 10,533$ and maximum $h\left(30,\dfrac{100}{3}\right)\approx 10,587$. On L_3: $x=30$, $h(30,y)=180y+64,000/y+8000/3$, $30\le y\le \dfrac{100}{3}$.

$h'(30,y)=180-64,000/y^2>0$ for $30\le y\le \dfrac{100}{3}$, so $h(30,y)$ is an increasing function of y with minimum $h(30,30)=10,200$ and maximum $h\left(30,\dfrac{100}{3}\right)\approx 10,587$. Thus the absolute minimum of h is $h(30,30)=10,200$, and the dimensions of the building that minimize heat loss are walls 30 m in length and height $\dfrac{4000}{30^2}=\dfrac{40}{9}\approx 4.44$ m.

(c) From part (b), the only critical point of h, which gives a local (and absolute) minimum, is approximately $h(25.54,20.43)\approx 9396$. So a building of volume 4000 m^2 with dimensions $x\approx 25.54$ m, $y\approx 20.43$ m,

$z\approx \dfrac{4000}{(25.54)(20.43)}\approx 7.67$ m has the least amount of heat loss.

51. Let x,y , z be the dimensions of the rectangular box. Then the volume of the box is xyz and $L=\sqrt{x^2+y^2+z^2}\Rightarrow L^2=x^2+y^2+z^2\Rightarrow z=\sqrt{L^2-x^2-y^2}$. Substituting, we have volume $V(x,y)=xy\sqrt{L^2-x^2-y^2}$, x,y>0 .

$V_x=xy\cdot\frac{1}{2}\left(L^2-x^2-y^2\right)^{-1/2}(-2x)+y\sqrt{L^2-x^2-y^2}=y\sqrt{L^2-x^2-y^2}-\dfrac{x^2y}{\sqrt{L^2-x^2-y^2}}$,

$V_y=x\sqrt{L^2-x^2-y^2}-\dfrac{xy^2}{\sqrt{L^2-x^2-y^2}}$,

$V_x=0$ implies $y\left(L^2-x^2-y^2\right)=x^2y\Rightarrow y\left(L^2-2x^2-y^2\right)=0\Rightarrow 2x^2+y^2=L^2$ (since $y>0$), and $V_y=0$ implies $x\left(L^2-x^2-y^2\right)=xy^2\Rightarrow x\left(L^2-x^2-2y^2\right)=0\Rightarrow x^2+2y^2=L^2$ (since $x>0$). Substituting $y^2=L^2-2x^2$ into $x^2+2y^2=L^2$ gives $x^2+2L^2-4x^2=L^2\Rightarrow 3x^2=L^2\Rightarrow x=L/\sqrt{3}$ (since $x>0$) and then $y=\sqrt{L^2-2\left(L/\sqrt{3}\right)^2}=L/\sqrt{3}$. So the only critical point is $\left(L/\sqrt{3},L/\sqrt{3}\right)$ which, from the geometrical nature of the problem, must give an absolute maximum. Thus the maximum volume is

$V\left(L/\sqrt{3},L/\sqrt{3}\right)=\left(L/\sqrt{3}\right)^2\sqrt{L^2-\left(L/\sqrt{3}\right)^2-\left(L/\sqrt{3}\right)^2}=L^3/\left(3\sqrt{3}\right)$ cubic units.

52. Since $p+q+r=1$ we can substitute $p=1-r-q$ into P giving $P=P(q,r)=2(1-r-q)q+2(1-r-q)r+2rq=2q-2q^2+2r-2r^2-2rq$. Since p , q and r represent proportions and $p+q+r=1$, we know $q\geq 0$, $r\geq 0$, and $q+r\leq 1$. Thus, we want to find the absolute maximum of the continuous function $P(q,r)$ on the closed set D enclosed by the lines $q=0$, $r=0$, and $q+r=1$. To find any critical points, we set the partial derivatives equal to zero: $P_q(q,r)=2-4q-2r=0$ and $P_r(q,r)=2-4r-2q=0$. The first equation gives $r=1-2q$, and substituting into the second equation we have

$2-4(1-2q)-2q=0\Rightarrow q=\frac{1}{3}$. Then we have one critical point, $\left(\frac{1}{3},\frac{1}{3}\right)$, where $P\left(\frac{1}{3},\frac{1}{3}\right)=\frac{2}{3}$. Next we find the maximum values of P on the boundary of D which consists of three line segments. For the segment given by $r=0$, $0\leq q\leq 1$, $P(q,r)=P(q,0)=2q-2q^2$, $0\leq q\leq 1$. This represents a parabola with maximum value $P\left(\frac{1}{2},0\right)=\frac{1}{2}$. On the segment $q=0$, $0\leq r\leq 1$ we have $P(0,r)=2r-2r^2$, $0\leq r\leq 1$. This represents a parabola with maximum value $P\left(0,\frac{1}{2}\right)=\frac{1}{2}$. Finally, on the segment $q+r=1$, $0\leq q\leq 1$,

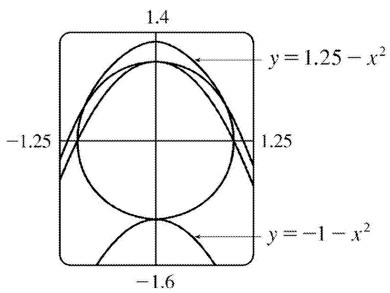
$P(\mathrm{q,r})=P(\mathrm{q},1-q)=2q-2q^2$, $0\leq q\leq 1$ which has a maximum value of $P\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2}$. Comparing these values with the value of P at the critical point, we see that the absolute maximum value of $P(\mathrm{q,r})$ on D is $\frac{2}{3}$.

53. Note that here the variables are m and b , and $f(\mathrm{m,b})=\sum_{i=1}^{n}\left[y_i-\left(mx_i+b\right)\right]^2$. Then $f_m=\sum_{i=1}^{n}-2x_i\left[y_i-\left(mx_i+b\right)\right]=0$ implies $\sum_{i=1}^{n}\left(x_iy_i-mx_i^2-bx_i\right)=0$ or $\sum_{i=1}^{n}x_iy_i=m\sum_{i=1}^{n}x_i^2+b\sum_{i=1}^{n}x_i$ and $f_b=\sum_{i=1}^{n}-2\left[y_i-\left(mx_i+b\right)\right]=0$ implies $\sum_{i=1}^{n}y_i=m\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}b=m\left(\sum_{i=1}^{n}x_i\right)+nb$. Thus we have the two desired equations. Now $f_{mm}=\sum_{i=1}^{n}2x_i^2$, $f_{bb}=\sum_{i=1}^{n}2=2n$ and $f_{mb}=\sum_{i=1}^{n}2x_i$. And $f_{mm}(\mathrm{m,b})>0$ always and $D(\mathrm{m,b})=4n\left(\sum_{i=1}^{n}x_i^2\right)-4\left(\sum_{i=1}^{n}x_i\right)^2=4\left[n\left(\sum_{i=1}^{n}x_i^2\right)-\left(\sum_{i=1}^{n}x_i\right)^2\right]>0$ always so the solutions of these two equations do indeed minimize $\sum_{i=1}^{n}d_i^2$.

54. Any such plane must cut out a tetrahedron in the first octant. We need to minimize the volume of the tetrahedron that passes through the point $(1,2,3)$. Writing the equation of the plane as $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, the volume of the tetrahedron is given by $V=\frac{abc}{6}$. But $(1,2,3)$ must lie on the plane, so we need $\frac{1}{a}+\frac{2}{b}+\frac{3}{c}=1$ (*) and thus can think of c as a function of a and b . Then $V_a=\frac{b}{6}\left(c+a\frac{\partial c}{\partial a}\right)$ and $V_b=\frac{a}{6}\left(c+b\frac{\partial c}{\partial b}\right)$. Differentiating (*) with respect to a we get $-a^{-2}-3c^{-2}\frac{\partial c}{\partial a}=0\Rightarrow\frac{\partial c}{\partial a}=\frac{-c^2}{3a^2}$, and differentiating (*) with respect to b gives $-2b^{-2}-3c^{-2}\frac{\partial c}{\partial b}=0\Rightarrow$ $\frac{\partial c}{\partial b}=\frac{-2c^2}{3b^2}$. Then $V_a=\frac{b}{6}\left(c+a\frac{-c^2}{3a^2}\right)=0\Rightarrow c=3a$, and $V_b=\frac{a}{6}\left(c+b\frac{-2c^2}{3b^2}\right)=0\Rightarrow c=\frac{3}{2}b$. Thus $3a=\frac{3}{2}b$ or $b=2a$. Putting these into (*) gives $\frac{3}{a}=1$ or $a=3$ and then $b=6$, $c=9$. Thus the equation of the required plane is $\frac{x}{3}+\frac{y}{6}+\frac{z}{9}=1$ or $6x+3y+2z=18$.

1. At the extreme values of f , the level curves of f just touch the curve $g(x,y)=8$ with a common tangent line. (See Figure 1 and the accompanying discussion.) We can observe several such occurrences on the contour map, but the level curve $f(x,y)=c$ with the largest value of c which still intersects the curve $g(x,y)=8$ is approximately $c=59$, and the smallest value of c corresponding to a level curve which intersects $g(x,y)=8$ appears to be $c=30$. Thus we estimate the maximum value of f subject to the constraint $g(x,y)=8$ to be about 59 and the minimum to be 30 .

2. **(a)** The values $c=\pm 1$ and $c=1.25$ seem to give curves which are tangent to the circle. These values represent possible extreme values of the function x^2+y subject to the constraint $x^2+y^2=1$.



(b) $\nabla f=\langle 2x,1\rangle$, $\lambda\nabla g=\langle 2\lambda x,2\lambda y\rangle$. So $2x=2\lambda x\Rightarrow$ either $\lambda=1$ or $x=0$. If $\lambda=1$, then $y=\frac{1}{2}$ and so $x=\pm\frac{\sqrt{3}}{2}$ (from the constraint). If $x=0$, then $y=\pm 1$. Therefore f has possible extreme values at the points $(0,\pm 1)$ and $\left(\pm\frac{\sqrt{3}}{2},\frac{1}{2}\right)$. We calculate $f\left(\pm\frac{\sqrt{3}}{2},\frac{1}{2}\right)=\frac{5}{4}$ (the maximum value), $f(0,1)=1$, and $f(0,-1)=-1$ (the minimum value). These are our answers from (a).

3. $f(x,y)=x^2-y^2$, $g(x,y)=x^2+y^2=1\Rightarrow\nabla f=\langle 2x,-2y\rangle$, $\lambda\nabla g=\langle 2\lambda x,2\lambda y\rangle$. Then $2x=2\lambda x$ implies $x=0$ or $\lambda=1$. If $x=0$, then $x^2+y^2=1$ implies $y=\pm 1$ and if $\lambda=1$, then $-2y=2\lambda y$ implies $y=0$ and thus $x=\pm 1$. Thus the possible points for the extreme values of f are $(\pm 1,0)$, $(0,\pm 1)$. But $f(\pm 1,0)=1$ while $f(0,\pm 1)=-1$ so the maximum value of f on $x^2+y^2=1$ is $f(\pm 1,0)=1$ and the minimum value is $f(0,\pm 1)=-1$.

4. $f(x,y)=4x+6y$, $g(x,y)=x^2+y^2=13\Rightarrow\nabla f=\langle 4,6\rangle$, $\lambda\nabla g=\langle 2\lambda x,2\lambda y\rangle$. Then $2\lambda x=4$ and $2\lambda y=6$ imply $x=\frac{2}{\lambda}$ and $y=\frac{3}{\lambda}$. But $13=x^2+y^2=\left(\frac{2}{\lambda}\right)^2+\left(\frac{3}{\lambda}\right)^2\Rightarrow 13=\frac{13}{\lambda^2}\Rightarrow\lambda=\pm 1$, so f has possible extreme values at the points $(2,3)$, $(-2,-3)$. We compute $f(2,3)=26$ and $f(-2,-3)=-26$, so the maximum value of f on $x^2+y^2=13$ is $f(2,3)=26$ and the minimum value is $f(-2,-3)=-26$.

5. $f(x,y)=x^2y$, $g(x,y)=x^2+2y^2=6 \Rightarrow \nabla f=\left\langle 2xy,x^2\right\rangle$, $\lambda\nabla g=\langle 2\lambda x,4\lambda y\rangle$. Then $2xy=2\lambda x$ implies $x=0$ or $\lambda=y$. If $x=0$, then $x^2=4\lambda y$ implies $\lambda=0$ or $y=0$. However, if $y=0$ then $g(x,y)=0$, a contradiction. So $\lambda=0$ and then $g(x,y)=6 \Rightarrow y=\pm\sqrt{3}$. If $\lambda=y$, then $x^2=4\lambda y$ implies $x^2=4y^2$, and so $g(x,y)=6 \Rightarrow$ $4y^2+2y^2=6 \Rightarrow y^2=1 \Rightarrow y=\pm 1$. Thus f has possible extreme values at the points $(0,\pm\sqrt{3})$, $(\pm 2,1)$, and $(\pm 2,-1)$. After evaluating f at these points, we find the maximum value to be $f(\pm 2,1)=4$ and the minimum to be $f(\pm 2,-1)=-4$.

6. $f(x,y)=x^2+y^2$, $g(x,y)=x^4+y^4=1 \Rightarrow \nabla f=\langle 2x,2y\rangle$, $\lambda\nabla g=\left\langle 4\lambda x^3,4\lambda y^3\right\rangle$. Then $x=2\lambda x^3$ implies $x=0$ or $\lambda=\frac{1}{2x^2}$. If $x=0$, then $x^4+y^4=1$ implies $y=\pm 1$. But $y=2\lambda y^3$ implies $y=0$ so $x=\pm 1$ or $\lambda=\frac{1}{2y^2}$ and $x^2=y^2$ and $2x^4=1$ so $x=\pm\frac{1}{\sqrt[4]{2}}$. Hence the possible points are $(0,\pm 1)$, $(\pm 1,0)$, $\left(\pm\frac{1}{\sqrt{[4]2}},\pm\frac{1}{\sqrt[4]{2}}\right)$, with the maximum value of f on $x^4+y^4=1$ being $f\left(\pm\frac{1}{\sqrt[4]{2}},\pm\frac{1}{\sqrt[4]{2}}\right)=\frac{2}{\sqrt{2}}=\sqrt{2}$ and the minimum value being $f(0,\pm 1)=f(\pm 1,0)=1$.

7. $f(x,y,z)=2x+6y+10z$, $g(x,y,z)=x^2+y^2+z^2=35 \Rightarrow \nabla f=\langle 2,6,10\rangle$, $\lambda\nabla g=\langle 2\lambda x,2\lambda y,2\lambda z\rangle$. Then $2\lambda x=2$, $2\lambda y=6$, $2\lambda z=10$ imply $x=\frac{1}{\lambda}$, $y=\frac{3}{\lambda}$, and $z=\frac{5}{\lambda}$. But $35=x^2+y^2+z^2=\left(\frac{1}{\lambda}\right)^2+\left(\frac{3}{\lambda}\right)^2+\left(\frac{5}{\lambda}\right)^2 \Rightarrow$ $35=\frac{35}{\lambda^2} \Rightarrow \lambda=\pm 1$, so f has possible extreme values

at the points $(1,3,5)$, $(-1,-3,-5)$. The maximum value of f on $x^2+y^2+z^2=35$ is $f(1,3,5)=70$, and the minimum is $f(-1,-3,-5)=-70$.

8. $f(x,y,z)=8x-4z$, $g(x,y,z)=x^2+10y^2+z^2=5 \Rightarrow \nabla f=\langle 8,0,-4\rangle$, $\lambda\nabla g=\langle 2\lambda x,20\lambda y,2\lambda z\rangle$. Then $2\lambda x=8$, $20\lambda y=0$, $2\lambda z=-4$ imply $x=\frac{4}{\lambda}$, $y=0$, and $z=-\frac{2}{\lambda}$. But $5=x^2+10y^2+z^2=\left(\frac{4}{\lambda}\right)^2+10(0)^2+\left(-\frac{2}{\lambda}\right)^2 \Rightarrow$ $5=\frac{20}{\lambda^2} \Rightarrow \lambda=\pm 2$, so f has possible extreme values at the points $(2,0,-1)$, $(-2,0,1)$. The maximum of f on $x^2+10y^2+z^2=5$ is $f(2,0,-1)=20$, and the minimum is $f(-2,0,1)=-20$.

9. $f(x,y,z)=xyz$, $g(x,y,z)=x^2+2y^2+3z^2=6 \Rightarrow \nabla f=\langle yz,xz,xy\rangle$, $\lambda\nabla g=\langle 2\lambda x,4\lambda y,6\lambda z\rangle$. Then $\nabla f=\lambda\nabla g$ implies $\lambda=(yz)/(2x)=(xz)/(4y)=(xy)/(6z)$ or $x^2=2y^2$ and $z^2=\frac{2}{3}y^2$. Thus $x^2+2y^2+3z^2=6$ implies $6y^2=6$ or

$y=\pm 1$. Then the possible points are $\left(\sqrt{2},\pm 1,\sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2},\pm 1,-\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2},\pm 1,\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2},\pm 1,-\sqrt{\frac{2}{3}}\right)$. The maximum value of f on the ellipsoid is $\frac{2}{\sqrt{3}}$, occurring when all coordinates are positive or exactly two are negative and the minimum is $-\frac{2}{\sqrt{3}}$ occurring when 1 or 3 of the coordinates are negative.

10. $f(x,y,z)=x^2y^2z^2$, $g(x,y,z)=x^2+y^2+z^2=1\Rightarrow \nabla f=\left\langle 2xy^2z^2,2yx^2z^2,2zx^2y^2\right\rangle$, $\lambda\nabla g=\langle 2\lambda x,2\lambda y,2\lambda z\rangle$. Then $\nabla f=\lambda\nabla g$ implies (1) $\lambda=y^2z^2=x^2z^2=x^2y^2$ and $\lambda\neq 0$, or (2) $\lambda=0$ and one or two (but not three) of the coordinates are 0 . If (1) then $x^2=y^2=z^2=\frac{1}{3}$. The minimum value of f on the sphere occurs in case (2) with a value of 0 and the maximum value is $\frac{1}{27}$ which arises from all the points from (1), that is, the points $\left(\pm\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$.

11. $f(x,y,z)=x^2+y^2+z^2$, $g(x,y,z)=x^4+y^4+z^4=1\Rightarrow$
$\nabla f=\langle 2x,2y,2z\rangle$, $\lambda\nabla g=\left\langle 4\lambda x^3,4\lambda y^3,4\lambda z^3\right\rangle$.
Case 1: If $x\neq 0$, $y\neq 0$ and $z\neq 0$, then $\nabla f=\lambda\nabla g$ implies $\lambda=1/\left(2x^2\right)=1/\left(2y^2\right)=1/\left(2z^2\right)$ or $x^2=y^2=z^2$ and $3x^4=1$ or $x=\pm\frac{1}{\sqrt[4]{3}}$ giving the points $\left(\pm\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}},-\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}},-\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}},-\frac{1}{\sqrt[4]{3}},-\frac{1}{\sqrt[4]{3}}\right)$ all with an f –value of $\sqrt{3}$. *Case 2:* If one of the variables equals zero and the other two are not zero, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{\sqrt{2}}$ and corresponding f value of $\sqrt{2}$. *Case 3:* If exactly two of the variables are zero, then the third variable has value ± 1 with the corresponding f value of 1 . Thus on $x^4+y^4+z^4=1$, the maximum value of f is $\sqrt{3}$ and the minimum value is 1 .

12. $f(x,y,z)=x^4+y^4+z^4$, $g(x,y,z)=x^2+y^2+z^2=1\Rightarrow$
$\nabla f=\left\langle 4x^3,4y^3,4z^3\right\rangle$, $\lambda\nabla g=\langle 2\lambda x,2\lambda y,2\lambda z\rangle$.
Case 1: If $x\neq 0$, $y\neq 0$ and $z\neq 0$ then $\nabla f=\lambda\nabla g$ implies $\lambda=2x^2=2y^2=2z^2$ or $x^2=y^2=z^2=\frac{1}{3}$ yielding 8 points each with an f –value of $\frac{1}{3}$.
Case 2:

If one of the variables is 0 and the other two are not, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{2}$ and the corresponding f –value is $\frac{1}{2}$.
Case 3: If exactly two of the variables are 0 , then the third variable has value ± 1 with corresponding f –value of 1 . Thus on $x^2+y^2+z^2=1$, the maximum value of f is 1 and the minimum value is $\frac{1}{3}$.

13. $f(x,y,z,t)=x+y+z+t$, $g(x,y,z,t)=x^2+y^2+z^2+t^2=1 \Rightarrow \langle 1,1,1,1\rangle=\langle 2\lambda x,2\lambda y,2\lambda z,2\lambda t\rangle$, so $\lambda=1/(2x)=1/(2y)=1/(2z)=1/(2t)$ and $x=y=z=t$. But $x^2+y^2+z^2+t^2=1$, so the possible points are $\left(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2}\right)$. Thus the maximum value of f is $f\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)=2$ and the minimum value is $f\left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)=-2$.
14. $f\left(x_1,x_2,\ldots,x_n\right)=x_1+x_2+\cdots+x_n$, $g\left(x_1,x_2,\ldots,x_n\right)=x_1^2+x_2^2+\cdots+x_n^2=1\Rightarrow$ $\langle 1,1,\ldots,1\rangle=\left\langle 2\lambda x_1,2\lambda x_2,\ldots,2\lambda x_n\right\rangle$, so $\lambda=1/\left(2x_1\right)=1/\left(2x_2\right)=\cdots=1/\left(2x_n\right)$ and $x_1=x_2=\cdots=x_n$. But $x_1^2+x_2^2+\cdots+x_n^2=1$, so $x_i=\pm 1/\sqrt{n}$ for $i=1$, … , n . Thus the maximum value of f is $f\left(1/\sqrt{n},1/\sqrt{n},\ldots,1/\sqrt{n}\right)=\sqrt{n}$ and the minimum value is $f\left(-1/\sqrt{n},-1/\sqrt{n},\ldots,-1/\sqrt{n}\right)=-\sqrt{n}$.
15. $f(x,y,z)=x+2y$, $g(x,y,z)=x+y+z=1$, $h(x,y,z)=y^2+z^2=4\Rightarrow \nabla f=\langle 1,2,0\rangle$, $\lambda\nabla g=\langle\lambda,\lambda,\lambda\rangle$ and $\mu\nabla h=\langle 0,2\mu y,2\mu z\rangle$. Then $1=\lambda$, $2=\lambda+2\mu y$ and $0=\lambda+2\mu z$ so $\mu y=\frac{1}{2}=-\mu z$ or $y=1/(2\mu)$, $z=-1/(2\mu)$. Thus $x+y+z=1$ implies $x=1$ and $y^2+z^2=4$ implies $\mu=\pm\frac{1}{2\sqrt{2}}$. Then the possible points are $\left(1,\pm\sqrt{2},\mp\sqrt{2}\right)$ and the maximum value is $f\left(1,\sqrt{2},-\sqrt{2}\right)=1+2\sqrt{2}$ and the minimum value is $f\left(1,-\sqrt{2},\sqrt{2}\right)=1-2\sqrt{2}$.
16. $f(x,y,z)=3x-y-3z$, $g(x,y,z)=x+y-z=0$, $h(x,y,z)=x^2+2z^2=1\Rightarrow\nabla f=\langle 3,-1,-3\rangle$, $\lambda\nabla g=\langle\lambda,\lambda,-\lambda\rangle$, $\mu\nabla h=(2\mu x,0,4\mu z)$. Then $3=\lambda+2\mu x$, $-1=\lambda$ and $-3=-\lambda+4\mu z$, so $\lambda=-1$, $\mu z=-1$, $\mu x=2$. Thus $h(x,y,z)=1$ implies $\frac{4}{\mu^2}+2\left(\frac{1}{\mu^2}\right)=1$ or $\mu=\pm\sqrt{6}$, so $z=\mp\frac{1}{\sqrt{6}}$; $x=\pm\frac{2}{\sqrt{6}}$; and $g(x,y,z)=0$ implies $y=\mp\frac{3}{\sqrt{6}}$. Hence the maximum of f subject to the constraints is $f\left(\frac{\sqrt{6}}{3},-\frac{\sqrt{6}}{2},-\frac{\sqrt{6}}{6}\right)=2\sqrt{6}$ and the minimum is $f\left(-\frac{\sqrt{6}}{3},\frac{\sqrt{6}}{2},\frac{\sqrt{6}}{6}\right)=-2\sqrt{6}$.

17. $f(x,y,z)=yz+xy, g(x,y,z)=xy=1$, $h(x,y,z)=y^2+z^2=1\Rightarrow\nabla f=\langle y,x+z,y\rangle$, $\lambda\nabla g=\langle\lambda y,\lambda x,0\rangle$,

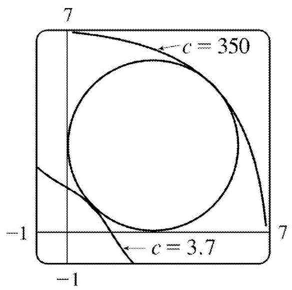
$\mu\nabla h=\langle 0,2\mu y,2\mu z\rangle$. Then $y=\lambda y$ implies $\lambda=1$, $x+z=\lambda x+2\mu y$ and $y=2\mu z$. Thus $\mu=z/(2y)=y/(2y)$ or $y^2=z^2$, and so $y^2+z^2=1$ implies $y=\pm\frac{1}{\sqrt{2}}$, $z=\pm\frac{1}{\sqrt{2}}$. Then $xy=1$ implies $x=\pm\sqrt{2}$ and the possible points are $\left(\pm\sqrt{2},\pm\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, $\left(\pm\sqrt{2},\pm\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$. Hence the maximum of f subject to the constraints is $f\left(\pm\sqrt{2},\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)=\frac{3}{2}$ and the minimum is $f\left(\pm\sqrt{2},\pm\frac{1}{\sqrt{2}},\mp\frac{1}{\sqrt{2}}\right)=\frac{1}{2}$.
Note: Since $xy=1$ is one of the constraints we could have solved the problem by solving $f(y,z)=yz+1$ subject to $y^2+z^2=1$.

18. $f(x,y)=2x^2+3y^2-4x-5\Rightarrow\nabla f=\langle 4x-4,6y\rangle=\langle 0,0\rangle\Rightarrow x=1$, $y=0$. Thus $(1,0)$ is the only critical point of f , and it lies in the region $x^2+y^2<16$. On the boundary, $g(x,y)=x^2+y^2=16\Rightarrow\lambda\nabla g=\langle 2\lambda x,2\lambda y\rangle$, so $6y=2\lambda y\Rightarrow$ either $y=0$ or $\lambda=3$. If $y=0$, then $x=\pm 4$; if $\lambda=3$, then $4x-4=2\lambda x\Rightarrow x=-2$ and $y=\pm 2\sqrt{3}$. Now $f(1,0)=-7$, $f(4,0)=11$, $f(-4,0)=43$, and $f\left(-2,\pm 2\sqrt{3}\right)=47$. Thus the maximum value of $f(x,y)$ on the disk $x^2+y^2\leq 16$ is $f\left(-2,\pm 2\sqrt{3}\right)=47$, and the minimum value is $f(1,0)=-7$.

19. $f(x,y)=e^{-xy}$. For the interior of the region, we find the critical points: $f_x=-ye^{-xy}$, $f_y=-xe^{-xy}$, so the only critical point is $(0,0)$, and $f(0,0)=1$. For the boundary, we use Lagrange multipliers. $g(x,y)=x^2+4y^2=1\Rightarrow\lambda\nabla g=\langle 2\lambda x,8\lambda y\rangle$, so setting $\nabla f=\lambda\nabla g$ we get $-ye^{-xy}=2\lambda x$ and $-xe^{-xy}=8\lambda y$. The first of these gives $e^{-xy}=-2\lambda x/y$, and then the second gives $-x(-2\lambda x/y)=8\lambda y\Rightarrow x^2=4y^2$. Solving this last equation with the constraint $x^2+4y^2=1$ gives $x=\pm\frac{1}{\sqrt{2}}$ and $y=\pm\frac{1}{2\sqrt{2}}$. Now $f\left(\pm\frac{1}{\sqrt{2}},\mp\frac{1}{2\sqrt{2}}\right)=e^{1/4}\approx 1.284$ and $f\left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{2\sqrt{2}}\right)=e^{-1/4}\approx 0.779$. The former are the maxima on the region and the latter are the minima.

20. **(a)** The graphs of $f(x,y)=3.7$ and $f(x,y)=350$ seem to be tangent to the circle, and so 3.7 and 350 are the approximate minimum and maximum values of the function $f(x,y)$ subject to the constraint $(x-3)^2+(y-3)^2=9$.
(b) Let $g(x,y)=(x-3)^2+(y-3)^2$. We calculate $f_x(x,y)=3x^2+3y$, $f_y(x,y)=3y^2+3x$, $g_x(x,y)=2x-6$, and $g_y(x,y)=2y-6$,
and use a CAS to search for solutions to the equations $g(x,y)=(x-3)^2+(y-3)^2=9$, $f_x=\lambda g_x$, and $f_y=\lambda g_y$ .The solutions are $(x,y)=\left(3-\frac{3}{2}\sqrt{2},3-\frac{3}{2}\sqrt{2}\right)\approx(0.879,0.879)$ and

$(x,y)=\left(3+\frac{3}{2}\sqrt{2},3+\frac{3}{2}\sqrt{2}\right)\approx(5.121,5.121)$. These give
$f\left(3-\frac{3}{2}\sqrt{2},3-\frac{3}{2}\sqrt{2}\right)=\frac{351}{2}-\frac{243}{2}\sqrt{2}\approx 3.673$ and
$f\left(3+\frac{3}{2}\sqrt{2},3+\frac{3}{2}\sqrt{2}\right)=\frac{351}{2}+\frac{243}{2}\sqrt{2}\approx 347.33$, in accordance with part (a).



21. $P(L,K)=bL^{\alpha}K^{1-\alpha}$, $g(L,K)=mL+nK=p\Rightarrow\nabla P=\left\langle \alpha bL^{\alpha-1}K^{1-\alpha},(1-\alpha)bL^{\alpha}K^{-\alpha}\right\rangle$, $\lambda\nabla g=\langle\lambda m,\lambda n\rangle$. Then $\alpha b(K/L)^{1-\alpha}=\lambda m$ and $(1-\alpha)b(L/K)^{\alpha}=\lambda n$ and $mL+nK=p$, so $\alpha b(K/L)^{1-\alpha}/m=(1-\alpha)b(L/K)^{\alpha}/n$ or $n\alpha/[m(1-\alpha)]=(L/K)^{\alpha}(L/K)^{1-\alpha}$ or $L=Kn\alpha/[m(1-\alpha)]$. Substituting into $mL+nK=p$ gives $K=(1-\alpha)p/n$ and $L=\alpha p/m$ for the maximum production.

22. $C(L,K)=mL+nK$, $g(L,K)=bL^{\alpha}K^{1-\alpha}=Q\Rightarrow\nabla C=\langle m,n\rangle$, $\lambda\nabla g=\left\langle \lambda\alpha bL^{\alpha-1}K^{1-\alpha},\lambda(1-\alpha)bL^{\alpha}K^{-\alpha}\right\rangle$.
Then $\frac{m}{\alpha b}\left(\frac{L}{K}\right)^{1-\alpha}=\frac{n}{(1-\alpha)b}\left(\frac{K}{L}\right)^{\alpha}$ and
$bL^{\alpha}K^{1-\alpha}=Q\Rightarrow\frac{n\alpha}{m(1-\alpha)}=\left(\frac{L}{K}\right)^{1-\alpha}\left(\frac{L}{K}\right)^{\alpha}\Rightarrow L=\frac{Kn\alpha}{m(1-\alpha)}$ and so $b\left[\frac{Kn\alpha}{m(1-\alpha)}\right]^{\alpha}K^{1-\alpha}=Q$.
Hence $K=\frac{Q}{b(n\alpha/[m(1-\alpha)])^{\alpha}}=\frac{Qm^{\alpha}(1-\alpha)^{\alpha}}{bn^{\alpha}\alpha^{\alpha}}$ and $L=\frac{Qm^{\alpha-1}(1-\alpha)^{\alpha-1}}{bn^{\alpha-1}\alpha^{\alpha-1}}=\frac{Qn^{1-\alpha}\alpha^{1-\alpha}}{bm^{1-\alpha}(1-\alpha)^{1-\alpha}}$
minimizes cost.

23. Let the sides of the rectangle be x and y . Then $f(x,y)=xy$, $g(x,y)=2x+2y=p\Rightarrow\nabla f(x,y)=\langle y,x\rangle$, $\lambda\nabla g=\langle 2\lambda,2\lambda\rangle$. Then $\lambda=\frac{1}{2}y=\frac{1}{2}x$ implies $x=y$ and the rectangle with maximum area is a square with side length $\frac{1}{4}p$.

24. Let $f(x,y,z)=s(s-x)(s-y)(s-z)$, $g(x,y,z)=x+y+z$. Then $\nabla f=\langle -s(s-y)(s-z),-s(s-x)(s-z),-s(s-x)(s-y)\rangle$, $\lambda\nabla g=\langle\lambda,\lambda,\lambda\rangle$. \ Thus (1) $(s-y)(s-z)=(s-x)(s-z)$ and (2) $(s-x)(s-z)=(s-x)(s-y)$. (1) implies $x=y$ while (2) implies $y=z$, so $x=y=z=p/3$ and the triangle with maximum area is equilateral.

25. Let $f(\mathrm{x,y,z})=d^2=(x-2)^2+(y-1)^2+(z+1)^2$, then we want to minimize f subject to the constraint $g(\mathrm{x,y,z})=x+y-z=1$. $\nabla f=\lambda\nabla g\Rightarrow\left\langle 2(x-2),2(y-1),2(z+1)\right\rangle=\lambda\left\langle 1,1,-1\right\rangle$, so $x=(\lambda+4)/2$, $y=(\lambda+2)/2$, $z=-(\lambda+2)/2$. Substituting into the constraint equation gives $\dfrac{\lambda+4}{2}+\dfrac{\lambda+2}{2}+\dfrac{\lambda+2}{2}=1\Rightarrow 3\lambda+8=2\Rightarrow\lambda=-2$, so $x=1$, $y=0$, and $z=0$. This must correspond to a minimum, so the shortest distance is $d=\sqrt{(1-2)^2+(0-1)^2+(0+1)^2}=\sqrt{3}$.

26. Let $f(\mathrm{x,y,z})=d^2=(x-1)^2+(y-2)^2+(z-3)^2$, then we want to minimize f subject to the constraint $g(\mathrm{x,y,z})=x-y+z=4$. $\nabla f=\lambda\nabla g\Rightarrow\left\langle 2(x-1),2(y-2),2(z-3)\right\rangle=\lambda\left\langle 1,-1,1\right\rangle$, so $x=(\lambda+2)/2$, $y=(4-\lambda)/2$, $z=(\lambda+6)/2$. Substituting into the constraint equation gives $\dfrac{\lambda+2}{2}-\dfrac{4-\lambda}{2}+\dfrac{\lambda+6}{2}=4\Rightarrow\lambda=\dfrac{4}{3}$, so $x=\dfrac{5}{3}$, $y=\dfrac{4}{3}$, and $z=\dfrac{11}{3}$. This must correspond to a minimum, so the point on the plane closest to the point $(1,2,3)$ is $\left(\dfrac{5}{3},\dfrac{4}{3},\dfrac{11}{3}\right)$.

27. $f(\mathrm{x,y,z})=x^2+y^2+z^2$, $g(\mathrm{x,y,z})=z^2-xy-1=0\Rightarrow\nabla f=\left\langle 2\mathrm{x},2\mathrm{y},2z\right\rangle=\lambda\nabla g=\left\langle-\lambda\mathrm{y},-\lambda\mathrm{x},2\lambda z\right\rangle$. Then $2z=2\lambda z$ implies $z=0$ or $\lambda=1$. If $z=0$ then $g(\mathrm{x,y,z})=1$ implies $xy=-1$ or $x=-1/y$. Thus $2x=-\lambda y$ and $2y=-\lambda x$ imply $\lambda=2/y^2=2y^2$ or $y=\pm1$, $x=\pm1$. If $\lambda=1$, then $2x=-y$ and $2y=-x$ imply $x=y=0$, so $z=\pm1$. Hence the possible points are $(\pm1,\mp1,0)$, $(0,0,\pm1)$ and the minimum value of f is $f(0,0,\pm1)=1$, so the points closest to the origin are $(0,0,\pm1)$.

28. $f(\mathrm{x,y,z})=x^2+y^2+z^2$, $g(\mathrm{x,y,z})=x^2y^2z=1\Rightarrow\nabla f=\left\langle 2\mathrm{x},2\mathrm{y},2z\right\rangle=\lambda\nabla g=\left\langle 2\lambda xy^2\mathrm{z},2\lambda x^2\mathrm{yz},\lambda x^2y^2\right\rangle$. Then $\lambda y^2z=1$, $\lambda x^2z=1$ and $\lambda x^2y^2=2z$ so $y^2z=x^2z$ and $x=\pm y$. Also $2z/1=\lambda x^2y^2/\left(\lambda x^2z\right)$ so $2z^2=y^2$ and $y=\pm\sqrt{2}\,z$. But $x^2y^2z=1$ implies $z>0$ and $4z^5=1$. Thus the points are $(\pm2^{1/10},\pm2^{1/10},2^{-2/5})$, and the minimum distance is attained at each of these.

29. $f(\mathrm{x,y,z})=xyz$, $g(\mathrm{x,y,z})=x+y+z=100\Rightarrow\nabla f=\left\langle\mathrm{yz,xz,xy}\right\rangle=\lambda\nabla g=\left\langle\lambda,\lambda,\lambda\right\rangle$. Then $\lambda=yz=xz=xy$ implies $x=y=z=\dfrac{100}{3}$.

30. $f(\mathrm{x,y,z})=x^ay^bz^c$, $g(\mathrm{x,y,z})=x+y+z=100\Rightarrow$
$\nabla f=\left\langle ax^{a-1}y^bz^c,\mathrm{b}x^ay^{b-1}z^c,\mathrm{c}x^ay^bz^{c-1}\right\rangle=\lambda\nabla g=\left\langle\lambda,\lambda,\lambda\right\rangle$. Then $\lambda=ax^{a-1}y^bz^c=bx^ay^{b-1}z^c=cx^ay^bz^{c-1}$ or $ayz=bxz=cxy$. Thus $x=\dfrac{ay}{b}$, $z=\dfrac{cy}{b}$, and $\dfrac{ay}{b}+y+\dfrac{cy}{b}=100$ implies that $y=\dfrac{100b}{a+b+c}$, $x=\dfrac{100a}{a+b+c}$ and

$z=\dfrac{100c}{a+b+c}$ gives the maximum.

31. If the dimensions are $2x$, $2y$ and $2z$, then $f(\mathrm{x,y,z})=8xyz$ and $g(\mathrm{x,y,z})=9x^2+36y^2+4z^2=36\Rightarrow$ $\nabla f=\langle 8\mathrm{yz},8\mathrm{xz},8xy\rangle=\lambda\nabla g=\langle 18\lambda\mathrm{x},72\lambda\mathrm{y},8\lambda z\rangle$. Thus $18\lambda x=8yz$, $72\lambda y=8xz$, $8\lambda z=8xy$ so $x^2=4y^2$, $z^2=9y^2$ and $36y^2+36y^2+36y^2=36$ or $y=\dfrac{1}{\sqrt{3}}$ ($y>0$). Thus the volume of the largest such box is $8\left(\dfrac{1}{\sqrt{3}}\right)\left(\dfrac{2}{\sqrt{3}}\right)\left(\dfrac{3}{\sqrt{3}}\right)=16\sqrt{3}$.

32. $f(\mathrm{x,y,z})=8xyz$, $g(\mathrm{x,y,z})=b^2c^2x+a^2c^2y+a^2b^2z=a^2b^2c^2\Rightarrow$ $\nabla f=\langle 8\mathrm{yz},8\mathrm{xz},8xy\rangle=\lambda\nabla g=\left\langle 2\lambda b^2c^2\mathrm{x},2\lambda a^2c^2\mathrm{y},2\lambda a^2b^2z\right\rangle$. Then $4yz=\lambda b^2c^2x$, $4xz=\lambda a^2c^2y$, $4xy=\lambda a^2b^2z$ imply $\lambda=\dfrac{4yz}{b^2c^2x}=\dfrac{4xz}{a^2c^2y}=\dfrac{4xy}{a^2b^2z}$ or $\dfrac{y}{b^2x}=\dfrac{x}{a^2y}$ and $\dfrac{z}{c^2y}=\dfrac{y}{b^2z}$. Thus $x=\dfrac{ay}{b}$, $z=\dfrac{cy}{b}$, and $a^2c^2y^2+c^2a^2y^2+a^2c^2y^2=a^2b^2c^2$, or $y=\dfrac{b}{\sqrt{3}}$, $x=\dfrac{a}{\sqrt{3}}$, $z=\dfrac{c}{\sqrt{3}}$ and the volume is $\dfrac{8}{3\sqrt{3}}abc$.

33. $f(\mathrm{x,y,z})=xyz$, $g(\mathrm{x,y,z})=x+2y+3z=6\Rightarrow\nabla f=\langle\mathrm{yz,xz,xy}\rangle=\lambda\nabla g=\langle\lambda,2\lambda,3\lambda\rangle$. Then $\lambda=yz=\dfrac{1}{2}xz=\dfrac{1}{3}xy$ implies $x=2y$, $z=\dfrac{2}{3}y$. But $2y+2y+2y=6$ so $y=1$, $x=2$, $z=\dfrac{2}{3}$ and the volume is $V=\dfrac{4}{3}$.

34. $f(\mathrm{x,y,z})=xyz$, $g(\mathrm{x,y,z})=xy+yz+xz=32\Rightarrow\nabla f=\langle\mathrm{yz,xz,xy}\rangle=\lambda\nabla g=\langle\lambda(y+z),\lambda(x+z),\lambda(x+y)\rangle$. Then (1) $\lambda(y+z)=yz$, (2) $\lambda(x+z)=xz$ and (3) $\lambda(x+y)=xy$. And (1) minus (2) implies $\lambda(y-x)=z(y-x)$ so $x=y$ or $\lambda=z$. If $\lambda=z$, then (1) implies $z(y+z)=yz$ or $z=0$ which is false. Thus $x=y$. Similarly (2) minus (3) implies $\lambda(z-y)=x(z-y)$ so $y=z$ or $\lambda=x$. As above, $\lambda\neq x$, so $x=y=z$ and $3x^2=32$ or $x=y=z=\dfrac{8}{\sqrt{6}}$ cm.

35. $f(\mathrm{x,y,z})=xyz$, $g(\mathrm{x,y,z})=4(x+y+z)=c\Rightarrow\nabla f=\langle\mathrm{yz,xz,xy}\rangle$, $\lambda\nabla g=\langle 4\lambda,4\lambda,4\lambda\rangle$. Thus $4\lambda=yz=xz=xy$ or $x=y=z=\dfrac{1}{12}c$ are the dimensions giving the maximum volume.

36. $C(\mathrm{x,y,z})=5xy+2xz+2yz$, $g(\mathrm{x,y,z})=xyz=V\Rightarrow$ $\nabla C=\langle 5y+2\mathrm{z},5x+2\mathrm{z},2x+2y\rangle=\lambda\nabla g=\langle\lambda\mathrm{yz},\lambda\mathrm{xz},\lambda xy\rangle$. Then (1) $\lambda yz=5y+2z$, (2) $\lambda xz=5x+2z$, (3) $\lambda xy=2(x+y)$ and (4) $xyz=V$. Now (1) – (2) implies $\lambda z(y-x)=5(y-x)$, so $x=y$ or $\lambda=5/z$, but z can't be 0 , so $x=y$. Then twice (2) minus five times (3) together with $x=y$ implies $\lambda y(2x-5y)=2(2z-5y)$ which gives $z=\dfrac{5}{2}y$. Hence $\dfrac{5}{2}y^3=V$ and the dimensions which minimize cost are $x=y=\sqrt[3]{\dfrac{2}{5}V}$ units,

$z=V^{1/3}\left(\frac{5}{2}\right)^{2/3}$ units.

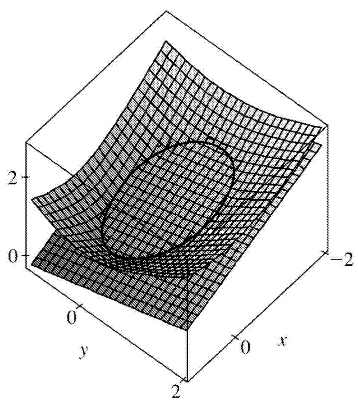
37. If the dimensions of the box are given by x , y , and z , then we need to find the maximum value of $f(x,y,z)=xyz$ ($x,y,z>0$) subject to the constraint $L=\sqrt{x^2+y^2+z^2}$ or $g(x,y,z)=x^2+y^2+z^2=L^2$. $\nabla f=\lambda\nabla g\Rightarrow\langle yz,xz,xy\rangle=\lambda\langle 2x,2y,2z\rangle$, so $yz=2\lambda x\Rightarrow\lambda=\frac{yz}{2x}$, $xz=2\lambda y\Rightarrow\lambda=\frac{xz}{2y}$, and $xy=2\lambda z\Rightarrow\lambda=\frac{xy}{2z}$. Thus $\lambda=\frac{yz}{2x}=\frac{xz}{2y}\Rightarrow x^2=y^2\Rightarrow x=y$ and $\lambda=\frac{yz}{2x}=\frac{xy}{2z}\Rightarrow x=z$. Substituting into the constraint equation gives $x^2+x^2+x^2=L^2\Rightarrow x^2=L^2/3\Rightarrow x=L/\sqrt{3}=y=z$ and the maximum volume is $(L/\sqrt{3})^3=L^3/(3\sqrt{3})$.

38. Let the dimensions of the box be x , y , and z , so its volume is $f(x,y,z)=xyz$, its surface area is $g(x,y,z)=xy+yz+xz=750$ and its total edge length is $h(x,y,z)=x+y+z=50$. Then $\nabla f=\langle yz,xz,xy\rangle=\lambda\nabla g+\mu\nabla h=\langle\lambda(y+z),\lambda(x+z),\lambda(x+y)\rangle+\langle\mu,\mu,\mu\rangle$. So (1) $yz=\lambda(y+z)+\mu$, (2) $xz=\lambda(x+z)+\mu$, and (3) $xy=\lambda(x+y)+\mu$. Notice that the box can't be a cube or else $x=y=z=\frac{50}{3}$ but then $xy+yz+xz=\frac{2500}{3}\neq 750$. Assume x is the distinct side, that is, $x\neq y$, $x\neq z$. Then (1) minus (2) implies $z(y-x)=\lambda(y-x)$ or $\lambda=z$, and (1) minus (3) implies $y(z-x)=\lambda(z-x)$ or $\lambda=y$. So $y=z=\lambda$ and $x+y+z=50$ implies $x=50-2\lambda$; also $xy+yz+xz=750$ implies $x(2\lambda)+\lambda^2=750$. Hence $50-2\lambda=\frac{750-\lambda^2}{2\lambda}$ or $3\lambda^2-100\lambda+750=0$ and $\lambda=\frac{50\pm 5\sqrt{10}}{3}$, giving the points $\left(\frac{1}{3}(50\mp 10\sqrt{10}),\frac{1}{3}(50\pm 5\sqrt{10}),\frac{1}{3}(50\pm 5\sqrt{10})\right)$. Thus the minimum of f is $f\left(\frac{1}{3}(50-10\sqrt{3}),\frac{1}{3}(50+5\sqrt{10}),\frac{1}{3}(50+5\sqrt{10})\right)=\frac{1}{27}(87{,}500-2500\sqrt{10})$, and its maximum is $f\left(\frac{1}{3}(50+10\sqrt{10}),\frac{1}{3}(50-5\sqrt{10}),\frac{1}{3}(50-5\sqrt{10})\right)=\frac{1}{27}(87{,}500+2500\sqrt{10})$.
Note: If either y or z is the distinct side, then symmetry gives the same result.

39. We need to find the extreme values of $f(x,y,z)=x^2+y^2+z^2$ subject to the two constraints $g(x,y,z)=x+y+2z=2$ and $h(x,y,z)=x^2+y^2-z=0$. $\nabla f=\langle 2x,2y,2z\rangle$, $\lambda\nabla g=\langle\lambda,\lambda,2\lambda\rangle$ and $\mu\nabla h=\langle 2\mu x,2\mu y,-\mu\rangle$. Thus we need (1) $2x=\lambda+2\mu x$, (2) $2y=\lambda+2\mu y$, (3) $2z=2\lambda-\mu$, (4) $x+y+2z=2$, and (5) $x^2+y^2-z=0$. From (1) and (2), $2(x-y)=2\mu(x-y)$, so if $x\neq y$, $\mu=1$. Putting this in (3) gives $2z=2\lambda-1$ or $\lambda=z+\frac{1}{2}$, but putting $\mu=1$ into (1) says $\lambda=0$. Hence $z+\frac{1}{2}=0$ or $z=-\frac{1}{2}$. Then (4) and (5)

become $x+y-3=0$ and $x^2+y^2+\frac{1}{2}=0$. The last equation cannot be true, so this case gives no solution. So we must have $x=y$. Then (4) and (5) become $2x+2z=2$ and $2x^2-z=0$ which imply $z=1-x$ and $z=2x^2$. Thus $2x^2=1-x$ or $2x^2+x-1=(2x-1)(x+1)=0$ so $x=\frac{1}{2}$ or $x=-1$. The two points to check are $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ and $(-1,-1,2)$: $f\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)=\frac{3}{4}$ and $f(-1,-1,2)=6$. Thus $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ is the point on the ellipse nearest the origin and $(-1,-1,2)$ is the one farthest from the origin.

40. **(a)** Parametric equations for the ellipse are easiest to determine using cylindrical coordinates. The cone is given by $z=r$, and the plane is $4r\cos\theta-3r\sin\theta+8z=5$. Substituting $z=r$ into the plane equation gives $4r\cos\theta-3r\sin\theta+8r=5\Rightarrow r=\dfrac{5}{4\cos\theta-3\sin\theta+8}$. Since $z=r$ on the ellipse, parametric equations (in cylindrical coordinates) are $\theta=t$, $r=z=\dfrac{5}{4\cos t-3\sin t+8}$, $0\le t\le 2\pi$.



(b) We need to find the extreme values of $f(x,y,z)=z$ subject to the two constraints $g(x,y,z)=4x-3y+8z=5$ and $h(x,y,z)=x^2+y^2-z^2=0$. $\nabla f=\lambda\nabla g+\mu\nabla h\Rightarrow$ $\langle 0,0,1\rangle=\lambda\langle 4,-3,8\rangle+\mu\langle 2x,2y,-2z\rangle$, so we need (1) $4\lambda+2\mu x=0\Rightarrow x=-\dfrac{2\lambda}{\mu}$, (2) $-3\lambda+2\mu y=0\Rightarrow y=\dfrac{3\lambda}{2\mu}$, (3) $8\lambda-2\mu z=1\Rightarrow z=\dfrac{8\lambda-1}{2\mu}$, (4) $4x-3y+8z=5$, and (5) $x^2+y^2=z^2$. Substituting (1), (2), and (3) into (4) gives $4\left(-\dfrac{2\lambda}{\mu}\right)-3\left(\dfrac{3\lambda}{2\mu}\right)+8\left(\dfrac{8\lambda-1}{2\mu}\right)=5\Rightarrow\mu=\dfrac{39\lambda-8}{10}$ and into (5) gives $\left(-\dfrac{2\lambda}{\mu}\right)^2+\left(\dfrac{3\lambda}{2\mu}\right)^2=\left(\dfrac{8\lambda-1}{2\mu}\right)^2\Rightarrow 16\lambda^2+9\lambda^2=(8\lambda-1)^2\Rightarrow 39\lambda^2-16\lambda+1=0\Rightarrow$ $\lambda=\dfrac{1}{13}$ or $\lambda=\dfrac{1}{3}$. If $\lambda=\dfrac{1}{13}$ then $\mu=-\dfrac{1}{2}$ and $x=\dfrac{4}{13}$, $y=-\dfrac{3}{13}$, $z=\dfrac{5}{13}$. If $\lambda=\dfrac{1}{3}$ then $\mu=\dfrac{1}{2}$ and $x=-\dfrac{4}{3}$, $y=1$, $z=\dfrac{5}{3}$. Thus the highest point on the ellipse is $\left(-\dfrac{4}{3},1,\dfrac{5}{3}\right)$ and the lowest point is

$\left(\frac{4}{13}, -\frac{3}{13}, \frac{5}{13} \right)$.

41. $f(x,y,z)=ye^{x-z}$, $g(x,y,z)=9x^2+4y^2+36z^2=36$, $h(x,y,z)=xy+yz=1$.
$\nabla f=\lambda \nabla g+\mu \nabla h \Rightarrow \left\langle ye^{x-z}, e^{x-z}, -ye^{x-z} \right\rangle=\lambda \langle 18x,8y,72z \rangle+\mu \langle y,x+z,y \rangle$, so $ye^{x-z}=18\lambda x+\mu y$, $e^{x-z}=8\lambda y+\mu(x+z)$, $-ye^{x-z}=72\lambda z+\mu y$, $9x^2+4y^2+36z^2=36$, $xy+yz=1$. Using a CAS to solve these 5 equations simultaneously for x , y , z , λ , and μ (in Maple, use the allvalues command), we get 4 real–valued solutions:

$x\approx 0.222444$, $y\approx -2.157012$, $z\approx -0.686049$, $\lambda\approx -0.200401$, $\mu\approx 2.108584$
$x\approx -1.951921$, $y\approx -0.545867$, $z\approx 0.119973$, $\lambda\approx 0.003141$, $\mu\approx -0.076238$
$x\approx 0.155142$, $y\approx 0.904622$, $z\approx 0.950293$, $\lambda\approx -0.012447$, $\mu\approx 0.489938$
$x\approx 1.138731$, $y\approx 1.768057$, $z\approx -0.573138$, $\lambda\approx 0.317141$, $\mu\approx 1.862675$

Substituting these values into f gives $f(0.222444,-2.157012,-0.686049)\approx -5.3506$, $f(-1.951921,-0.545867,0.119973)\approx -0.0688$, $f(0.155142,0.904622,0.950293)\approx 0.4084$, $f(1.138731,1.768057,-0.573138)\approx 9.7938$. Thus the maximum is approximately 9.7938 , and the mininum is approximately -5.3506 .

42. $f(x,y,z)=x+y+z$, $g(x,y,z)=x^2-y^2-z=0$, $h(x,y,z)=x^2+z^2=4$.
$\nabla f=\lambda \nabla g+\mu \nabla h \Rightarrow \langle 1,1,1 \rangle=\lambda \langle 2x,-2y,-1 \rangle+\mu \langle 2x,0,2z \rangle$, so $1=2\lambda x+2\mu x$, $1=-2\lambda y$, $1=-\lambda+2\mu z$, $x^2-y^2=z$, $x^2+z^2=4$. Using a CAS to solve these 5 equations simultaneously for x , y , z , λ , and μ , we get 4 real–valued solutions:

$x\approx -1.652878$, $y\approx -1.964194$, $z\approx -1.126052$, $\lambda\approx 0.254557$, $\mu\approx -0.557060$
$x\approx -1.502800$, $y\approx 0.968872$, $z\approx 1.319694$, $\lambda\approx -0.516064$, $\mu\approx 0.183352$
$x\approx -0.992513$, $y\approx 1.649677$, $z\approx -1.736352$, $\lambda\approx -0.303090$, $\mu\approx -0.200682$
$x\approx 1.895178$, $y\approx 1.718347$, $z\approx 0.638984$, $\lambda\approx -0.290977$, $\mu\approx 0.554805$

Substituting these values into f gives $f(-1.652878,-1.964194,-1.126052)\approx -4.7431$, $f(-1.502800,0.968872,1.319694)\approx 0.7858$, $f(-0.992513,1.649677,-1.736352)\approx -1.0792$, $f(1.895178,1.718347,0.638984)\approx 4.2525$. Thus the maximum is approximately 4.2525 , and the mininum is approximately -4.7431 .

43. **(a)** We wish to maximize $f\left(x_1,x_2,\dots,x_n\right)=\sqrt[n]{x_1x_2\cdots x_n}$ subject to $g\left(x_1,x_2,\dots,x_n\right)=x_1+x_2+\cdots+x_n=c$ and $x_i>0$. ∇f

$$=\left\langle \frac{1}{n}\left(x_1x_2\cdots x_n\right)^{\frac{1}{n}-1}\left(x_2\cdots x_n\right), \frac{1}{n}\left(x_1x_2\cdots x_n\right)^{\frac{1}{n}-1}\left(x_1x_3\cdots x_n\right), \ldots, \frac{1}{n}\left(x_1x_2\cdots x_n\right)^{\frac{1}{n}-1}\left(x_1\cdots x_{n-1}\right)\right\rangle$$

and

$\lambda\nabla g=\langle\lambda,\lambda,\ldots,\lambda\rangle$, so we need to solve the system of equations

$$\frac{1}{n}\left(x_1x_2\cdots x_n\right)^{\frac{1}{n}-1}\left(x_2\cdots x_n\right)=\lambda \quad\Rightarrow\quad x_1^{1/n}x_2^{1/n}\cdots x_n^{1/n}=n\lambda x_1$$

$$\frac{1}{n}\left(x_1x_2\cdots x_n\right)^{\frac{1}{n}-1}\left(x_1x_3\cdots x_n\right)=\lambda \quad\Rightarrow\quad x_1^{1/n}x_2^{1/n}\cdots x_n^{1/n}=n\lambda x_2$$

$$\cdots$$

$$\cdots$$

$$\cdots$$

$$\frac{1}{n}\left(x_1x_2\cdots x_n\right)^{\frac{1}{n}-1}\left(x_1\cdots x_{n-1}\right)=\lambda \quad\Rightarrow\quad x_1^{1/n}x_2^{1/n}\cdots x_n^{1/n}=n\lambda x_n$$

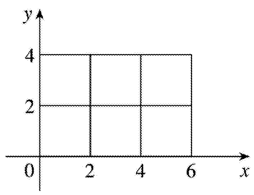
This implies $n\lambda x_1=n\lambda x_2=\cdots=n\lambda x_n$. Note $\lambda\neq 0$, otherwise we can't have all $x_i>0$. Thus $x_1=x_2=\cdots=x_n$. But $x_1+x_2+\cdots+x_n=c\Rightarrow nx_1=c\Rightarrow x_1=\frac{c}{n}=x_2=x_3=\cdots=x_n$. Then the only point where f can have an extreme value is $\left(\frac{c}{n},\frac{c}{n},\ldots,\frac{c}{n}\right)$. Since we can choose values for $\left(x_1,x_2,\ldots,x_n\right)$ that make f as close to zero (but not equal) as we like, f has no minimum value. Thus the maximum value is $f\left(\frac{c}{n},\frac{c}{n},\ldots,\frac{c}{n}\right)=\sqrt[n]{\frac{c}{n}\cdot\frac{c}{n}\cdot\cdots\cdot\frac{c}{n}}=\frac{c}{n}$.

(b) From part (a), $\frac{c}{n}$ is the maximum value of f . Thus $f\left(x_1,x_2,\ldots,x_n\right)=\sqrt[n]{x_1x_2\cdots x_n}\leq\frac{c}{n}$. But $x_1+x_2+\cdots+x_n=c$, so $\sqrt[n]{x_1x_2\cdots x_n}\leq\frac{x_1+x_2+\cdots+x_n}{n}$. These two means are equal when f attains its maximum value $\frac{c}{n}$, but this can occur only at the point $\left(\frac{c}{n},\frac{c}{n},\ldots,\frac{c}{n}\right)$ we found in part (a). So the means are equal only when $x_1=x_2=x_3=\cdots=x_n=\frac{c}{n}$.

44. **(a)** Let $f\left(x_1,\ldots,x_n,y_1,\ldots,y_n\right)=\Sigma_{i=1}^{n}x_iy_i$, $g\left(x_1,\ldots,x_n\right)=\Sigma_{i=1}^{n}x_i^2$, and $h\left(x_1,\ldots,x_n\right)=\Sigma_{i=1}^{n}y_i^2$. Then $\nabla f=\nabla\Sigma_{i=1}^{n}x_iy_i=\left\langle y_1,y_2,\ldots,y_n,x_1,x_2,\ldots,x_n\right\rangle$, $\nabla g=\nabla\Sigma_{i=1}^{n}x_i^2=\left\langle 2x_1,2x_2,\ldots,2x_n,0,0,\ldots,0\right\rangle$ and

$\nabla h = \nabla \sum_{i=1}^{n} y_i^2 = \left\langle 0,0,\ldots,0,2y_1,2y_2,\ldots,2y_n \right\rangle$. So $\nabla f = \lambda \nabla g + \mu \nabla h \Leftrightarrow y_i = 2\lambda x_i$ and $x_i = 2\mu y_i$, $1 \le i \le n$.
Then $1 = \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} 4\lambda^2 x_i^2 = 4\lambda^2 \sum_{i=1}^{n} x_i^2 = 4\lambda^2 \Rightarrow \lambda = \pm \frac{1}{2}$.
If $\lambda = \frac{1}{2}$ then $y_i = 2\left(\frac{1}{2}\right) x_i = x_i$, $1 \le i \le n$. Thus $\sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} x_i^2 = 1$. Similarly if $\lambda = -\frac{1}{2}$ we get $y_i = -x_i$ and $\sum_{i=1}^{n} x_i y_i = -1$. Similarly we get $\mu = \pm \frac{1}{2}$ giving $y_i = \pm x_i$, $1 \le i \le n$, and $\sum_{i=1}^{n} x_i y_i = \pm 1$. Thus the maximum value of $\sum_{i=1}^{n} x_i y_i$ is 1 .
(b) Here we assume $\sum_{i=1}^{n} a_i^2 \ne 0$ and $\sum_{i=1}^{n} b_i^2 \ne 0$. (If $\sum_{i=1}^{n} a_i^2 = 0$, then each $a_i = 0$ and so the inequality is trivially true.) $x_i = \dfrac{a_i}{\sqrt{\sum a_j^2}} \Rightarrow \sum x_i^2 = \dfrac{\sum a_i^2}{\sum a_j^2} = 1$, and $y_i = \dfrac{b_i}{\sqrt{\sum b_j^2}} \Rightarrow \sum y_i^2 = \dfrac{\sum b_i^2}{\sum b_j^2} = 1$. Therefore, from (a), $\sum x_i y_i = \sum \dfrac{a_i b_i}{\sqrt{\sum a_j^2}\sqrt{\sum b_j^2}} \le 1 \Leftrightarrow \sum a_i b_i \le \sqrt{\sum a_j^2}\sqrt{\sum b_j^2}$.

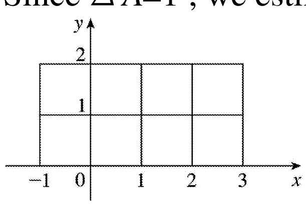
1. **(a)** The subrectangles are shown in the figure.
The surface is the graph of $f(x,y)=xy$ and $\Delta A=4$,
so we estimate



$$V \approx \sum_{i=1}^{3}\sum_{j=1}^{2} f\left(x_i,y_j\right)\Delta A$$
$$=f(2,2)\Delta A+f(2,4)\Delta A+f(4,2)\Delta A+f(4,4)\Delta A+f(6,2)\Delta A+f(6,4)\Delta A$$
$$=4(4)+8(4)+8(4)+16(4)+12(4)+24(4)=288$$

(b)
$$V \approx \sum_{i=1}^{3}\sum_{j=1}^{2} f\left(\overline{x}_i,\overline{y}_j\right)\Delta A$$
$$=f(1,1)\Delta A+f(1,3)\Delta A+f(3,1)\Delta A+f(3,3)\Delta A+f(5,1)\Delta A+f(5,3)\Delta A$$
$$=1(4)+3(4)+3(4)+9(4)+5(4)+15(4)=144$$

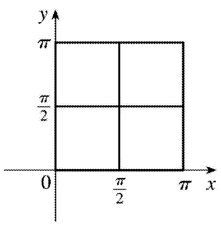
2. The subrectangles are shown in the figure.
Since $\Delta A=1$, we estimate



$$\iint_R \left(y^2-2x^2\right)dA \approx \sum_{i=1}^{4}\sum_{j=1}^{2} f\left(x_{ij}^*,y_{ij}^*\right)\Delta A$$
$$=f(-1,1)\Delta A+f(-1,2)\Delta A+f(0,1)\Delta A+f(0,2)\Delta A$$
$$+f(1,1)\Delta A+f(1,2)\Delta A+f(2,1)\Delta A+f(2,2)\Delta A$$
$$=-1(1)+2(1)+1(1)+4(1)-1(1)+2(1)-7(1)-4(1)=-4$$

3. **(a)** The subrectangles are shown in the figure. Since $\Delta A=\pi^2/4$, we estimate
$$\iint_R \sin(x+y)dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(x_{ij}^*,y_{ij}^*\right)\Delta A$$
$$=f(0,0)\Delta A+f\left(0,\frac{\pi}{2}\right)\Delta A+f\left(\frac{\pi}{2},0\right)\Delta A+f\left(\frac{\pi}{2},\frac{\pi}{2}\right)\Delta A$$

$$=0\left(\frac{\pi^2}{4}\right)+1\left(\frac{\pi^2}{4}\right)+1\left(\frac{\pi^2}{4}\right)+0\left(\frac{\pi^2}{4}\right)=\frac{\pi^2}{2}\approx 4.935$$



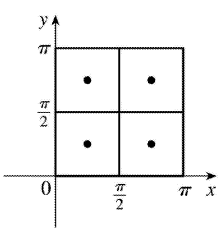
(b)

$$\iint_R \sin(x+y)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\bar{x}_i,\bar{y}_j)\,\Delta A$$

$$=f\left(\frac{\pi}{4},\frac{\pi}{4}\right)\Delta A+f\left(\frac{\pi}{4},\frac{3\pi}{4}\right)\Delta A$$

$$+f\left(\frac{3\pi}{4},\frac{\pi}{4}\right)\Delta A+f\left(\frac{3\pi}{4},\frac{3\pi}{4}\right)\Delta A$$

$$=1\left(\frac{\pi^2}{4}\right)+0\left(\frac{\pi^2}{4}\right)+0\left(\frac{\pi^2}{4}\right)+(-1)\left(\frac{\pi^2}{4}\right)=0$$



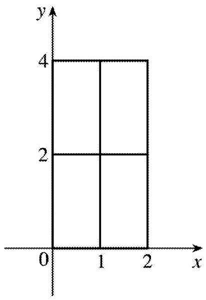
4. **(a)** The subrectangles are shown in the figure.

The surface is the graph of $f(x,y)=x+2y^2$ and $\Delta A=2$, so we estimate

$$V=\iint_R (x+2y^2)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(x_{ij}^*,y_{ij}^*\right)\Delta A$$

$$=f(1,0)\,\Delta A+f(1,2)\,\Delta A+f(2,0)\,\Delta A+f(2,2)\,\Delta A$$

$$=1(2)+9(2)+2(2)+10(2)=44$$

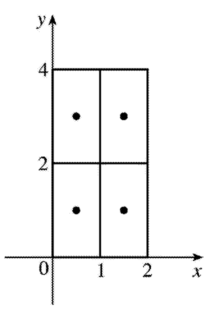


(b)

$$V = \iint_R (x+2y^2)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\bar{x}_i,\bar{y}_j)\,\Delta A$$

$$= f\left(\frac{1}{2},1\right)\Delta A + f\left(\frac{1}{2},3\right)\Delta A + f\left(\frac{3}{2},1\right)\Delta A + f\left(\frac{3}{2},3\right)\Delta A$$

$$= \frac{5}{2}(2) + \frac{37}{2}(2) + \frac{7}{2}(2) + \frac{39}{2}(2) = 88$$



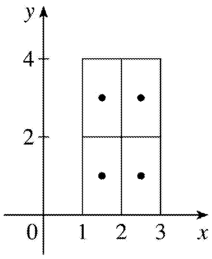
5. **(a)** Each subrectangle and its midpoint are shown in the figure. The area of each subrectangle is $\Delta A=2$, so we evaluate f at each midpoint and estimate

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\bar{x}_i,\bar{y}_j\right)\Delta A$$

$$= f(1.5,1)\,\Delta A + f(1.5,3)\,\Delta A$$

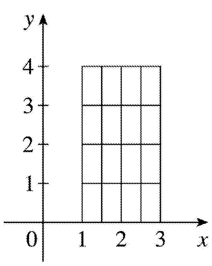
$$+ f(2.5,1)\,\Delta A + f(2.5,3)\,\Delta A$$

$$= 1\,(2) + (-8)\,(2) + 5\,(2) + (-1)\,(2) = -6$$



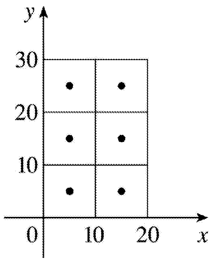
(b) The subrectangles are shown in the figure. In each subrectangle, the sample point farthest from the origin is the upper right corner, and the area of each subrectangle is $\Delta A=\frac{1}{2}$. Thus we estimate

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{4}\sum_{j=1}^{4} f\left(x_i,y_j\right)\Delta A$$
$$=f(1.5,1)\Delta A+f(1.5,2)\Delta A+f(1.5,3)\Delta A+f(1.5,4)\Delta A$$
$$+f(2,1)\Delta A+f(2,2)\Delta A+f(2,3)\Delta A+f(2,4)\Delta A$$
$$+f(2.5,1)\Delta A+f(2.5,2)\Delta A+f(2.5,3)\Delta A+f(2.5,4)\Delta A$$
$$+f(3,1)\Delta A+f(3,2)\Delta A+f(3,3)\Delta A+f(3,4)\Delta A$$



$$=1\left(\frac{1}{2}\right)+(-4)\left(\frac{1}{2}\right)+(-8)\left(\frac{1}{2}\right)+(-6)\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)+0\left(\frac{1}{2}\right)+(-5)\left(\frac{1}{2}\right)+(-8)\left(\frac{1}{2}\right)$$
$$+5\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)+(-1)\left(\frac{1}{2}\right)+(-4)\left(\frac{1}{2}\right)+8\left(\frac{1}{2}\right)+6\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)+0\left(\frac{1}{2}\right)$$
$$=-3.5$$

6. To approximate the volume, let R be the planar region corresponding to the surface of the water in the pool, and place R on coordinate axes so that x and y correspond to the dimensions given. Then we define $f(x,y)$ to be the depth of the water at (x,y) , so the volume of water in the pool is the volume of the solid that lies above the rectangle $R=[0,20]\times[0,30]$ and below the graph of $f(x,y)$. We can estimate this volume using the Midpoint Rule with $m=2$ and $n=3$, so $\Delta A=100$. Each subrectangle with its midpoint is shown in the figure. Then



$$
\begin{aligned}
V &\approx \sum_{i=1}^{2}\sum_{j=1}^{3} f\left(\bar{x}_i,\bar{y}_j\right)\Delta A \\
&= \Delta A\left[f(5,5)+f(5,15)+f(5,25)+f(15,5)+f(15,15)+f(15,25)\right] \\
&= 100(3+7+10+3+5+8)=3600
\end{aligned}
$$

Thus, we estimate that the pool contains 3600 cubic feet of water.
Alternatively, we can approximate the volume with a Riemann sum where $m=4$, $n=6$ and the sample points are taken to be, for example, the upper right corner of each subrectangle. Then $\Delta A=25$ and

$$
\begin{aligned}
V &\approx \sum_{i=1}^{4}\sum_{j=1}^{6} f\left(x_i,y_j\right)\Delta A \\
&= 25\left[3+4+7+8+10+8+4+6+8+10+12+10+3+4\right. \\
&\quad \left.+5+6+8+7+2+2+2+3+4+4\right] \\
&= 25(140)=3500
\end{aligned}
$$

So we estimate that the pool contains 3500 ft^3 of water.

7. The values of $f(x,y)=\sqrt{52-x^2-y^2}$ get smaller as we move farther from the origin, so on any of the subrectangles in the problem, the function will have its largest value at the lower left corner of the subrectangle and its smallest value at the upper right corner, and any other value will lie between these two. So using these subrectangles we have $U<V<L$. (Note that this is true no matter how R is divided into subrectangles.)

8. From the level curves we see that $f\left(\frac{1}{2},\frac{1}{2}\right)\approx 11$. So, using the Midpoint Rule with only one subrectangle, we get $\iint_R f(x,y)\,dA\approx 1\cdot f\left(\frac{1}{2},\frac{1}{2}\right)\approx 11$. Dividing R into four squares of equal size, we get

$$\iint_R f(x,y)\,dA\approx \frac{1}{4}\left[f\left(\frac{1}{4},\frac{1}{4}\right)+f\left(\frac{1}{4},\frac{3}{4}\right)+f\left(\frac{3}{4},\frac{1}{4}\right)+f\left(\frac{3}{4},\frac{3}{4}\right)\right]\approx \frac{1}{4}(11+13+9.5+11)\approx 11$$

.
Using sixteen squares we get the same result. So $\iint_R f(x,y)\,dA\approx 11$.

9. **(a)** With $m=n=2$, we have $\Delta A=4$. Using the contour map to estimate the value of f at the center of each subrectangle, we have

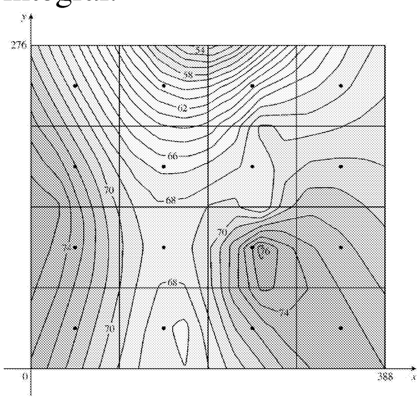
$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\bar{x}_i,\bar{y}_j\right)\Delta A=\Delta A\left[f(1,1)+f(1,3)+f(3,1)+f(3,3)\right]$$
$$\approx 4(27+4+14+17)=248$$

(b) $f_{\text{ave}}=\dfrac{1}{A(R)}\iint_R f(x,y)\,dA\approx \dfrac{1}{16}(248)=15.5$

10. As in Example 4, we place the origin at the southwest corner of the state. Then $R=[0,388]\times[0,276]$ (in miles) is the rectangle corresponding to Colorado and we define $f(x,y)$ to be the temperature at the location (x,y) . The average temperature is given by

$$f_{\text{ave}}=\frac{1}{A(R)}\iint_R f(x,y)\,dA=\frac{1}{388\cdot 276}\iint_R f(x,y)\,dA$$

We can use the Midpoint Rule with $m=n=4$ to give a reasonable estimate of the value of the double integral.



Thus, we divide R into 16 regions of equal size, as shown in the figure, with the center of each subrectangle indicated. The area of each subrectangle is $\Delta A=\dfrac{388}{4}\cdot\dfrac{276}{4}=6693$, so using the contour map to estimate the function values at each midpoint, we have

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{4}\sum_{j=1}^{4} f\left(\bar{x}_i,\bar{y}_j\right)\Delta A$$
$$\approx \Delta A$$
$$+72.0+74.9+68.4+63.7+73.2+72.3+70.3+67.7]$$
$$=6693(1111.5)$$

Therefore,

$f_{\text{ave}} \approx \dfrac{6693 \cdot 1111.5}{388 \cdot 276} \approx 69.5$, so the average temperature in Colorado on May 1, 1996, was approximately $69.5°$.

Alternatively, we can use the Midpoint Rule with $m=n=2$ which is easier computationally but will most likely be less accurate since we have fewer subrectangles. In this case, $\Delta A = \dfrac{388}{2} \cdot \dfrac{276}{2} = 26,772$ and we can use the same grid to estimate the function values at the midpoints of the four subrectangles. Then

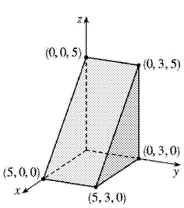
$$\begin{aligned} \iint_R f(x,y)\,dA &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\bar{x}_i,\bar{y}_j\right)\Delta A \approx 26,772[70.0+66.5+74.3+68.5] \\ &= 26,772 \cdot 279.3 \end{aligned}$$

and $f_{\text{ave}} \approx \dfrac{26,772 \cdot 279.3}{388 \cdot 276} \approx 69.8°$.

11. $z=3>0$, so we can interpret the integral as the volume of the solid S that lies below the plane $z=3$ and above the rectangle $[-2,2]\times[1,6]$. S is a rectangular solid, thus $\iint_R 3\,dA = 4 \cdot 5 \cdot 3 = 60$.

12. $z=5-x \geq 0$ for $0 \leq x \leq 5$, so we can interpret the integral as the volume of the solid S that lies below the plane $z=5-x$ and above the rectangle $[0,5]\times[0,3]$. S is a triangular cylinder whose volume is $3\,(\text{area of triangle}) = 3\left(\dfrac{1}{2} \cdot 5 \cdot 5\right) = 37.5$. Thus,

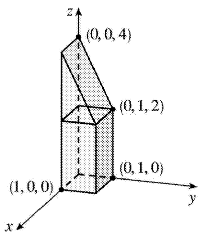
$$\iint_R (5-x)\,dA = 37.5$$



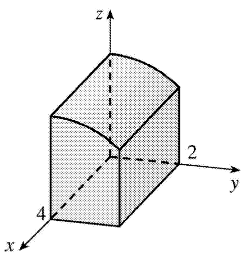
13. $z=f(x,y)=4-2y \geq 0$ for $0 \leq y \leq 1$. Thus the integral represents the volume of that part of the rectangular solid $[0,1]\times[0,1]\times[0,4]$ which lies below the plane $z=4-2y$.

So

$$\iint_R (4-2y)\,dA = (1)(1)(2) + \frac{1}{2}(1)(1)(2) = 3$$



14. Here $z=\sqrt{9-y^2}$, so $z^2+y^2=9$, $z\geq 0$. Thus the integral represents the volume of the top half of the part of the circular cylinder $z^2+y^2=9$ that lies above the rectangle $[0,4]\times[0,2]$.



15. To calculate the estimates using a programmable calculator, we can use an algorithm similar to that of Exercise 5.1.7 [ET 5.1.7]. In Maple, we can define the function $f(x,y)=e^{-x^2-y^2}$ (calling it f), load the student package, and then use the command
middlesum(middlesum(f,x=0..1,m),
y=0..1,m);

to get the estimate with $n=m^2$ squares of equal size. Mathematica has no special Riemann sum command, but we can define f and then use nested Sum commands to calculate the estimates.

n	estimate
1	0.6065
4	0.5694
16	0.5606
64	0.5585
256	0.5579
1024	0.5578

16.

n	estimate
1	0.9922

4	0.9262
16	0.8797

n	estimate
64	0.8660
256	0.8625
1024	0.8616

17. If we divide R into mn subrectangles, $\iint_R k\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^{*}, y_{ij}^{*}\right)\Delta A$ for any choice of sample points $\left(x_{ij}^{*}, y_{ij}^{*}\right)$. But $f\left(x_{ij}^{*}, y_{ij}^{*}\right)=k$ always and $\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta A=$ area of $R=(b-a)(d-c)$. Thus, no matter how we choose the sample points, $\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^{*}, y_{ij}^{*}\right)\Delta A=k\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta A=k(b-a)(d-c)$ and so

$$\iint_R k\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^{*}, y_{ij}^{*}\right)\Delta A=\lim_{m,n\to\infty} k\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta A$$

$$= \lim_{m,n\to\infty} k(b-a)(d-c)=k(b-a)(d-c)$$

18. On R , $0\le x+y\le 2<\pi$ and $\sin\theta\ge 0$ for $0\le\theta\le\pi$. Thus $f(x,y)=\sin(x+y)\ge 0$ for all $(x,y)\in R$. Since $0\le \sin(x+y)\le 1$, Property (9) gives $\iint_R 0\,dA\le\iint_R \sin(x+y)\,dA\le\iint_R 1\,dA$, so by Exercise 17 we have $0\le\iint_R \sin(x+y)\,dA\le 1$.

1. $\int_0^3 \left(2x+3x^2y\right)dx=\left[x^2+x^3y\right]_{x=0}^{x=3}=(9+27y)-(0+0)=9+27y$,

$\int_0^4 \left(2x+3x^2y\right)dy=\left[2xy+3x^2\frac{y^2}{2}\right]_{y=0}^{y=4}=\left(8x+3x^2\cdot\frac{16}{2}\right)-(0+0)=8x+24x^2$

2. $\int_0^3 \frac{y}{x+2}\,dx=\left[y\ln|x+2|\right]_{x=0}^{x=3}=y\ln 5-y\ln 2=y\ln\frac{5}{2}$,

$\int_0^4 \frac{y}{x+2}\,dy=\frac{1}{x+2}\left[\frac{y^2}{2}\right]_{y=0}^{y=4}=\frac{1}{x+2}\left(\frac{16}{2}-0\right)=\frac{8}{x+2}$

3. $\int_1^3\int_0^1 (1+4xy)\,dx\,dy=\int_1^3\left[x+2x^2y\right]_{x=0}^{x=1}dy=\int_1^3(1+2y)\,dy=\left[y+y^2\right]_1^3=(3+9)-(1+1)=10$

4.

$\int_2^4\int_{-1}^1\left(x^2+y^2\right)dy\,dx=\int_2^4\left[x^2y+\frac{1}{3}y^3\right]_{y=-1}^{y=1}dx=\int_2^4\left[\left(x^2+\frac{1}{3}\right)-\left(-x^2-\frac{1}{3}\right)\right]dx$

$=\int_2^4\left(2x^2+\frac{2}{3}\right)dx=\left[\frac{2}{3}x^3+\frac{2}{3}x\right]_2^4=\left(\frac{128}{3}+\frac{8}{3}\right)-\left(\frac{16}{3}+\frac{4}{3}\right)=\frac{116}{3}$

5. $\int_0^2\int_0^{\pi/2} x\sin y\,dy\,dx=\int_0^2 x\,dx\int_0^{\pi/2}\sin y\,dy$ [as in Example 5] $=\left[\frac{x^2}{2}\right]_0^2\left[-\cos y\right]_0^{\pi/2}=(2-0)(0+1)=2$.

6.

$\int_1^4\int_0^2\left(x+\sqrt{y}\right)dx\,dy=\int_1^4\left[\frac{1}{2}x^2+x\sqrt{y}\right]_{x=0}^{x=2}dy=\int_1^4\left(2+2\sqrt{y}\right)dy$

$=\left[2y+2\cdot\frac{2}{3}y^{3/2}\right]_1^4=\left(8+\frac{4}{3}\cdot 8\right)-\left(2+\frac{4}{3}\right)=\frac{46}{3}$

7.

$\int_0^2\int_0^1(2x+y)^8\,dx\,dy=\int_0^2\left[\frac{1}{2}\frac{(2x+y)^9}{9}\right]_{x=0}^{x=1}dy$

$$=\frac{1}{18}\int_0^2[(2+y)^9-(0+y)^9]\,dy=\frac{1}{18}\left[\frac{(2+y)^{10}}{10}-\frac{y^{10}}{10}\right]_0^2$$

$$=\frac{1}{180}[(4^{10}-2^{10})-(2^{10}-0^{10})]=\frac{1{,}046{,}528}{180}=\frac{261{,}632}{45}$$

8.

$$\int_0^1\int_1^2\frac{xe^x}{y}\,dy\,dx=\int_0^1 xe^x\,dx\int_1^2\frac{1}{y}\,dy \text{ [as in Example 5]}$$

$$=\left[xe^x-e^x\right]_0^1\left[\ln|y|\right]_1^2 \text{ [by integrating by parts]}$$

$$=[(e-e)-(0-1)](\ln 2-0)=\ln 2$$

9.

$$\int_1^4\int_1^2\left(\frac{x}{y}+\frac{y}{x}\right)dy\,dx=\int_1^4\left[x\ln|y|+\frac{1}{x}\cdot\frac{1}{2}y^2\right]_{y=1}^{y=2}dx=\int_1^4\left(x\ln 2+\frac{3}{2x}\right)dx$$

$$=\left[\frac{1}{2}x^2\ln 2+\frac{3}{2}\ln|x|\right]_1^4=8\ln 2+\frac{3}{2}\ln 4-\frac{1}{2}\ln 2$$

$$=\frac{15}{2}\ln 2+3\ln 4^{1/2}=\frac{21}{2}\ln 2$$

10.

$$\int_1^2\int_0^1(x+y)^{-2}\,dx\,dy=\int_1^2\left[-(x+y)^{-1}\right]_{x=0}^{x=1}dy=\int_1^2\left[y^{-1}-(1+y)^{-1}\right]dy$$

$$=\left[\ln y-\ln(1+y)\right]_1^2=\ln 2-\ln 3-0+\ln 2=\ln\frac{4}{3}$$

11.

$$\int_0^{\ln 2}\int_0^{\ln 5}e^{2x-y}\,dx\,dy=\left(\int_0^{\ln 5}e^{2x}\,dx\right)\left(\int_0^{\ln 2}e^{-y}\,dy\right)=\left[\frac{1}{2}e^{2x}\right]_0^{\ln 5}\left[-e^{-y}\right]_0^{\ln 2}$$

$$=\left(\frac{25}{2}-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)=6$$

12.

$$\int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\,dydx = \int_0^1\left[x\sqrt{x^2+y^2+1}\,\right]_{y=0}^{y=1}dx = \int_0^1 x\left(\sqrt{x^2+2}-\sqrt{x^2+1}\,\right)dx$$

$$= \frac{1}{3}\left[(x^2+2)^{3/2}-(x^2+1)^{3/2}\right]_0^1 = \frac{1}{3}\left[(3^{3/2}-2^{3/2})-(2^{3/2}-1)\right]$$

$$= \frac{1}{3}\left(3\sqrt{3}-4\sqrt{2}+1\right)$$

13.

$$\iint_R (6x^2y^3-5y^4)\,dA = \int_0^3\int_0^1 (6x^2y^3-5y^4)\,dydx = \int_0^3\left[\frac{3}{2}x^2y^4-y^5\right]_{y=0}^{y=1}dx$$

$$= \int_0^3\left(\frac{3}{2}x^2-1\right)dx = \left[\frac{1}{2}x^3-x\right]_0^3 = \frac{27}{2}-3 = \frac{21}{2}$$

14.

$$\iint_R \cos(x+2y)\,dA = \int_0^{\pi}\int_0^{\pi/2}\cos(x+2y)\,dydx$$

$$= \int_0^{\pi}\left[\frac{1}{2}\sin(x+2y)\right]_{y=0}^{y=\pi/2}dx = \frac{1}{2}\int_0^{\pi}(\sin(x+\pi)-\sin x)\,dx$$

$$= \frac{1}{2}\left[-\cos(x+\pi)+\cos x\right]_0^{\pi} = \frac{1}{2}\left[-\cos 2\pi+\cos\pi-(-\cos\pi+\cos 0)\right]$$

$$= \frac{1}{2}(-1-1-(1+1)) = -2$$

15.

$$\iint_R \frac{xy^2}{x^2+1}\,dA = \int_0^1\int_{-3}^3 \frac{xy^2}{x^2+1}\,dydx = \int_0^1 \frac{x}{x^2+1}\,dx\int_{-3}^3 y^2\,dy$$

$$= \left[\frac{1}{2}\ln(x^2+1)\right]_0^1\left[\frac{1}{3}y^3\right]_{-3}^3 = \frac{1}{2}(\ln 2-\ln 1)\cdot\frac{1}{3}(27+27) = 9\ln 2$$

16.

$$\iint_R \frac{1+x^2}{1+y^2}\,dA = \int_0^1\int_0^1 \frac{1+x^2}{1+y^2}\,dydx = \int_0^1(1+x^2)\,dx\int_0^1\frac{1}{1+y^2}\,dy$$

$$=\left[x+\frac{1}{3}x^{3}\right]_{0}^{1}\left[\tan^{-1}y\right]_{0}^{1}=\left(1+\frac{1}{3}-0\right)\left(\frac{\pi}{4}-0\right)=\frac{\pi}{3}$$

17. $\int_{0}^{\pi/6}\int_{0}^{\pi/3} x\sin(x+y)\,dy\,dx$

$=\int_{0}^{\pi/6}\left[-x\cos(x+y)\right]_{y=0}^{y=\pi/3}dx=\int_{0}^{\pi/6}\left[x\cos x-x\cos\left(x+\frac{\pi}{3}\right)\right]dx$

$=x\left[\sin x-\sin\left(x+\frac{\pi}{3}\right)\right]_{0}^{\pi/6}-\int_{0}^{\pi/6}\left[\sin x-\sin\left(x+\frac{\pi}{3}\right)\right]dx$

[by integrating by parts separately for each term]

$=\frac{\pi}{6}\left[\frac{1}{2}-1\right]-\left[-\cos x+\cos\left(x+\frac{\pi}{3}\right)\right]_{0}^{\pi/6}=-\frac{\pi}{12}-\left[-\frac{\sqrt{3}}{2}+0-\left(-1+\frac{1}{2}\right)\right]$

$=\frac{\sqrt{3}-1}{2}-\frac{\pi}{12}$

18. $\iint_{R}\frac{x}{1+xy}\,dA=\int_{0}^{1}\int_{0}^{1}\frac{x}{1+xy}\,dy\,dx$

$=\int_{0}^{1}\left[\ln(1+xy)\right]_{y=0}^{y=1}dx=\int_{0}^{1}\left[\ln(1+x)-\ln 1\right]dx$

$=\int_{0}^{1}\ln(1+x)\,dx=\left[(1+x)\ln(1+x)-x\right]_{0}^{1}$

[by integrating by parts]

$=(2\ln 2-1)-(\ln 1-0)=2\ln 2-1$

19.

$\iint_{R}xye^{x^{2}y}\,dA=\int_{0}^{2}\int_{0}^{1}xye^{x^{2}y}\,dx\,dy=\int_{0}^{2}\left[\frac{1}{2}e^{x^{2}y}\right]_{x=0}^{x=1}dy=\frac{1}{2}\int_{0}^{2}(e^{y}-1)\,dy$

$$=\frac{1}{2}\left[e^{y}-y\right]_{0}^{2}=\frac{1}{2}\left[(e^{2}-2)-(1-0)\right]=\frac{1}{2}(e^{2}-3)$$

20. $\int_{0}^{1}\int_{1}^{2}\frac{x}{x^{2}+y^{2}}\,dx\,dy=\int_{0}^{1}\left[\frac{1}{2}\ln(x^{2}+y^{2})\right]_{x=1}^{x=2}dy=\frac{1}{2}\int_{0}^{1}\left[\ln(4+y^{2})-\ln(1+y^{2})\right]dy$

To evaluate the first term, we integrate by parts with $u=\ln(4+y^{2})\Rightarrow$

$du = \dfrac{2y}{4+y^2}\,dy$ and

$dv = dy \Rightarrow v = y$. Then

$$\int \ln(4+y^2)dy = y\ln(4+y^2) - \int \frac{2y^2}{4+y^2}\,dy = y\ln(4+y^2) - \int\left(2 - \frac{8}{4+y^2}\right)dy$$

$$= y\ln(4+y^2) - 2y + 8\cdot\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) = y\ln(4+y^2) - 2y + 4\tan^{-1}\left(\frac{y}{2}\right)$$

Similarly, $\int \ln(1+y^2)dy = y\ln(1+y^2) - 2y + 2\tan^{-1}y$. Thus,

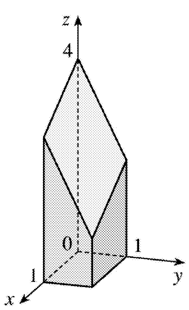
$$\int_0^1\int_1^2 \frac{x}{x^2+y^2}\,dx\,dy = \frac{1}{2}\int_0^1\left[\ln(4+y^2) - \ln(1+y^2)\right]dy$$

$$= \frac{1}{2}\left[y\ln(4+y^2) - 2y + 4\tan^{-1}\left(\frac{y}{2}\right) - y\ln(1+y^2) + 2y - 2\tan^{-1}y\right]_0^1$$

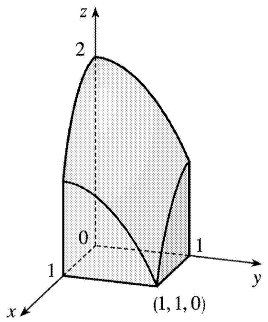
$$= \frac{1}{2}\left[\left(\ln 5 + 4\tan^{-1}\left(\frac{1}{2}\right) - \ln 2 - 2\tan^{-1}1\right) - 0\right]$$

$$= \frac{1}{2}\left[\ln 5 - \ln 2 + 4\tan^{-1}\left(\frac{1}{2}\right) - 2\left(\frac{\pi}{4}\right)\right] = \frac{1}{2}\ln\frac{5}{2} + 2\tan^{-1}\left(\frac{1}{2}\right) - \frac{\pi}{4}$$

21. $z = f(x,y) = 4 - x - 2y \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$.
So the solid is the region in the first octant which lies below the plane $z = 4 - x - 2y$ and above $[0,1]\times[0,1]$.



22. $z = 2 - x^2 - y^2 \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. So the solid is the region in the first octant which lies below the circular paraboloid $z = 2 - x^2 - y^2$ and above $[0,1]\times[0,1]$.



23.

$$V=\iint_R (12-3x-2y)\,dA=\int_{-2}^{3}\int_{0}^{1}(12-3x-2y)\,dx\,dy=\int_{-2}^{3}\left[12x-\frac{3}{2}x^2-2xy\right]_{x=0}^{x=1}dy$$

$$=\int_{-2}^{3}\left(\frac{21}{2}-2y\right)dy=\left[\frac{21}{2}y-y^2\right]_{-2}^{3}=\frac{95}{2}$$

24.

$$V=\iint_R (4+x^2-y^2)\,dA=\int_{-1}^{1}\int_{0}^{2}(4+x^2-y^2)\,dy\,dx=\int_{-1}^{1}\left[4y+x^2y-\frac{1}{3}y^3\right]_{y=0}^{y=2}dx$$

$$=\int_{-1}^{1}\left(2x^2+\frac{16}{3}\right)dx=\left[\frac{2}{3}x^3+\frac{16}{3}x\right]_{-1}^{1}=\frac{2}{3}+\frac{16}{3}+\frac{2}{3}+\frac{16}{3}=12$$

25.

$$V=\int_{-2}^{2}\int_{-1}^{1}\left(1-\frac{1}{4}x^2-\frac{1}{9}y^2\right)dx\,dy=4\int_{0}^{2}\int_{0}^{1}\left(1-\frac{1}{4}x^2-\frac{1}{9}y^2\right)dx\,dy$$

$$=4\int_{0}^{2}\left[x-\frac{1}{12}x^3-\frac{1}{9}y^2x\right]_{x=0}^{x=1}dy=4\int_{0}^{2}\left(\frac{11}{12}-\frac{1}{9}y^2\right)dy=4\left[\frac{11}{12}y-\frac{1}{27}y^3\right]_{0}^{2}=4\cdot\frac{83}{54}=\frac{166}{27}$$

26.

$$V=\int_{-1}^{1}\int_{0}^{\pi}(1+e^x\sin y)\,dy\,dx=\int_{-1}^{1}\left[y-e^x\cos y\right]_{y=0}^{y=\pi}dx=\int_{-1}^{1}(\pi+e^x-0+e^x)\,dx$$

$$=\int_{-1}^{1}(\pi+2e^x)\,dx=\left[\pi x+2e^x\right]_{-1}^{1}=2\pi+2e-\frac{2}{e}$$

27. Here we need the volume of the solid lying under the surface $z=x\sqrt{x^2+y}$ and above the square $R=[0,1]\times[0,1]$ in the xy –plane.

$$V \int_0^1\int_0^1 x\sqrt{x^2+y}\,dx\,dy=\int_0^1 \frac{1}{3}\left[(x^2+y)^{3/2}\right]_{x=0}^{x=1}dy=\frac{1}{3}\int_0^1\left[(1+y)^{3/2}-y^{3/2}\right]dy$$

$$=\frac{1}{3}\cdot\frac{2}{5}\left[(1+y)^{5/2}-y^{5/2}\right]_0^1=\frac{4}{15}\left(2\sqrt{2}-1\right)$$

28. Here we need the volume of the solid lying under the surface $z=1+(x-1)^2+4y^2$ and above the rectangle $R=[0,3]\times[0,2]$ in the xy –plane.

$$V = \int_0^3\int_0^2\left[1+(x-1)^2+4y^2\right]dy\,dx=\int_0^3\left[y+(x-1)^2y+\frac{4}{3}y^3\right]_{y=0}^{y=2}dx$$

$$=\int_0^3\left[2+2(x-1)^2+\frac{32}{3}\right]dx=\left[\frac{38}{3}x+\frac{2}{3}(x-1)^3\right]_0^3=44$$

29. In the first octant, $z\geq 0\Rightarrow y\leq 3$, so

$$V=\int_0^3\int_0^2(9-y^2)\,dx\,dy=\int_0^3\left[9x-y^2x\right]_{x=0}^{x=2}dy=\int_0^3(18-2y^2)\,dy=\left[18y-\frac{2}{3}y^3\right]_0^3=36$$

30. **(a)** Here we need the volume of the solid lying under the surface $z=6-xy$ and above the rectangle $R=[-2,2]\times[0,3]$ in the xy –plane.

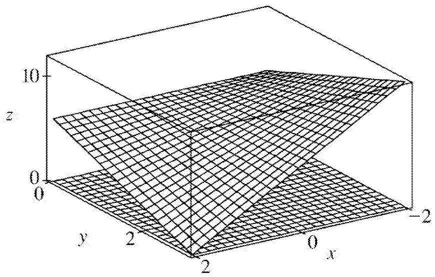
$$V = \int_{-2}^2\int_0^3(6-xy)\,dy\,dx$$

$$=\int_{-2}^2\left[6y-\frac{1}{2}xy^2\right]_{y=0}^{y=3}dx$$

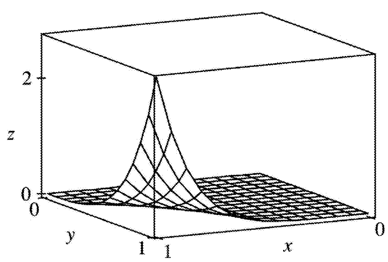
$$=\int_{-2}^2\left(18-\frac{9}{2}x\right)dx$$

$$=\left[18x-\frac{9}{4}x^2\right]_{-2}^2=72$$

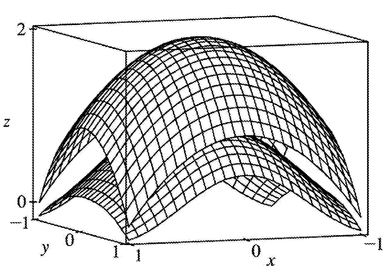
(b) The solid occupies the region between the two surfaces shown.



31. In Maple, we can calculate the integral by defining the integrand as f and then using the command int(int(f,x=0..1),y=0..1);.
In Mathematica, we can use the command Integrate[Integrate[f,{x,0,1}], {y,0,1}]. We find that $\iint_R x^5 y^3 e^{xy}\, dA = 21e - 57 \approx 0.0839$. We can use plot3d (in Maple) or Plot3d (in Mathematica) to graph the function.



32. In Maple, we can calculate the integral by defining f:=exp(−x^2)*cos(x^2+y^2); and g:=2−x^2−y^2; and then using the command evalf(int(int(g−f,x=−1..1),y=−1..1),5);. In Mathematica, we can use the command N[Integrate[Integrate[f,{x,0,1}],{y,0,1}],5].



In each of these commands, the 5 indicates that we want only five significant digits; this speeds up the calculation considerably. We find that $\iint_R \left[(2-x^2-y^2) - \left(e^{-x^2} \cos (x^2+y^2) \right) \right] dA \approx 3.0271$. We can use the plot3d command (in Maple) or Plot3d (in Mathematica) to graph both functions on the same screen.

33. R is the rectangle $[-1,1]\times[0,5]$. Thus, $A(R)=2\cdot 5=10$ and

$$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x,y)\,dA = \frac{1}{10}\int_0^5\int_{-1}^1 x^2 y\,dx\,dy = \frac{1}{10}\int_0^5\left[\frac{1}{3}x^3 y\right]_{x=-1}^{x=1} dy = \frac{1}{10}\int_0^5 \frac{2}{3}y\,dy$$

$$= \frac{1}{10}\left[\frac{1}{3}y^2\right]_0^5 = \frac{5}{6}$$

34. $A(R)=4\cdot 1=4$, so

$$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x,y)\,dA = \frac{1}{4}\int_0^4\int_0^1 e^y\sqrt{x+e^y}\,dy\,dx = \frac{1}{4}\int_0^4\left[\frac{2}{3}(x+e^y)^{3/2}\right]_{y=0}^{y=1} dx$$

$$= \frac{1}{4}\cdot\frac{2}{3}\int_0^4\left[(x+e)^{3/2}-(x+1)^{3/2}\right]dx = \frac{1}{6}\left[\frac{2}{5}(x+e)^{5/2}-\frac{2}{5}(x+1)^{5/2}\right]_0^4$$

$$= \frac{1}{6}\cdot\frac{2}{5}\left[(4+e)^{5/2}-5^{5/2}-e^{5/2}+1\right] = \frac{1}{15}\left[(4+e)^{5/2}-e^{5/2}-5^{5/2}+1\right]\approx 3.327$$

35. Let $f(x,y)=\dfrac{x-y}{(x+y)^3}$. Then a CAS gives $\int_0^1\int_0^1 f(x,y)\,dy\,dx=\frac{1}{2}$ and $\int_0^1\int_0^1 f(x,y)\,dx\,dy=-\frac{1}{2}$.
To explain the seeming violation of Fubini's Theorem, note that f has an infinite discontinuity at $(0,0)$ and thus does not satisfy the conditions of Fubini's Theorem. In fact, both iterated integrals involve improper integrals which diverge at their lower limits of integration.

36. **(a)** Loosely speaking, Fubini's Theorem says that the order of integration of a function of two variables does not affect the value of the double integral, while Clairaut's Theorem says that the order of differentiation of such a function does not affect the value of the second–order derivative. Also, both theorems require continuity (though Fubini's allows a finite number of smooth curves to contain discontinuities).
(b) To find g_{xy} , we first hold y constant and use the single–variable Fundamental Theorem of Calculus, Part 1: $g_x=\dfrac{d}{dx}\,g(x,y)=\dfrac{d}{dx}\int_a^x\left(\int_c^y f(s,t)\,dt\right)ds=\int_c^y f(x,t)\,dt$. Now we use the Fundamental Theorem again: $g_{xy}=\dfrac{d}{dy}\int_c^y f(x,t)\,dt=f(x,y)$.

To find g_{yx} , we first use Fubini's Theorem to find that $\int_a^x\int_c^y f(s,t)\,dt\,ds=\int_c^y\int_a^x f(s,t)\,dt\,ds$, and then use the Fundamental Theorem twice, as above, to get $g_{yx}=f(x,y)$. So $g_{xy}=g_{yx}=f(x,y)$.

1.

$$\int_0^1\int_0^{x^2}(x+2y)\,dy\,dx=\int_0^1\left[xy+y^2\right]_{y=0}^{y=x^2}dx=\int_0^1\left[x(x^2)+(x^2)^2-0-0\right]dx$$

$$=\int_0^1(x^3+x^4)\,dx=\left[\frac{1}{4}x^4+\frac{1}{5}x^5\right]_0^1=\frac{9}{20}$$

2.

$$\int_1^2\int_y^2 xy\,dx\,dy=\int_1^2\left[\frac{1}{2}x^2y\right]_{x=y}^{x=2}dy=\int_1^2\frac{1}{2}y(4-y^2)\,dy=\frac{1}{2}\int_1^2(4y-y^3)\,dy$$

$$=\frac{1}{2}\left[2y^2-\frac{1}{4}y^4\right]_1^2=\frac{1}{2}\left(8-4-2+\frac{1}{4}\right)=\frac{9}{8}$$

3.

$$\int_0^1\int_y^{e^y}\sqrt{x}\,dx\,dy=\int_0^1\left[\frac{2}{3}x^{3/2}\right]_{x=y}^{x=e^y}dy=\frac{2}{3}\int_0^1(e^{3y/2}-y^{3/2})\,dy=\frac{2}{3}\left[\frac{2}{3}e^{3y/2}-\frac{2}{5}y^{5/2}\right]_0^1$$

$$=\frac{2}{3}\left(\frac{2}{3}e^{3/2}-\frac{2}{5}-\frac{2}{3}e^0+0\right)=\frac{4}{9}e^{3/2}-\frac{32}{45}$$

4.

$$\int_0^1\int_x^{2-x}(x^2-y)\,dy\,dx=\int_0^1\left[x^2y-\frac{1}{2}y^2\right]_{y=x}^{y=2-x}dx=\int_0^1\left[x^2(2-x)-\frac{1}{2}(2-x)^2-x^2(x)+\frac{1}{2}x^2\right]dx$$

$$=\int_0^1(-2x^3+2x^2+2x-2)\,dx=\left[-\frac{1}{2}x^4+\frac{2}{3}x^3+x^2-2x\right]_0^1=-\frac{5}{6}$$

5.

$$\int_0^{\pi/2}\int_0^{\cos\theta}e^{\sin\theta}\,dr\,d\theta=\int_0^{\pi/2}\left[re^{\sin\theta}\right]_{r=0}^{r=\cos\theta}d\theta=\int_0^{\pi/2}(\cos\theta)e^{\sin\theta}\,d\theta=e^{\sin\theta}\Big]_0^{\pi/2}$$

$$=e^{\sin(\pi/2)}-e^0=e-1$$

6.

$$\int_0^1\int_0^v \sqrt{1-v^2}\,du\,dv = \int_0^1 \Big[u\sqrt{1-v^2}\Big]_{u=0}^{u=v} dv = \int_0^1 v\sqrt{1-v^2}\,dv = -\frac{1}{3}(1-v^2)^{3/2}\Big]_0^1$$

$$= -\frac{1}{3}(0-1) = \frac{1}{3}$$

7.

$$\iint_D x^3y^2\,dA = \int_0^2\int_{-x}^{x} x^3y^2\,dy\,dx = \int_0^2\left[\frac{1}{3}x^3y^3\right]_{y=-x}^{y=x} dx = \frac{1}{3}\int_0^2 2x^6\,dx$$

$$= \frac{2}{3}\left[\frac{1}{7}x^7\right]_0^2 = \frac{2}{21}\left[2^7-0\right] = \frac{256}{21}$$

8.

$$\iint_D \frac{4y}{x^3+2}\,dA = \int_1^2\int_0^{2x} \frac{4y}{x^3+2}\,dy\,dx = \int_1^2\left[\frac{2y^2}{x^3+2}\right]_{y=0}^{y=2x} dx = \int_1^2 \frac{8x^2}{x^3+2}\,dx$$

$$= \frac{8}{3}\ln\left|x^3+2\right|\Big]_1^2 = \frac{8}{3}(\ln 10-\ln 3) = \frac{8}{3}\ln\frac{10}{3}$$

9.

$$\int_0^1\int_0^{\sqrt{x}} \frac{2y}{x^2+1}\,dy\,dx = \int_0^1\left[\frac{y^2}{x^2+1}\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 \frac{x}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln\left|x^2+1\right|\Big]_0^1 = \frac{1}{2}(\ln 2-\ln 1) = \frac{1}{2}\ln 2$$

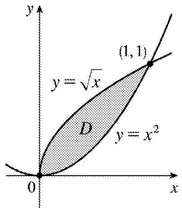
10. $\displaystyle\int_0^1\int_0^y e^{y^2}\,dx\,dy = \int_0^1\Big[xe^{y^2}\Big]_{x=0}^{x=y} dy = \int_0^1 ye^{y^2}\,dy = \frac{1}{2}e^{y^2}\Big]_0^1 = \frac{1}{2}(e-1)$

11. $\displaystyle\int_1^2\int_y^{y^3} e^{x/y}\,dx\,dy = \int_1^2\Big[ye^{x/y}\Big]_{x=y}^{x=y^3} dy = \int_1^2\left(ye^{y^2}-ey\right)dy = \left[\frac{1}{2}e^{y^2}-\frac{1}{2}ey^2\right]_1^2 = \frac{1}{2}(e^4-4e)$

12. $\displaystyle\int_0^1\int_0^y x\sqrt{y^2-x^2}\,dx\,dy = \int_0^1\left[-\frac{1}{3}(y^2-x^2)^{3/2}\right]_{x=0}^{x=y} dy = \frac{1}{3}\int_0^1 y^3\,dy = \frac{1}{3}\cdot\frac{1}{4}y^4\Big]_0^1 = \frac{1}{12}$

13. $\displaystyle\int_0^1\int_0^{x^2} x\cos y\,dy\,dx=\int_0^1\left[x\sin y\right]_{y=0}^{y=x^2}dx=\int_0^1 x\sin x^2\,dx= -\frac{1}{2}\cos x^2\Big]_0^1=\frac{1}{2}(1-\cos 1)$

14.

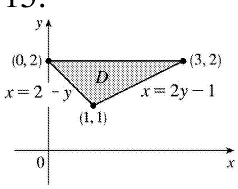


$$\int_0^1\int_{x^2}^{\sqrt{x}}(x+y)\,dy\,dx=\int_0^1\left[xy+\frac{1}{2}y^2\right]_{y=x^2}^{y=\sqrt{x}}dx$$

$$=\int_0^1\left(x^{3/2}+\frac{1}{2}x-x^3-\frac{1}{2}x^4\right)dx$$

$$=\left[\frac{2}{5}x^{5/2}+\frac{1}{4}x^2-\frac{1}{4}x^4-\frac{1}{10}x^5\right]_0^1=\frac{3}{10}$$

15.



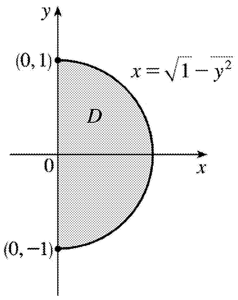
$$\int_1^2\int_{2-y}^{2y-1}y^3\,dx\,dy=\int_1^2\left[xy^3\right]_{x=2-y}^{x=2y-1}dy$$

$$=\int_1^2\left[(2y-1)-(2-y)\right]y^3\,dy$$

$$=\int_1^2(3y^4-3y^3)\,dy=\left[\frac{3}{5}y^5-\frac{3}{4}y^4\right]_1^2$$

$$=\frac{96}{5}-12-\frac{3}{5}+\frac{3}{4}=\frac{147}{20}$$

16.



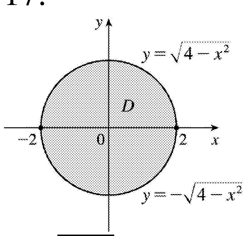
$$\iint_D xy^2\,dA = \int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} xy^2\,dx\,dy$$

$$= \int_{-1}^{1} y^2\left[\frac{1}{2}x^2\right]_{x=0}^{x=\sqrt{1-y^2}} dy = \frac{1}{2}\int_{-1}^{1} y^2(1-y^2)\,dy$$

$$= \frac{1}{2}\int_{-1}^{1}(y^2-y^4)\,dy = \frac{1}{2}\left[\frac{1}{3}y^3-\frac{1}{5}y^5\right]_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}+\frac{1}{3}-\frac{1}{5}\right) = \frac{2}{15}$$

17.



$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(2x-y)\,dy\,dx$$

$$= \int_{-2}^{2}\left[2xy-\frac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} dx$$

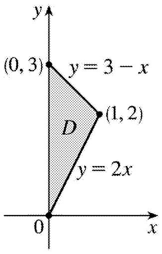
$$= \int_{-2}^{2}\left[2x\sqrt{4-x^2}-\frac{1}{2}\left(4-x^2\right)+2x\sqrt{4-x^2}+\frac{1}{2}\left(4-x^2\right)\right]dx$$

$$= \int_{-2}^{2}4x\sqrt{4-x^2}\,dx = -\frac{4}{3}\left(4-x^2\right)^{3/2}\Big]_{-2}^{2} = 0$$

(Or, note that $4x\sqrt{4-x^2}$ is an odd function, so

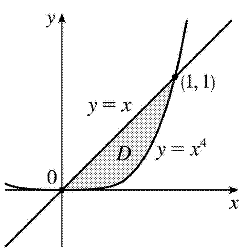
$\int_{-2}^{2} 4x\sqrt{4-x^2}\,dx=0$.)

18.



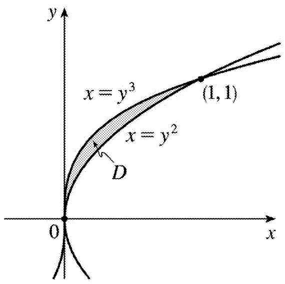
$$\iint_D xy\,dA=\int_0^1\int_{2x}^{3-x} 2xy\,dy\,dx=\int_0^1\left[xy^2\right]_{y=2x}^{y=3-x}dx$$
$$=\int_0^1 x[(3-x)^2-(2x)^2]\,dx$$
$$=\int_0^1 (-3x^3-6x^2+9x)\,dx$$
$$=\left[-\frac{3}{4}x^4-2x^3+\frac{9}{2}x^2\right]_0^1=-\frac{3}{4}-2+\frac{9}{2}=\frac{7}{4}$$

19.



$$V=\int_0^1\int_{x^4}^{x}(x+2y)\,dy\,dx$$
$$=\int_0^1\left[xy+y^2\right]_{y=x^4}^{y=x}dx=\int_0^1(2x^2-x^5-x^8)dx$$
$$=\left[\frac{2}{3}x^3-\frac{1}{6}x^6-\frac{1}{9}x^9\right]_0^1=\frac{2}{3}-\frac{1}{6}-\frac{1}{9}=\frac{7}{18}$$

20.

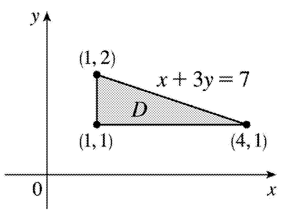


$$V = \int_0^1 \int_{y^3}^{y^2} (2x+y^2)\,dx\,dy$$

$$= \int_0^1 \Big[x^2 + xy^2 \Big]_{x=y^3}^{x=y^2} dy = \int_0^1 (2y^4 - y^6 - y^5)\,dy$$

$$= \left[\frac{2}{5}y^5 - \frac{1}{7}y^7 - \frac{1}{6}y^6 \right]_0^1 = \frac{19}{210}$$

21.

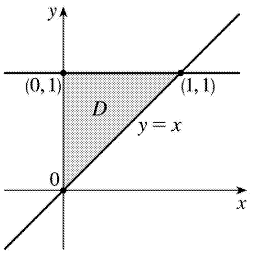


$$V = \int_1^2 \int_1^{7-3y} xy\,dx\,dy = \int_1^2 \left[\frac{1}{2}x^2y \right]_{x=1}^{x=7-3y} dy$$

$$= \frac{1}{2}\int_1^2 (48y - 42y^2 + 9y^3)\,dy$$

$$= \frac{1}{2}\left[24y^2 - 14y^3 + \frac{9}{4}y^4 \right]_1^2 = \frac{31}{8}$$

22.

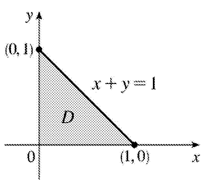


$$V=\int_0^1\int_x^1(x^2+3y^2)\,dy\,dx$$

$$=\int_0^1\left[x^2y+y^3\right]_{y=x}^{y=1}dx=\int_0^1(x^2+1-2x^3)\,dx$$

$$=\left[\frac{1}{3}x^3+x-\frac{1}{2}x^4\right]_0^1=\frac{5}{6}$$

23.



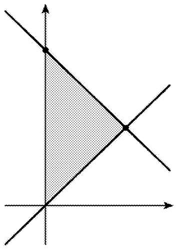
$$V=\int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx$$

$$=\int_0^1\left[y-xy-\frac{y^2}{2}\right]_{y=0}^{y=1-x}dx$$

$$=\int_0^1\left[(1-x)^2-\frac{1}{2}(1-x)^2\right]dx$$

$$=\int_0^1\frac{1}{2}(1-x)^2\,dx=\left[-\frac{1}{6}(1-x)^3\right]_0^1=\frac{1}{6}$$

24.

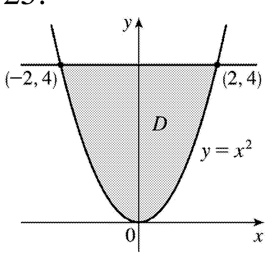


$$V = \int_0^1 \int_x^{2-x} x\,dy\,dx$$

$$= \int_0^1 x[\,y\,]_{y=x}^{y=2-x} dx = \int_0^1 (2x - 2x^2)\,dx$$

$$= \left[x^2 - \frac{2}{3}x^3 \right]_0^1 = \frac{1}{3}$$

25.

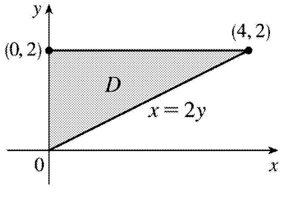


$$V = \int_{-2}^{2} \int_{x^2}^{4} x^2\,dy\,dx$$

$$= \int_{-2}^{2} x^2[y]_{y=x^2}^{y=4} dx = \int_{-2}^{2} (4x^2 - x^4)\,dx$$

$$= \left[\frac{4}{3}x^3 - \frac{1}{5}x^5 \right]_{-2}^{2} = \frac{32}{3} - \frac{32}{5} + \frac{32}{3} - \frac{32}{5} = \frac{128}{15}$$

26.

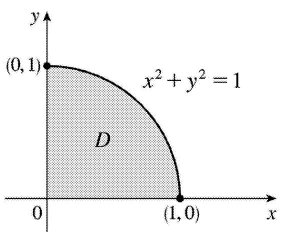


$$V=\int_0^2\int_0^{2y}\sqrt{4-y^2}\,dx\,dy$$

$$=\int_0^2\left[x\sqrt{4-y^2}\,\right]_{x=0}^{x=2y}dy=\int_0^2 2y\sqrt{4-y^2}\,dy$$

$$=\left[-\frac{2}{3}\left(4-y^2\right)^{3/2}\right]_0^2=0+\frac{16}{3}=\frac{16}{3}$$

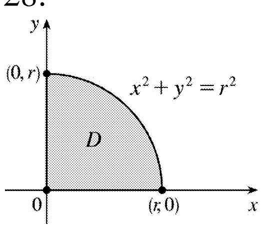
27.



$$V=\int_0^1\int_0^{\sqrt{1-x^2}}y\,dy\,dx=\int_0^1\left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{1-x^2}}dx$$

$$=\int_0^1\frac{1-x^2}{2}\,dx=\frac{1}{2}\left[x-\frac{1}{3}x^3\right]_0^1=\frac{1}{3}$$

28.



By symmetry, the desired volume V is 8 times the volume V_1 in the first octant. Now

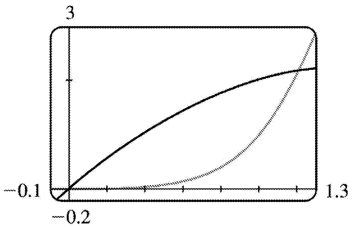
$$V_1=\int_0^r\int_0^{\sqrt{r^2-y^2}}\sqrt{r^2-y^2}\,dx\,dy$$

$$=\int_0^r\left[x\sqrt{r^2-y^2}\,\right]_{x=0}^{x=\sqrt{r^2-y^2}}dy$$

$$= \int_0^r (r^2 - y^2)\,dy = \left[r^2 y - \frac{1}{3} y^3 \right]_0^r = \frac{2}{3} r^3$$

Thus $V = \frac{16}{3} r^3$.

29.



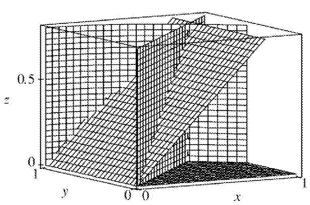
From the graph, it appears that the two curves intersect at $x=0$ and at $x \approx 1.213$. Thus the desired integral is

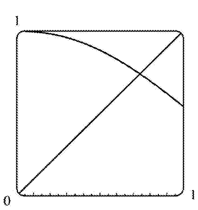
$$\iint_D x\,dA \approx \int_0^{1.213} \int_{x^4}^{3x-x^2} x\,dy\,dx = \int_0^{1.213} [xy]_{y=x^4}^{y=3x-x^2} dx$$

$$= \int_0^{1.213} (3x^2 - x^3 - x^5)\,dx = \left[x^3 - \frac{1}{4} x^4 - \frac{1}{6} x^6 \right]_0^{1.213}$$

$$\approx 0.713$$

30.





The desired solid is shown in the first graph. From the second graph, we estimate that $y=\cos x$ intersects $y=x$ at $x \approx 0.7391$. Therefore the volume of the solid is

$$V \approx \int_0^{0.7391} \int_x^{\cos x} z\,dy\,dx = \int_0^{0.7391} \int_x^{\cos x} x\,dy\,dx$$

$$= \int_0^{0.7391} [xy]_{y=x}^{y=\cos x} dx = \int_0^{0.7391} (x\cos x - x^2)dx$$

$$= \left[\cos x + x\sin x - \frac{1}{3}x^3 \right]_0^{0.7391} \approx 0.1024$$

Note: There is a different solid which can also be construed to satisfy the conditions stated in the exercise. This is the solid bounded by all of the given surfaces, as well as the plane $y=0$. In case you calculated the volume of this solid and want to check your work, its volume is

$$V \approx \int_0^{0.7391} \int_0^x x\,dy\,dx + \int_{0.7391}^{\pi/2} \int_0^{\cos x} x\,dy\,dx \approx 0.4684 .$$

31. The two bounding curves $y=1-x^2$ and $y=x^2-1$ intersect at $(\pm 1,0)$ with $1-x^2 \geq x^2-1$ on $[-1,1]$. Within this region, the plane $z=2x+2y+10$ is above the plane $z=2-x-y$, so

$$V = \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (2x+2y+10)dydx - \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (2-x-y)dydx$$

$$= \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (2x+2y+10-(2-x-y))dydx = \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (3x+3y+8)dydx$$

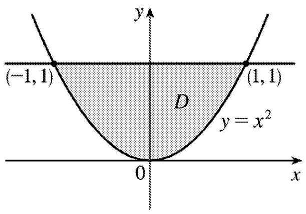
$$= \int_{-1}^{1} \left[3xy + \frac{3}{2}y^2 + 8y \right]_{y=x^2-1}^{y=1-x^2} dx$$

$$= \int_{-1}^{1} \left[3x(1-x^2) + \frac{3}{2}(1-x^2)^2 + 8(1-x^2) - 3x(x^2-1) - \frac{3}{2}(x^2-1)^2 - 8(x^2-1) \right] dx$$

$$= \int_{-1}^{1} (-6x^3 - 16x^2 + 6x + 16)dx = \left[-\frac{3}{2}x^4 - \frac{16}{3}x^3 + 3x^2 + 16x \right]_{-1}^{1}$$

$$= -\frac{3}{2} - \frac{16}{3} + 3 + 16 + \frac{3}{2} - \frac{16}{3} - 3 + 16 = \frac{64}{3}$$

32. The two planes intersect in the line $y=1$, $z=3$, so the region of integration is the plane region enclosed by the parabola $y=x^2$ and the line $y=1$. We have $2+y \geq 3y$ for $0 \leq y \leq 1$, so the solid region is bounded above by $z=2+y$ and bounded below by $z=3y$.



$$V=\int_{-1}^{1}\int_{x^2}^{1}(2+y)\,dy\,dx-\int_{-1}^{1}\int_{x^2}^{1}(3y)\,dy\,dx=\int_{-1}^{1}\int_{x^2}^{1}(2+y-3y)\,dy\,dx=\int_{-1}^{1}\int_{x^2}^{1}(2-2y)\,dy\,dx$$

$$=\int_{-1}^{1}\Big[2y-y^2\Big]_{y=x^2}^{y=1}dx=\int_{-1}^{1}(1-2x^2+x^4)\,dx=\Big[x-\frac{2}{3}x^3+\frac{1}{5}x^5\Big]_{-1}^{1}=\frac{16}{15}$$

33. The two bounding curves $y=x^3-x$ and $y=x^2+x$ intersect at the origin and at $x=2$, with $x^2+x>x^3-x$ on $(0,2)$. Using a CAS, we find that the volume is

$$V=\int_{0}^{2}\int_{x^3-x}^{x^2+x}z\,dy\,dx=\int_{0}^{2}\int_{x^3-x}^{x^2+x}(x^3y^4+xy^2)\,dy\,dx=\frac{13{,}984{,}735{,}616}{14{,}549{,}535}$$

34. For $|x|\le 1$ and $|y|\le 1$, $2x^2+y^2<8-x^2-2y^2$. Also, the cylinder is described by the inequalities $-1\le x\le 1$, $-\sqrt{1-x^2}\le y\le\sqrt{1-x^2}$. So the volume is given by

$$V=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\Big[(8-x^2-2y^2)-(2x^2+y^2)\Big]dy\,dx=\frac{13\pi}{2}$$

[using a CAS]

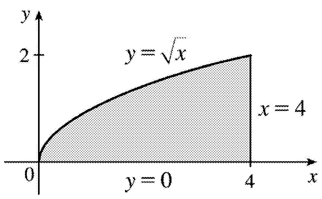
35. The two surfaces intersect in the circle $x^2+y^2=1$, $z=0$ and the region of integration is the disk D : $x^2+y^2\le 1$. Using a CAS, the volume is $\iint_D(1-x^2-y^2)\,dA=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(1-x^2-y^2)\,dy\,dx=\frac{\pi}{2}$.

36. The projection onto the xy –plane of the intersection of the two surfaces is the circle $x^2+y^2=2y\Rightarrow$ $x^2+y^2-2y=0\Rightarrow x^2+(y-1)^2=1$, so the region of integration is given by $-1\le x\le 1$,

$1-\sqrt{1-x^2} \le y \le 1+\sqrt{1-x^2}$. In this region, $2y \ge x^2+y^2$ so, using a CAS, the volume is

$$V=\int_{-1}^{1}\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} [2y-(x^2+y^2)]\,dy\,dx=\frac{\pi}{2} \ .$$

37.



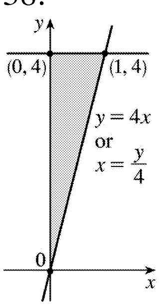
Because the region of integration is

$$D=\{(x,y)\mid 0\le y\le\sqrt{x}, 0\le x\le 4\}$$
$$=\{(x,y)\mid y^2\le x\le 4, 0\le y\le 2\}$$

we have

$$\int_0^4\int_0^{\sqrt{x}} f(x,y)\,dy\,dx=\iint_D f(x,y)\,dA=\int_0^2\int_{y^2}^4 f(x,y)\,dx\,dy \ .$$

38.



Because the region of integration is

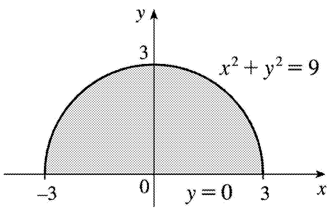
$$D=\{(x,y)\mid 4x\le y\le 4, 0\le x\le 1\}$$
$$=\left\{(x,y)\mid 0\le x\le \frac{y}{4}, 0\le y\le 4\right\}$$

we have

$$\int_0^1\int_{4x}^4 f(x,y)\,dy\,dx=\iint_D f(x,y)\,dA$$

$$=\int_0^4 \int_0^{y/4} f(\text{x,y})\,dx\,dy$$

39.



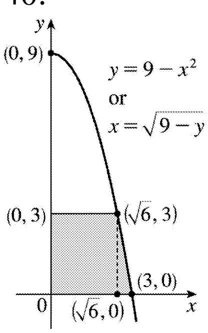
Because the region of integration is

$$D = \left\{ (\text{x,y}) \mid -\sqrt{9-y^2} \le x \le \sqrt{9-y^2}, 0 \le y \le 3 \right\}$$
$$= \left\{ (\text{x,y}) \mid 0 \le y \le \sqrt{9-x^2}, -3 \le x \le 3 \right\}$$

we have

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \text{f(x,y) dx dy} = \iint_D \text{f(x,y) dA} = \int_{-3}^3 \int_0^{\sqrt{9-x^2}} \text{f(x,y) dy dx}$$

40.



To reverse the order, we must break the region into two separate type I regions. Because the region of integration is

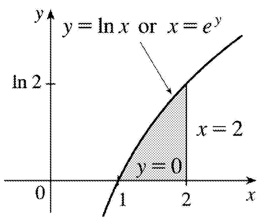
$$D = \left\{ (\text{x,y}) \mid 0 \le x \le \sqrt{9-y}, 0 \le y \le 3 \right\}$$
$$= \left\{ (\text{x,y}) \mid 0 \le y \le 3, 0 \le x \le \sqrt{6} \right\} \cup \left\{ (\text{x,y}) \mid 0 \le y \le 9-x^2, \sqrt{6} \le x \le 3 \right\}$$

we have

$$\int_0^3 \int_0^{\sqrt{9-y}} f(\text{x,y})\,dx\,dy = \iint_D f(\text{x,y})\,dA$$

$$= \int_0^{\sqrt{6}} \int_0^3 f(x,y)\,dy\,dx + \int_{\sqrt{6}}^3 \int_0^{9-x^2} f(x,y)\,dy\,dx$$

41.



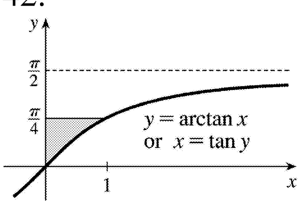
Because the region of integration is

$$D = \{(x,y) \mid 0 \le y \le \ln x, 1 \le x \le 2\}$$
$$= \left\{ (x,y) \mid e^y \le x \le 2, 0 \le y \le \ln 2 \right\}$$

we have

$$\int_1^2 \int_0^{\ln x} f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA$$
$$= \int_0^{\ln 2} \int_{e^y}^2 f(x,y)\,dx\,dy$$

42.



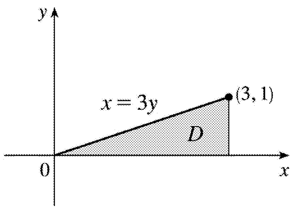
Because the region of integration is

$$D = \left\{ (x,y) \mid \arctan x \le y \le \frac{\pi}{4}, 0 \le x \le 1 \right\}$$
$$= \left\{ (x,y) \mid 0 \le x \le \tan y, 0 \le y \le \frac{\pi}{4} \right\}$$

we have

$$\int_0^1 \int_{\arctan x}^{\pi/4} f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA$$
$$= \int_0^{\pi/4} \int_0^{\tan y} f(x,y)\,dx\,dy$$

43.

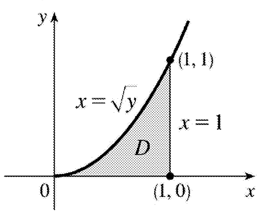


$$\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy = \int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx$$

$$= \int_0^3 \left[e^{x^2} y \right]_{y=0}^{y=x/3} dx = \int_0^3 \left(\frac{x}{3} \right) e^{x^2}\,dx$$

$$= \frac{1}{6}\ e^{x^2} \Big]_0^3 = \frac{e^9 - 1}{6}$$

44.

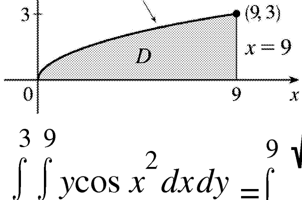


$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy = \int_0^1 \int_0^{x^2} \sqrt{x^3+1}\,dy\,dx$$

$$= \int_0^1 \left[\sqrt{x^3+1}\ y \right]_{y=0}^{y=x^2} dx = \int_0^1 x^2 \sqrt{x^3+1}\,dx$$

$$= \frac{2}{9}(x^3+1)^{3/2} \Big]_0^1 = \frac{2}{9}\left(2^{3/2} - 1\right)$$

45.

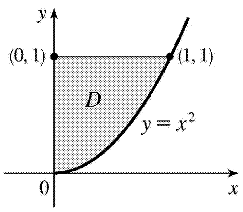


$$\int_0^3 \int_{y^2}^9 y\cos x^2\,dx\,dy = \int_0^9 \int_0^{\sqrt{x}} y\cos x^2\,dy\,dx$$

$$= \int_0^9 \cos x^2 \left[\frac{y^2}{2} \right]_{y=0}^{y=\sqrt{x}} dx = \int_0^9 \frac{1}{2} x \cos x^2 \, dx$$

$$= \frac{1}{4} \sin x^2 \Big]_0^9 = \frac{1}{4} \sin 81$$

46.



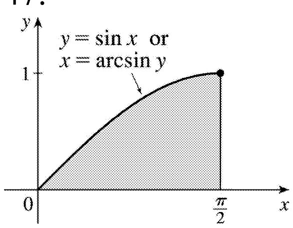
$$\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx = \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3)\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^4}{4} \sin(y^3) \right]_{x=0}^{x=\sqrt{y}} dy$$

$$= \int_0^1 \frac{1}{4} y^2 \sin(y^3)\, dy$$

$$= -\frac{1}{12} \cos(y^3) \Big]_0^1 = \frac{1}{12}(1 - \cos 1)$$

47.



$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1 + \cos^2 x}\, dx\, dy$$

$$= \int_0^{\pi/2} \int_0^{\sin x} \cos x \sqrt{1 + \cos^2 x}\, dy\, dx$$

$$= \int_0^{\pi/2} \cos x \sqrt{1 + \cos^2 x}\, [y]_{y=0}^{y=\sin x} dx$$

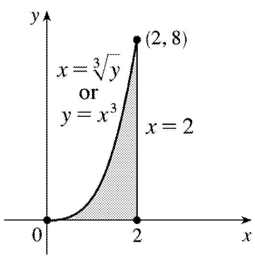
$$=\int_0^{\pi/2} \cos x\sqrt{1+\cos^2 x}\sin x\,dx$$

[Let $u=\cos x, du=-\sin x\,dx, dx=du/(-\sin x)$]

$$=\int_1^0 -u\sqrt{1+u^2}\,du= \left. -\frac{1}{3}\left(1+u^2\right)^{3/2}\right]_1^0$$

$$=\frac{1}{3}\left(\sqrt{8}-1\right)=\frac{1}{3}\left(2\sqrt{2}-1\right)$$

48.



$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy =\int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx$$

$$=\int_0^2 e^{x^4}\left[y\right]_{y=0}^{y=x^3} dx=\int_0^2 x^3 e^{x^4}\,dx$$

$$= \left. \frac{1}{4}e^{x^4}\right]_0^2=\frac{1}{4}(e^{16}-1)$$

49.

$$D=\{(x,y)|0\le x\le 1,-x+1\le y\le 1\}\cup\{(x,y)|-1\le x\le 0,x+1\le y\le 1\}$$
$$\cup\{(x,y)|0\le x\le 1,-1\le y\le x-1\}\cup\{(x,y)|-1\le x\le 0,-1\le y\le -x-1\} ,$$

all type I.

$$\iint_D x^2\,dA = \int_0^1 \int_{1-x}^1 x^2\,dy\,dx+\int_{-1}^0 \int_{x+1}^1 x^2\,dy\,dx+\int_0^1 \int_{-1}^{x-1} x^2\,dy\,dx+\int_{-1}^0 \int_{-1}^{-x-1} x^2\,dy\,dx$$

$$= 4\int_0^1 \int_{1-x}^1 x^2\,dy\,dx\text{[by symmetry of the regions and because } f(x,y)=x^2\ge 0]$$

$$= 4\int_0^1 x^3\,dx=4\left[\frac{1}{4}x^4\right]_0^1=1$$

50.

$D=\left\{ (x,y) \mid -1\le x\le 0, -1\le y\le 1+x^2\right\} \cup \left\{ (x,y) \mid 0\le x\le 1, \sqrt{x}\le y\le 1+x^2\right\}$
$\cup \left\{ (x,y) \mid 0\le x\le 1, -1\le y\le -\sqrt{x} \right\}$,

all type I.

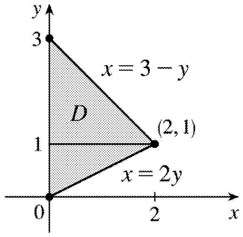
$$\iint_D xy\,dA = \int_{-1}^{0}\int_{-1}^{1+x^2} xy\,dy\,dx+\int_{0}^{1}\int_{\sqrt{x}}^{1+x^2} xy\,dy\,dx+\int_{0}^{1}\int_{-1}^{-\sqrt{x}} xy\,dy\,dx$$
$$= \int_{-1}^{0}\left[\frac{1}{2}xy^2\right]_{y=-1}^{y=1+x^2}dx+\int_0^1\left[\frac{1}{2}xy^2\right]_{y=\sqrt{x}}^{y=1+x^2}dx+\int_0^1\left[\frac{1}{2}xy^2\right]_{y=-1}^{y=-\sqrt{x}}dx$$
$$= \int_{-1}^{0}\left(x^3+\frac{1}{2}x^5\right)dx+\int_0^1\frac{1}{2}(x^5+2x^3-x^2+x)dx+\int_0^1\frac{1}{2}(x^2-x)dx$$
$$= \left[\frac{1}{4}x^4+\frac{1}{12}x^6\right]_{-1}^{0}+\frac{1}{2}\left[\frac{1}{6}x^6+\frac{1}{2}x^4-\frac{1}{3}x^3+\frac{1}{2}x^2\right]_0^1+\frac{1}{2}\left[\frac{1}{3}x^3-\frac{1}{2}x^2\right]_0^1$$
$$= -\frac{1}{3}+\frac{5}{12}-\frac{1}{12}=0$$

51. For $D=[0,1]\times[0,1]$, $0\le\sqrt{x^3+y^3}\le\sqrt{2}$ and $A(D)=1$, so $0\le\iint_D\sqrt{x^3+y^3}\,dA\le\sqrt{2}$.

52. Since $D=\left\{ (x,y) \mid x^2+y^2\le \frac{1}{4}\right\}$, $1=e^0\le e^{x^2+y^2}\le e^{1/4}$ and $A(D)=\frac{\pi}{4}$, so $\frac{\pi}{4}\le\iint_D e^{x^2+y^2}dA\le(e^{1/4})\frac{\pi}{4}$
.

53. Since $m\le f(x,y)\le M$, $\iint_D m\,dA\le\iint_D f(x,y)\,dA\le\iint_D M\,dA$ by (8) $\Rightarrow$
$m\iint_D 1\,dA\le\iint_D f(x,y)\,dA\le M\iint_D 1\,dA$ by (7) $\Rightarrow mA(D)\le\iint_D f(x,y)\,dA\le MA(D)$ by (10).

54.



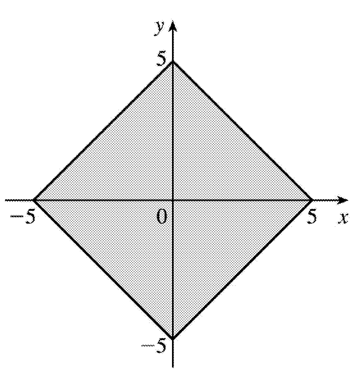
$$\iint_D f(x,y)\,dA = \int_0^1 \int_0^{2y} f(x,y)\,dx\,dy + \int_1^3 \int_0^{3-y} f(x,y)\,dx\,dy$$
$$= \int_0^2 \int_{x/2}^{3-x} f(x,y)\,dy\,dx$$

55. $\iint_D (x^2 \tan x + y^3 + 4)\,dA = \iint_D x^2 \tan x\,dA + \iint_D y^3\,dA + \iint_D 4\,dA$. But $x^2 \tan x$ is an odd function of x and D is symmetric with respect to the y –axis, so $\iint_D x^2 \tan x\,dA = 0$. Similarly, y^3 is an odd function of y and D is symmetric with respect to the x –axis, so $\iint_D y^3\,dA = 0$. Thus

$$\iint_D (x^2 \tan x + y^3 + 4)\,dA = 4\iint_D dA = 4(\text{area of } D) = 4\cdot\pi\left(\sqrt{2}\right)^2 = 8\pi$$

56. First, 0in0.19in] The region D , shown in the figure, is symmetric with respect to the y –axis and $3x$ is an odd function of x , so $\iint_D 3x\,dA = 0$. Similarly, $4y$ is an odd function of y and D is symmetric with respect to the x –axis, so $\iint_D 4y\,dA = 0$. Then

$$\iint_D (2 - 3x + 4y)\,dA = \iint_D 2\,dA = 2\iint_D dA$$
$$= 2(\text{area of } D) = 2(50)$$
$$= 100$$

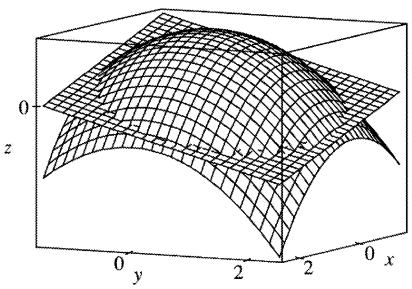


57. Since $\sqrt{1 - x^2 - y^2} \geq 0$, we can interpret $\iint_D \sqrt{1 - x^2 - y^2}\,dA$ as the volume of the solid that lies below

the graph of $z=\sqrt{1-x^2-y^2}$ and above the region D in the xy –plane. $z=\sqrt{1-x^2-y^2}$ is equivalent to $x^2+y^2+z^2=1$, $z\geq 0$ which meets the xy –plane in the circle $x^2+y^2=1$, the boundary of D . Thus, the solid is an upper hemisphere of radius 1 which has volume $\frac{1}{2}\left[\frac{4}{3}\pi(1)^3\right]=\frac{2}{3}\pi$.

58. To find the equations of the boundary curves, we require that the z –values of the two surfaces be the same. In Maple, we use the command solve(4–x^2–y^2=1–x–y,y);
and in Mathematica, we use Solve[4–x^2–y^ 2==1–x–y,y] . We find that the curves have equations

$$y=\frac{1\pm\sqrt{13+4x-4x^2}}{2} .$$



To find the two points of intersection of these curves, we use the CAS to solve $13+4x-4x^2=0$, finding that $x=\frac{1\pm\sqrt{14}}{2}$. So, using the CAS to evaluate the integral, the volume of intersection is

$$V=\int_{(1-\sqrt{14})/2}^{(1+\sqrt{14})/2}\int_{\left(1-\sqrt{13+4x-4x^2}\right)/2}^{\left(1+\sqrt{13+4x-4x^2}\right)/2}[(4-x^2-y^2)-(1-x-y)]\,dy\,dx=\frac{49\pi}{8} .$$

1. The region R is more easily described by polar coordinates: $R=\{(r,\theta)\,|\,0\le r\le 2, 0\le\theta\le 2\pi\}$. Thus $\iint_R f(x,y)\,dA=\int_0^{2\pi}\int_0^2 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

2. The region R is more easily described by rectangular coordinates: $R=\{(x,y)\,|\,0\le x\le 2, 0\le y\le 2-x\}$. Thus $\iint_R f(x,y)\,dA=\int_0^2\int_0^{2-x} f(x,y)\,dy\,dx$.

3. The region R is more easily described by rectangular coordinates: $R=\{(x,y)\,|\,-2\le x\le 2, x\le y\le 2\}$. Thus $\iint_R f(x,y)\,dA=\int_{-2}^2\int_x^2 f(x,y)\,dy\,dx$.

4. The region R is more easily described by polar coordinates: $R=\left\{(r,\theta)\,|\,1\le r\le 3, 0\le\theta\le\frac{\pi}{2}\right\}$. Thus $\iint_R f(x,y)\,dA=\int_0^{\pi/2}\int_1^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

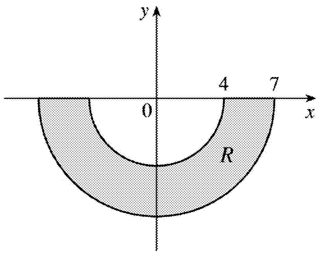
5. The region R is more easily described by polar coordinates: $R=\{(r,\theta)\,|\,2\le r\le 5, 0\le\theta\le 2\pi\}$. Thus $\iint_R f(x,y)\,dA=\int_0^{2\pi}\int_2^5 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

6. The region R is more easily described by polar coordinates: $R=\left\{(r,\theta)\,|\,0\le r\le 2\sqrt{2}, \frac{\pi}{4}\le\theta\le\frac{5\pi}{4}\right\}$. Thus $\iint_R f(x,y)\,dA=\int_{\pi/4}^{5\pi/4}\int_0^{2\sqrt{2}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

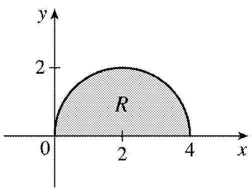
7. The integral $\int_\pi^{2\pi}\int_4^7 r\,dr\,d\theta$ represents the area of the region $R=\{(r,\theta)\,|\,4\le r\le 7, \pi\le\theta\le 2\pi\}$, the lower half of a ring.

$$\int_\pi^{2\pi}\int_4^7 r\,dr\,d\theta = \left(\int_\pi^{2\pi} d\theta\right)\left(\int_4^7 r\,dr\right)$$

$$= [\theta]_\pi^{2\pi}\left[\frac{1}{2}r^2\right]_4^7=\pi\cdot\frac{1}{2}(49-16)=\frac{33\pi}{2}$$



8. The integral $\int_0^{\pi/2}\int_0^{4\cos\theta} r\,dr\,d\theta$ represents the area of the region $R=\{(\mathrm{r},\theta)|0\le r\le 4\cos\theta, 0\le\theta\le\pi/2\}$. Since $r=4\cos\theta \Leftrightarrow r^2=4r\cos\theta \Leftrightarrow x^2+y^2=4x \Leftrightarrow (x-2)^2+y^2=4$, R is the portion in the first quadrant of a circle of radius 2 with center $(2,0)$.



$$\int_0^{\pi/2}\int_0^{4\cos\theta} r\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{1}{2}r^2\right]_{r=0}^{r=4\cos\theta} d\theta = \int_0^{\pi/2} 8\cos^2\theta\,d\theta$$

$$= \int_0^{\pi/2} 4(1+\cos 2\theta)d\theta = 4\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2\pi$$

9. The disk D can be described in polar coordinates as $D=\{(\mathrm{r},\theta)|0\le r\le 3, 0\le\theta\le 2\pi\}$. Then

$$\iint_D xy\,dA = \int_0^{2\pi}\int_0^3 (r\cos\theta)(r\sin\theta)r\,dr\,d\theta = \left(\int_0^{2\pi}\sin\theta\cos\theta\,d\theta\right)\left(\int_0^3 r^3\,dr\right)$$

$$=\left[\frac{1}{2}\sin^2\theta\right]_0^{2\pi}\left[\frac{1}{4}r^4\right]_0^3 = 0$$

10.

$$\iint_R (x+y)dA = \int_{\pi/2}^{3\pi/2}\int_1^2 (r\cos\theta+r\sin\theta)r\,dr\,d\theta = \int_{\pi/2}^{3\pi/2}\int_1^2 r^2(\cos\theta+\sin\theta\,dr\,d\theta$$

$$=\left(\int_{\pi/2}^{3\pi/2}(\cos\theta+\sin\theta\,d\theta\right)\left(\int_1^2 r^2\,dr\right) = [\sin\theta-\cos\theta]_{\pi/2}^{3\pi/2}\left[\frac{1}{3}r^3\right]_1^2$$

$$=(-1-0-1+0)\left(\frac{8}{3}-\frac{1}{3}\right) = -\frac{14}{3}$$

11.

$$\iint_R \cos(x^2+y^2)\,dA = \int_0^{\pi}\int_0^{3} \cos(r^2)\,r\,dr\,d\theta = \left(\int_0^{\pi} d\theta\right)\left(\int_0^{3} r\cos(r^2)\,dr\right)$$

$$=[\theta]_0^{\pi}\left[\frac{1}{2}\sin(r^2)\right]_0^3 = \pi\cdot\frac{1}{2}(\sin 9 - \sin 0) = \frac{\pi}{2}\sin 9$$

12.

$$\iint_R \sqrt{4-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{2}\sqrt{4-r^2}\,r\,dr\,d\theta = \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^{2} r\sqrt{4-r^2}\,dr\right)$$

$$=[\theta]_{-\pi/2}^{\pi/2}\left[-\frac{1}{2}\cdot\frac{2}{3}(4-r^2)^{3/2}\right]_0^2 = \left(\frac{\pi}{2}+\frac{\pi}{2}\right)\left(-\frac{1}{3}(0-4^{3/2})\right) = \frac{8}{3}\pi$$

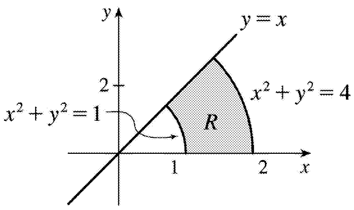
13.

$$\iint_D e^{-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{2} e^{-r^2}\,r\,dr\,d\theta = \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^{2} re^{-r^2}\,dr\right)$$

$$=[\theta]_{-\pi/2}^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^2 = \pi\left(-\frac{1}{2}\right)(e^{-4}-e^{0}) = \frac{\pi}{2}(1-e^{-4})$$

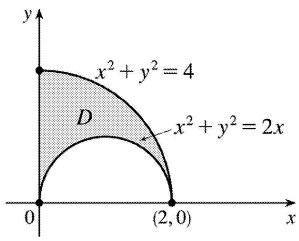
14. $\iint_R ye^{x}\,dA = \int_0^{\pi/2}\int_0^{5}(r\sin\theta)e^{r\cos\theta}\,r\,dr\,d\theta = \int_0^{5}\int_0^{\pi/2} r^2\sin\theta\, e^{r\cos\theta}\,d\theta\,dr$. First we integrate $\int_0^{\pi/2} r^2\sin\theta\, e^{r\cos\theta}\,d\theta$: Let $u=r\cos\theta \Rightarrow du=-r\sin\theta\,d\theta$, and $\int_0^{\pi/2} r^2\sin\theta\, e^{r\cos\theta}\,d\theta = \int_{u=r}^{u=0} -r\,e^{u}\,du = -r[e^{0}-e^{r}] = re^{r}-r$. Then $\int_0^{5}\int_0^{\pi/2} r^2\sin\theta\, e^{r\cos\theta}\,d\theta\,dr = \int_0^{5}(re^{r}-r)\,dr = \left[re^{r}-e^{r}-\frac{1}{2}r^2\right]_0^5 = 4e^{5}-\frac{23}{2}$, where we integrated by parts in the first term.

15. R is the region shown in the figure, and can be described by $R=\{(r,\theta)\,|\,0\le\theta\le\pi/4, 1\le r\le 2\}$.
Thus $\iint_R \arctan(y/x)\,dA = \int_0^{\pi/4}\int_1^{2}\arctan(\tan\theta)\,r\,dr\,d\theta$
since $y/x=\tan\theta$. Also, $\arctan(\tan\theta)=\theta$ for $0\le\theta\le\pi/4$,
so the integral becomes



$$\int_0^{\pi/4}\int_1^2 \theta\, r\, dr\, d\theta = \int_0^{\pi/4} \theta\, d\theta \int_1^2 r\, dr = \left[\frac{1}{2}\theta^2\right]_0^{\pi/4}\left[\frac{1}{2}r^2\right]_1^2 = \frac{\pi^2}{32}\cdot\frac{3}{2} = \frac{3}{64}\pi^2 .$$

16.



$$\iint_D x\, dA = \iint_{\substack{x^2+y^2\leq 4\\ x\geq 0, y\geq 0}} x\, dA - \iint_{\substack{(x-1)^2+y^2\leq 1\\ y\geq 0}} x\, dA$$

$$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\, dr\, d\theta - \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{3}(8\cos\theta)\, d\theta - \int_0^{\pi/2}\frac{1}{3}(8\cos^4\theta)\, d\theta$$

$$= \frac{8}{3} - \frac{8}{12}\left[\cos^3\theta\sin\theta + \frac{3}{2}(\theta + \sin\theta\cos\theta)\right]_0^{\pi/2}$$

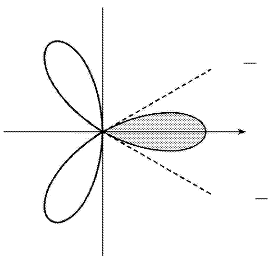
$$= \frac{8}{3} - \frac{2}{3}\left[0 + \frac{3}{2}\left(\frac{\pi}{2}\right)\right] = \frac{16-3\pi}{6}$$

17. One loop is given by the region
$D = \{(r,\theta) \mid -\pi/6 \leq \theta \leq \pi/6, 0 \leq r \square 3\theta\}$, so the area is

$$\iint_D dA = \int_{-\pi/6}^{\pi/6}\int_0^{\cos 3\theta} r\, dr\, d\theta = \int_{-\pi/6}^{\pi/6}\left[\frac{1}{2}r^2\right]_{r=0}^{r=\cos 3\theta} d\theta$$

$$= \int_{-\pi/6}^{\pi/6}\frac{1}{2}\cos^2 3\theta\, d\theta = 2\int_0^{\pi/6}\frac{1}{2}\left(\frac{1+\cos 6\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = \frac{\pi}{12}$$



18. $D = \{(r,\theta)\,|\,0 \le \theta \le 2\pi, 0 \le r \le 4+3\cos\theta\}$, so

$$A(D) = \iint_D dA = \int_0^{2\pi}\int_0^{4+3\cos\theta} r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{2}r^2\right]_{r=0}^{r=4+3\cos\theta} d\theta = \frac{1}{2}\int_0^{2\pi}(4+3\cos\theta)^2 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}(16+24\cos\theta+9\cos^2\theta)d\theta = \frac{1}{2}\int_0^{2\pi}\left(16+24\cos\theta+9\cdot\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\left[16\theta+24\sin\theta+\frac{9}{2}\theta+\frac{9}{4}\sin 2\theta\right]_0^{2\pi} = \frac{41}{2}\pi$$

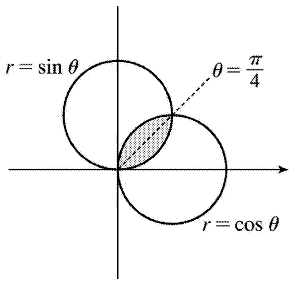
19. By symmetry,

$$A = 2\int_0^{\pi/4}\int_0^{\sin\theta} r\,dr\,d\theta = 2\int_0^{\pi/4}\left[\frac{1}{2}r^2\right]_{r=0}^{r=\sin\theta} d\theta$$

$$= \int_0^{\pi/4}\sin^2\theta\,d\theta = \int_0^{\pi/4}\frac{1}{2}(1-\cos 2\theta)d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2} - 0 + \frac{1}{2}\sin 0\right] = \frac{1}{8}(\pi-2)$$



20. $2=4\sin\theta$ implies that $\theta=\frac{\pi}{6}$ or $\frac{5\pi}{6}$, so

$$A = \int_{\pi/6}^{5\pi/6}\int_{2}^{4\sin\theta} r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6}\left[\frac{1}{2}r^2\right]_{r=2}^{r=4\sin\theta} d\theta = \int_{\pi/6}^{5\pi/6}(8\sin^2\theta-2)\,d\theta$$

$$= \int_{\pi/6}^{5\pi/6}[4(1-\cos 2\theta)-2]\,d\theta = [2\theta-2\sin 2\theta]_{\pi/6}^{5\pi/6} = \frac{4\pi}{3}+2\sqrt{3}\ .$$

21. $V=\iint_{x^2+y^2\le 9}(x^2+y^2)\,dA=\int_0^{2\pi}\int_0^3(r^2)\,r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_0^3 r^3\,dr=[\theta]_0^{2\pi}\left[\frac{1}{4}r^4\right]_0^3=2\pi\left(\frac{81}{4}\right)=\frac{81\pi}{2}$

22. The sphere $x^2+y^2+z^2=16$ intersects the xy –plane in the circle $x^2+y^2=16$, so

$$V = 2\iint_{4\le x^2+y^2\le 16}\sqrt{16-x^2-y^2}\,dA \text{ [by symmetry]}$$

$$= 2\int_0^{2\pi}\int_2^4\sqrt{16-r^2}\,r\,dr\,d\theta = 2\int_0^{2\pi}d\theta\int_2^4 r(16-r^2)^{1/2}\,dr$$

$$= 2[\theta]_0^{2\pi}\left[-\frac{1}{3}(16-r^2)^{3/2}\right]_2^4 = -\frac{2}{3}(2\pi)(0-12^{3/2}) = \frac{4\pi}{3}\left(12\sqrt{12}\right) = 32\sqrt{3}\,\pi$$

23. By symmetry,

$$V = 2\iint_{x^2+y^2\le a^2}\sqrt{a^2-x^2-y^2}\,dA = 2\int_0^{2\pi}\int_0^a\sqrt{a^2-r^2}\,r\,dr\,d\theta = 2\int_0^{2\pi}d\theta\int_0^a r\sqrt{a^2-r^2}\,dr$$

$$= 2[\theta]_0^{2\pi}\left[-\frac{1}{3}(a^2-r^2)^{3/2}\right]_0^a = 2(2\pi)\left(0+\frac{1}{3}a^3\right) = \frac{4\pi}{3}a^3$$

24. The paraboloid $z=10-3x^2-3y^2$ intersects the plane $z=4$ when $4=10-3x^2-3y^2$ or $x^2+y^2=2$. So

$$V = \iint_{x^2+y^2\le 2} \left[(10-3x^2-3y^2)-4\right]dA = \int_0^{2\pi}\int_0^{\sqrt{2}} (6-3r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} (6r-3r^3)\,dr = [\theta]_0^{2\pi}\left[3r^2-\frac{3}{4}r^4\right]_0^{\sqrt{2}} = 6\pi$$

25. The cone $z=\sqrt{x^2+y^2}$ intersects the sphere $x^2+y^2+z^2=1$ when $x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=1$ or $x^2+y^2=\frac{1}{2}$. So

$$V = \iint_{x^2+y^2\le 1/2} \left(\sqrt{1-x^2-y^2}-\sqrt{x^2+y^2}\right)dA = \int_0^{2\pi}\int_0^{1/\sqrt{2}} \left(\sqrt{1-r^2}-r\right)r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{1/\sqrt{2}} \left(r\sqrt{1-r^2}-r^2\right)dr = [\theta]_0^{2\pi}\left[-\frac{1}{3}(1-r^2)^{3/2}-\frac{1}{3}r^3\right]_0^{1/\sqrt{2}}$$

$$= 2\pi\left(-\frac{1}{3}\right)\left(\frac{1}{\sqrt{2}}-1\right) = \frac{\pi}{3}\left(2-\sqrt{2}\right)$$

26. The two paraboloids intersect when $3x^2+3y^2=4-x^2-y^2$ or $x^2+y^2=1$. So

$$V = \iint_{x^2+y^2\le 1} [(4-x^2-y^2)-3(x^2+y^2)]\,dA = \int_0^{2\pi}\int_0^{1} 4(1-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{1} (4r-4r^3)\,dr = [\theta]_0^{2\pi}\left[2r^2-r^4\right]_0^{1} = 2\pi$$

27. The given solid is the region inside the cylinder $x^2+y^2=4$ between the surfaces $z=\sqrt{64-4x^2-4y^2}$ and $z=-\sqrt{64-4x^2-4y^2}$. So

$$V = \iint_{x^2+y^2\le 4} \left[\sqrt{64-4x^2-4y^2}-\left(-\sqrt{64-4x^2-4y^2}\right)\right]dA$$

$$= \iint_{x^2+y^2\le 4} 2\sqrt{64-4x^2-4y^2}\,dA = 4\int_0^{2\pi}\int_0^{2}\sqrt{16-r^2}\,r\,dr\,d\theta$$

$$= 4\int_0^{2\pi} d\theta \int_0^2 r\sqrt{16-r^2}\,dr = 4[\,\theta\,]_0^{2\pi}\left[-\frac{1}{3}(16-r^2)^{3/2}\right]_0^2$$

$$= 8\pi\left(-\frac{1}{3}\right)(12^{3/2}-16^{2/3}) = \frac{8\pi}{3}\left(64-24\sqrt{3}\right)$$

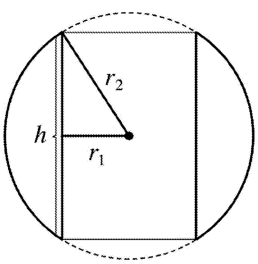
28. **(a)** Here the region in the xy –plane is the annular region $r_1^2 \le x^2+y^2 \le r_2^2$ and the desired volume is twice that above the xy –plane. Hence

$$V = 2\iint_{r_1^2\le x^2+y^2\le r_2^2}\sqrt{r_2^2-x^2-y^2}\,dA = 2\int_0^{2\pi}\int_{r_1}^{r_2}\sqrt{r_2^2-r^2}\,r\,dr\,d\theta$$

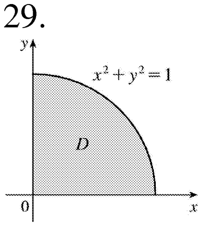
$$= 2\int_0^{2\pi} d\theta\int_{r_1}^{r_2}\sqrt{r_2^2-r^2}\,r\,dr = \frac{4\pi}{3}\left[-(r_2^2-r^2)^{3/2}\right]_{r_1}^{r_2} = \frac{4\pi}{3}(r_2^2-r_1^2)^{3/2}$$

(b) A cross–sectional cut is shown in the figure. So $r_2^2 = \left(\frac{1}{2}h\right)^2 + r_1^2$ or $\frac{1}{4}h^2 = r_2^2 - r_1^2$.

Thus the volume in terms of h is $V = \frac{4\pi}{3}\left(\frac{1}{4}h^2\right)^{3/2} = \frac{\pi}{6}h^3$.



29.

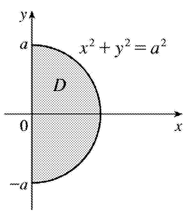


$$\int_0^1\int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx = \int_0^{\pi/2}\int_0^1 e^{r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^1 r e^{r^2} \, dr$$

$$= [\theta]_0^{\pi/2} \left[\frac{1}{2} e^{r^2} \right]_0^1 = \frac{1}{4} \pi (e-1)$$

30.

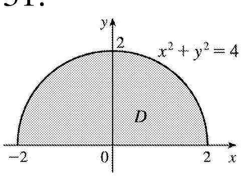


$$\int_{-\pi/2}^{\pi/2} \int_0^a (r^2)^{3/2} r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} d\theta \int_0^a r^4 \, dr$$

$$= [\theta]_{-\pi/2}^{\pi/2} \left[\frac{1}{5} r^5 \right]_0^a$$

$$= \frac{1}{5} \pi a^5$$

31.



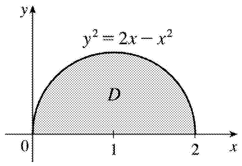
$$\int_0^{\pi} \int_0^2 (r\cos\theta)^2 (r\sin\theta)^2 r \, dr \, d\theta = \int_0^{\pi} (\sin\theta\cos\theta)^2 d\theta \int_0^2 r^5 \, dr$$

$$= \int_0^{\pi} \left(\frac{1}{2} \sin 2\theta \right)^2 d\theta \int_0^2 r^5 \, dr$$

$$= \frac{1}{4} \left[\frac{1}{2}\theta - \frac{1}{8} \sin 4\theta \right]_0^{\pi} \left[\frac{1}{6} r^6 \right]_0^2$$

$$= \frac{1}{4} \left(\frac{\pi}{2} \right) \left(\frac{64}{6} \right) = \frac{4\pi}{3}$$

32.



$$\int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{1}{3}r^3\right]_{r=0}^{r=2\cos\theta} d\theta$$
$$= \int_0^{\pi/2}\left(\frac{8}{3}\cos^3\theta\right)d\theta$$
$$= \frac{8}{3}\left[\sin\theta - \frac{1}{3}\sin^3\theta\right]_0^{\pi/2} = \frac{16}{9}$$

33. The surface of the water in the pool is a circular disk D with radius 20 ft. If we place D on coordinate axes with the origin at the center of D and define $f(x,y)$ to be the depth of the water at (x,y) , then the volume of water in the pool is the volume of the solid that lies above $D=\{(x,y)\mid x^2+y^2\le 400\}$ and below the graph of $f(x,y)$. We can associate north with the positive y –direction, so we are given that the depth is constant in the x –direction and the depth increases linearly in the y –direction from $f(0,-20)=2$ to $f(0,20)=7$. The trace in the yz –plane is a line segment from $(0,-20,2)$ to $(0,20,7)$. The slope of this line is $\frac{7-2}{20-(-20)} = \frac{1}{8}$, so an equation of the line is $z-7=\frac{1}{8}(y-20)\Rightarrow z=\frac{1}{8}y+\frac{9}{2}$. Since $f(x,y)$ is independent of x , $f(x,y)=\frac{1}{8}y+\frac{9}{2}$. Thus the volume is given by $\iint_D f(x,y)\,dA$, which is most conveniently evaluated using polar coordinates. Then $D=\{(r,\theta)\mid 0\le r\le 20, 0\le\theta\le 2\pi\}$ and substituting $x=r\cos\theta$, $y=r\sin\theta$ the integral becomes

$$\int_0^{2\pi}\int_0^{20}\left(\frac{1}{8}r\sin\theta+\frac{9}{2}\right)r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{24}r^3\sin\theta+\frac{9}{4}r^2\right]_{r=0}^{r=20}d\theta = \int_0^{2\pi}\left(\frac{1000}{3}\sin\theta+900\right)d\theta$$
$$= \left[-\frac{1000}{3}\cos\theta+900\theta\right]_0^{2\pi} = 1800\pi$$

Thus the pool contains $1800\pi\approx 5655$ ft^3 of water.

34. **(a)** The total amount of water supplied each hour to the region within R feet of the sprinkler is

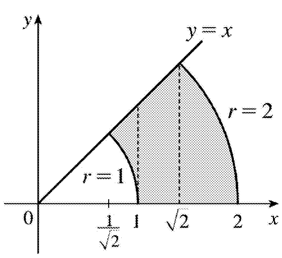
$$V = \int_0^{2\pi}\int_0^R e^{-r}r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^R re^{-r}\,dr = [\theta]_0^{2\pi}\left[-re^{-r}-e^{-r}\right]_0^R$$
$$= 2\pi[-Re^{-R}-e^{-R}+0+1] = 2\pi(1-Re^{-R}-e^{-R})\text{ft}^3$$

(b) The average amount of water per hour per square foot supplied to the region within R feet of the sprinkler is $\dfrac{V}{\text{areaofregion}} = \dfrac{V}{\pi R^2} = \dfrac{2\left(1-Re^{-R}-e^{-R}\right)}{R^2}$ ft^3 (per hour per square foot). See the definition of the average value of a function on page 1022 [ET 986].

35. $\displaystyle\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx+\int_{1}^{\sqrt{2}}\int_{0}^{x} xy\,dy\,dx+\int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xy\,dy\,dx$

$$=\int_0^{\pi/4}\int_1^2 r^3\cos\theta\sin\theta\,dr\,d\theta=\int_0^{\pi/4}\left[\frac{r^4}{4}\cos\theta\sin\theta\right]_{r=1}^{r=2}d\theta$$

$$=\frac{15}{4}\int_0^{\pi/4}\sin\theta\cos\theta\,d\theta=\frac{15}{4}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/4}=\frac{15}{16}$$



36. **(a)** $\displaystyle\iint_{D_a} e^{-(x^2+y^2)}\,dA=\int_0^{2\pi}\int_0^{a} re^{-r^2}\,dr\,d\theta=2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^a=\pi\left(1-e^{-a^2}\right)$ for each a . Then $\displaystyle\lim_{a\to\infty}\pi\left(1-e^{-a^2}\right)=\pi$ since $e^{-a^2}\to 0$ as $a\to\infty$. Hence $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dA=\pi$.

(b) $\displaystyle\iint_{S_a} e^{-(x^2+y^2)}\,dA=\int_{-a}^{a}\int_{-a}^{a} e^{-x^2}e^{-y^2}\,dx\,dy=\left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right)$ for each a .

Then, from (a), $\displaystyle\pi=\iint_{R^2} -(x^2+y^2)\,dA$, so

$$\pi=\lim_{a\to\infty}\iint_{S_a} e^{-(x^2+y^2)}\,dA=\lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right)=\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right).$$

To evaluate $\displaystyle\lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right)$, we are using the fact that these integrals are bounded.

This is true since on $[-1,1]$, $0<e^{-x^2}\le 1$ while on $(-\infty,-1)$, $0<e^{-x^2}\le e^{x}$ and on $(1,\infty)$, $0<e^{-x^2}<e^{-x}$.
Hence $0\le \int_{-\infty}^{\infty} e^{-x^2}dx\le \int_{-\infty}^{-1} e^{x}dx+\int_{-1}^{1} dx+\int_{1}^{\infty} e^{-x}dx=2(e^{-1}+1)$.

(c) Since $\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right)=\pi$ and y can be replaced by x , $\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2=\pi$ implies that $\int_{-\infty}^{\infty} e^{-x^2}dx=\pm\sqrt{\pi}$. But $e^{-x^2}\ge 0$ for all x , so $\int_{-\infty}^{\infty} e^{-x^2}dx=\sqrt{\pi}$.

(d) Letting $t=\sqrt{2}\,x$, $\int_{-\infty}^{\infty} e^{-x^2}dx=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2}}\left(e^{-t^2/2}\right)dt$, so that $\sqrt{\pi}=\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-t^2/2}dt$ or $\int_{-\infty}^{\infty} e^{-t^2/2}dt=\sqrt{2\pi}$.

37. **(a)** We integrate by parts with $u=x$ and $dv=xe^{-x^2}dx$. Then $du=dx$ and $v=-\frac{1}{2}e^{-x^2}$, so

$$\begin{aligned}\int_0^{\infty} x^2e^{-x^2}dx &= \lim_{t\to\infty}\int_0^t x^2e^{-x^2}dx=\lim_{t\to\infty}\left(-\frac{1}{2}xe^{-x^2}\Big]_0^t+\int_0^t\frac{1}{2}e^{-x^2}dx\right)\\ &= \lim_{t\to\infty}\left(-\frac{1}{2}te^{-t^2}\right)+\frac{1}{2}\int_0^{\infty}e^{-x^2}dx=0+\frac{1}{2}\int_0^{\infty}e^{-x^2}dx \text{ [by l'Hospital's Rule]}\\ &= \frac{1}{4}\int_{-\infty}^{\infty}e^{-x^2}dx \text{ [since } e^{-x^2} \text{ is an even function]}\\ &= \frac{1}{4}\sqrt{\pi} \text{ [by Exercise 36(c)]}\end{aligned}$$

(b) Let $u=\sqrt{x}$. Then $u^2=x\Rightarrow dx=2u\,du\Rightarrow$

$$\begin{aligned}\int_0^{\infty}\sqrt{x}\,e^{-x}dx &= \lim_{t\to\infty}\int_0^t\sqrt{x}\,e^{-x}dx=\lim_{t\to\infty}\int_0^{\sqrt{t}}ue^{-u^2}2u\,du=2\int_0^{\infty}u^2e^{-u^2}du\\ &= 2\left(\frac{1}{4}\sqrt{\pi}\right) \text{ [by part(a)]}=\frac{1}{2}\sqrt{\pi}\end{aligned}$$

1.

$$Q=\iint_D \sigma(x,y)\,dA=\int_1^3\int_0^2 (2xy+y^2)\,dy\,dx=\int_1^3\left[xy^2+\frac{1}{3}y^3\right]_{y=0}^{y=2}dx$$
$$=\int_1^3\left(4x+\frac{8}{3}\right)dx=\left[2x^2+\frac{8}{3}x\right]_1^3=16+\frac{16}{3}=\frac{64}{3}\text{ C}$$

2.

$$Q=\iint_D \sigma(x,y)\,dA=\iint_D (x+y+x^2+y^2)\,dA=\int_0^{2\pi}\int_0^2 (r\cos\theta+r\sin\theta+r^2)\,r\,dr\,d\theta$$
$$=\int_0^{2\pi}\int_0^2 dr\,d\theta=\int_0^{2\pi}\left[\frac{1}{3}r^3(\cos\theta+\sin\theta)+\frac{1}{4}r^4\right]_{r=0}^{r=2}d\theta$$
$$=\int_0^{2\pi}\left[\frac{8}{3}(\cos\theta+\sin\theta)+4\right]d\theta=\left[\frac{8}{3}(\sin\theta-\cos\theta)+4\theta\right]_0^{2\pi}=8\pi\text{C}$$

3. $m=\iint_D \rho(x,y)\,dA=\int_0^2\int_{-1}^1 xy^2\,dy\,dx=\int_0^2 x\,dx\int_{-1}^1 y^2\,dy=\left[\frac{1}{2}x^2\right]_0^2\left[\frac{1}{3}y^3\right]_{-1}^1=2\cdot\frac{2}{3}=\frac{4}{3}$,

$\bar{x}=\frac{1}{m}\iint_D x\rho(x,y)\,dA=\frac{3}{4}\int_0^2\int_{-1}^1 x^2y^2\,dy\,dx=\frac{3}{4}\int_0^2 x^2\,dx\int_{-1}^1 y^2\,dy=\frac{3}{4}\left[\frac{1}{3}x^3\right]_0^2\left[\frac{1}{3}y^3\right]_{-1}^1=\frac{3}{4}\cdot\frac{8}{3}\cdot\frac{2}{3}=\frac{4}{3}$,

$\bar{y}=\frac{1}{m}\iint_D y\rho(x,y)\,dA=\frac{3}{4}\int_0^2\int_{-1}^1 xy^3\,dy\,dx=\frac{3}{4}\int_0^2 x\,dx\int_{-1}^1 y^3\,dy=\frac{3}{4}\left[\frac{1}{2}x^2\right]_0^2\left[\frac{1}{4}y^4\right]_{-1}^1=\frac{3}{4}\cdot 2\cdot 0=0$.

Hence, $(\bar{x},\bar{y})=\left(\frac{4}{3},0\right)$.

4. $m=\iint_D \rho(x,y)\,dA=\int_0^a\int_0^b cxy\,dy\,dx=c\int_0^a x\,dx\int_0^b y\,dy=c\left[\frac{1}{2}x^2\right]_0^a\left[\frac{1}{2}y^2\right]_0^b=\frac{1}{4}a^2b^2c$,

$M_y=\iint_D x\rho(x,y)\,dA=\int_0^a\int_0^b cx^2y\,dy\,dx=c\int_0^a x^2\,dx\int_0^b y\,dy=c\left[\frac{1}{3}x^3\right]_0^a\left[\frac{1}{2}y^2\right]_0^b=\frac{1}{6}a^3b^2c$, and

$M_x=\iint_D y\rho(x,y)\,dA=\int_0^a\int_0^b cxy^2\,dy\,dx=c\int_0^a x\,dx\int_0^b y^2\,dy=c\left[\frac{1}{2}x^2\right]_0^a\left[\frac{1}{3}y^3\right]_0^b=\frac{1}{6}a^2b^3c$.

Hence, $(\bar{x},\bar{y})=\left(\frac{M_y}{m},\frac{M_x}{m}\right)=\left(\frac{2}{3}a,\frac{2}{3}b\right)$.

5.

$$m=\int_0^2\int_{x/2}^{3-x}(x+y)\,dy\,dx=\int_0^2\left[xy+\frac{1}{2}y^2\right]_{y=x/2}^{y=3-x}dx=\int_0^2\left[x\left(3-\frac{3}{2}x\right)+\frac{1}{2}(3-x)^2-\frac{1}{8}x^2\right]dx$$

$$=\int_0^2\left(-\frac{9}{8}x^2+\frac{9}{2}\right)dx=\left[-\frac{9}{8}\left(\frac{1}{3}x^3\right)+\frac{9}{2}x\right]_0^2=6\ ,$$

$$M_y=\int_0^2\int_{x/2}^{3-x}(x^2+xy)\,dy\,dx=\int_0^2\left[x^2y+\frac{1}{2}xy^2\right]_{y=x/2}^{y=3-x}dx=\int_0^2\left(\frac{9}{2}x-\frac{9}{8}x^3\right)dx=\frac{9}{2}$$, and

$$M_x=\int_0^2\int_{x/2}^{3-y}(xy+y^2)\,dy\,dx=\int_0^2\left[\frac{1}{2}xy^2+\frac{1}{3}y^3\right]_{y=x/2}^{y=3-x}dx=\int_0^2\left(9-\frac{9}{2}x\right)dx=9$$. Hence $m=6$,

$$(\overline{x},\overline{y})=\left(\frac{M_y}{m},\frac{M_x}{m}\right)=\left(\frac{3}{4},\frac{3}{2}\right)\ .$$

6. $$m=\int_0^1\int_y^{4-3y}x\,dx\,dy=\int_0^1\left[\frac{1}{2}(4-3y)^2-\frac{1}{2}y^2\right]dy=\left[-\frac{1}{18}(4-3y)^3-\frac{1}{6}y^3\right]_0^1=\frac{10}{3}\ ,$$

$$M_y=\int_0^1\int_y^{4-3y}x^2\,dx\,dy=\int_0^1\left[\frac{1}{3}(4-3y)^3-\frac{1}{3}y^3\right]dy=\left[-\frac{1}{36}(4-3y)^4-\frac{1}{12}y^4\right]_0^1=7\ ,$$

$$M_x=\int_0^1\int_y^{4-3y}xy\,dx\,dy=\int_0^1\left[\frac{1}{2}y(4-3y)^2-\frac{1}{2}y^3\right]dy=\int_0^1(8y-12y^2+4y^3)\,dy=1\ .$$

Hence $m=\frac{10}{3}$, $(\overline{x},\overline{y})=(2.1,0.3)$.

7. $$m=\int_0^1\int_0^{e^x}y\,dy\,dx=\int_0^1\left[\frac{1}{2}y^2\right]_{y=0}^{y=e^x}dx=\frac{1}{2}\int_0^1 e^{2x}dx=\left.\frac{1}{4}e^{2x}\right]_0^1=\frac{1}{4}(e^2-1)\ ,$$

$$M_y=\int_0^1\int_0^{e^x}xy\,dy\,dx=\frac{1}{2}\int_0^1 xe^{2x}dx=\frac{1}{2}\left[\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}\right]_0^1=\frac{1}{8}(e^2+1)$$, and

$$M_x=\int_0^1\int_0^{e^x}y^2\,dy\,dx=\int_0^1\left[\frac{1}{3}y^3\right]_{y=0}^{y=e^x}dx=\frac{1}{3}\int_0^1 e^{3x}dx=\frac{1}{3}\left[\frac{1}{3}e^{3x}\right]_0^1=\frac{1}{9}(e^3-1)\ .$$

Hence $m=\frac{1}{4}(e^2-1)$, $(\overline{x},\overline{y})=\left(\dfrac{\frac{1}{8}(e^2+1)}{\frac{1}{4}(e^2-1)},\dfrac{\frac{1}{9}(e^3-1)}{\frac{1}{4}(e^2-1)}\right)=\left(\dfrac{e^2+1}{2(e^2-1)},\dfrac{4(e^3-1)}{9(e^2-1)}\right)$.

8.

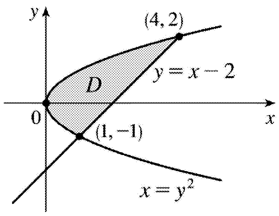
$m=\int_0^1\int_0^{\sqrt{x}} x\,dy\,dx=\int_0^1 x[y]_{y=0}^{y=\sqrt{x}}dx=\int_0^1 x^{3/2}dx=\left.\frac{2}{5}x^{5/2}\right]_0^1=\frac{2}{5}$,

$M_y=\int_0^1\int_0^{\sqrt{x}} x^2\,dy\,dx=\int_0^1 x[y]_{y=0}^{y=\sqrt{x}}dx=\int_0^1 x^{5/2}dx=\left.\frac{2}{7}x^{7/2}\right]_0^1=\frac{2}{7}$, and

$M_x=\int_0^1\int_0^{\sqrt{x}} yx\,dy\,dx=\int_0^1 x\left[\frac{1}{2}y^2\right]_{y=0}^{y=\sqrt{x}}dx=\frac{1}{2}\int_0^1 x^2dx=\frac{1}{2}\left[\frac{1}{3}x^3\right]_0^1=\frac{1}{6}$.

Hence $m=\frac{2}{5}$, $(\overline{x},\overline{y})=\left(\frac{2/7}{2/5},\frac{1/6}{2/5}\right)=\left(\frac{5}{7},\frac{5}{12}\right)$.

9.



$m=\int_{-1}^{2}\int_{y^2}^{y+2} 3\,dx\,dy=\int_{-1}^{2}(3y+6-3y^2)dy=\frac{27}{2}$,

$$M_y=\int_{-1}^{2}\int_{y^2}^{y+2} 3x\,dx\,dy=\int_{-1}^{2}\frac{3}{2}\left[(y+2)^2-y^4\right]dy$$
$$=\left[\frac{1}{2}(y+2)^3-\frac{3}{10}y^5\right]_{-1}^{2}=\frac{108}{5}$$

and

$$M_x=\int_{-1}^{2}\int_{y^2}^{y+2} 3y\,dx\,dy=\int_{-1}^{2}(3y^2+6y-3y^3)dy$$
$$=\left[y^3+3y^2-\frac{3}{4}y^4\right]_{-1}^{2}=\frac{27}{4}$$

Hence $m=\frac{27}{2}$, $(\overline{x},\overline{y})=\left(\frac{8}{5},\frac{1}{2}\right)$.

10. $m=\int_0^{\pi/2}\int_0^{\cos x} x\,dy\,dx=\int_0^{\pi/2} x\cos x\,dx=[x\sin x+\cos x]_0^{\pi/2}=\frac{\pi}{2}-1$,

$M_y=\int_0^{\pi/2}\int_0^{\cos x} x^2\,dy\,dx=\int_0^{\pi/2} x^2\cos x\,dx=\left[x^2\sin x+2x\cos x-2\sin x\right]_0^{\pi/2}=\frac{\pi^2}{4}-2$, and

$M_x = \int_0^{\pi/2}\int_0^{\cos x} xy\,dy\,dx = \int_0^{\pi/2} \frac{1}{2} x\cos^2 x\,dx = \frac{1}{2}\left[\frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x\right]_0^{\pi/2} = \frac{\pi^2}{32} - \frac{1}{8}$.

Hence $m = \frac{\pi - 2}{2}$, $(\bar{x}, \bar{y}) = \left(\frac{\pi^2 - 8}{2(\pi - 2)}, \frac{\pi + 2}{16}\right)$.

11. $\rho(x,y) = ky = kr\sin\theta$, $m = \int_0^{\pi/2}\int_0^1 kr^2 \sin\theta\,dr\,d\theta = \frac{1}{3}k\int_0^{\pi/2} \sin\theta\,d\theta = \frac{1}{3}k[-\cos\theta]_0^{\pi/2} = \frac{1}{3}k$,

$M_y = \int_0^{\pi/2}\int_0^1 kr^3 \sin\theta\cos\theta\,dr\,d\theta = \frac{1}{4}k\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta = \frac{1}{8}k[-\cos 2\theta]_0^{\pi/2} = \frac{1}{8}k$,

$M_x = \int_0^{\pi/2}\int_0^1 kr^3 \sin^2\theta\,dr\,d\theta = \frac{1}{4}k\int_0^{\pi/2} \sin^2\theta\,d\theta = \frac{1}{8}k[\theta + \sin 2\theta]_0^{\pi/2} = \frac{\pi}{16}k$.

Hence $(\bar{x}, \bar{y}) = \left(\frac{3}{8}, \frac{3\pi}{16}\right)$.

12. $\rho(x,y) = k(x^2 + y^2) = kr^2$, $m = \int_0^{\pi/2}\int_0^1 kr^3\,dr\,d\theta = \frac{\pi}{8}k$,

$M_y = \int_0^{\pi/2}\int_0^1 kr^4 \cos\theta\,dr\,d\theta = \frac{1}{5}k\int_0^{\pi/2} \cos\theta\,d\theta = \frac{1}{5}k[\sin\theta]_0^{\pi/2} = \frac{1}{5}k$,

$M_x = \int_0^{\pi/2}\int_0^1 kr^4 \sin\theta\,dr\,d\theta = \frac{1}{5}k\int_0^{\pi/2} \sin\theta\,d\theta = \frac{1}{5}k[-\cos\theta]_0^{\pi/2} = \frac{1}{5}k$.

Hence $(\bar{x}, \bar{y}) = \left(\frac{8}{5\pi}, \frac{8}{5\pi}\right)$.

13. Placing the vertex opposite the hypotenuse at $(0,0)$, $\rho(x,y) = k(x^2 + y^2)$. Then

$$m = \int_0^a\int_0^{a-x} k\left(x^2 + y^2\right)dy\,dx = k\int_0^a \left[ax^2 - x^3 + \frac{1}{3}(a-x)^3\right]dx$$

$$= k\left[\frac{1}{3}ax^3 - \frac{1}{4}x^4 - \frac{1}{12}(a-x)^4\right]_0^a = \frac{1}{6}ka^4$$

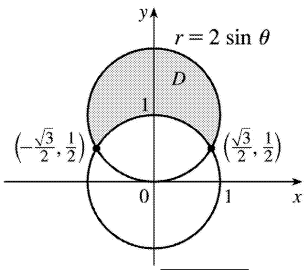
By symmetry,

$$M_y = M_x = \int_0^a\int_0^{a-x} ky(x^2 + y^2)\,dy\,dx = k\int_0^a \left[\frac{1}{2}(a-x)^2 x^2 + \frac{1}{4}(a-x)^4\right]dx$$

$$= k\left[\frac{1}{6}a^2x^3 - \frac{1}{4}ax^4 + \frac{1}{10}x^5 - \frac{1}{20}(a-x)^5\right]_0^a = \frac{1}{15}ka^5$$

Hence $\left(\overline{x},\overline{y}\right)=\left(\frac{2}{5}a,\frac{2}{5}a\right)$.

14.



$\rho(x,y)=k/\sqrt{x^2+y^2}=k/r$,

$$m=\int_{\pi/6}^{5\pi/6}\int_{1}^{2\sin\theta}\frac{k}{r}\,r\,dr\,d\theta=k\int_{\pi/6}^{5\pi/6}[(2\sin\theta)-1]\,d\theta$$

$$=k\left[-2\cos\theta-\theta\right]_{\pi/6}^{5\pi/6}=2k\left(\sqrt{3}-\frac{\pi}{3}\right)$$

By symmetry of D and $f(x)=x$, $M_y=0$, and

$$M_x=\int_{\pi/6}^{5\pi/6}\int_{1}^{2\sin\theta}kr\sin\theta\,dr\,d\theta=\frac{1}{2}k\int_{\pi/6}^{5\pi/6}(4\sin^3\theta-\sin\theta)\,d\theta$$

$$=\frac{1}{2}k\left[-3\cos\theta+\frac{4}{3}\cos^3\theta\right]_{\pi/6}^{5\pi/6}=\sqrt{3}\,k$$

Hence $\left(\overline{x},\overline{y}\right)=\left(0,\frac{3\sqrt{3}}{2\left(3\sqrt{3}-\pi\right)}\right)$.

15.

$$I_x=\iint_D y^2\rho(x,y)\,dA=\int_0^1\int_0^{e^x}y^2\cdot y\,dy\,dx=\int_0^1\left[\frac{1}{4}y^4\right]_{y=0}^{y=e^x}dx=\frac{1}{4}\int_0^1 e^{4x}\,dx$$

$$=\frac{1}{4}\left[\frac{1}{4}e^{4x}\right]_0^1=\frac{1}{16}(e^4-1)\ ,$$

$$I_y=\iint_D x^2\rho(x,y)\,dA=\int_0^1\int_0^{e^x}x^2y\,dy\,dx=\int_0^1 x^2\left[\frac{1}{2}y^2\right]_{y=0}^{y=e^x}dx=\frac{1}{2}\int_0^1 x^2e^{2x}\,dx$$

$$=\frac{1}{2}\left[\left(\frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{4}\right)e^{2x}\right]_0^1 \text{ [integrate by parts twice]}$$

$$=\frac{1}{8}(e^2-1),$$

and $I_0=I_x+I_y=\frac{1}{16}(e^4-1)+\frac{1}{8}(e^2-1)=\frac{1}{16}(e^4+2e^2-3)$.

16. $I_x=\int_0^{\pi/2}\int_0^1(r^2\sin^2\theta)(kr^2)r\,dr\,d\theta=\frac{1}{6}k\int_0^{\pi/2}\sin^2\theta\,d\theta=\frac{1}{6}k\left[\frac{1}{4}(2\theta-\sin 2\theta)\right]_0^{\pi/2}=\frac{\pi}{24}k$,

$I_y=\int_0^{\pi/2}\int_0^1(r^2\cos^2\theta)(kr^2)r\,dr\,d\theta=\frac{1}{6}k\int_0^{\pi/6}\cos^2\theta\,d\theta=\frac{1}{6}k\left[\frac{1}{4}(2\theta+\sin 2\theta)\right]_0^{\pi/2}=\frac{\pi}{24}k$, and

$I_0=I_x+I_y=\frac{\pi}{12}k$.

17. $I_x=\int_{-1}^{2}\int_{y^2}^{y+2}3y^2\,dx\,dy=\int_{-1}^{2}(3y^3+6y^2-3y^4)dy=\left[\frac{3}{4}y^4+2y^3-\frac{3}{5}y^5\right]_{-1}^{2}=\frac{189}{20}$,

$I_y=\int_{-1}^{2}\int_{y^2}^{y+2}3x^2\,dx\,dy=\int_{-1}^{2}\left[(y+2)^3-y^6\right]dy=\left[\frac{1}{4}(y+2)^4-\frac{1}{7}y^7\right]_{-1}^{2}=\frac{1269}{28}$, and $I_0=I_x+I_y=\frac{1917}{35}$.

18. If we find the moments of inertia about the x – and y –axes, we can determine in which direction rotation will be more difficult. (See the explanation following Example 4.) The moment of inertia about the x –axis is given by

$$I_x=\iint_D y^2\rho(\text{x,y})\,dA=\int_0^2\int_0^2 y^2(1+0.1x)\,dy\,dx=\int_0^2(1+0.1x)\left[\frac{1}{3}y^3\right]_{y=0}^{y=2}dx$$

$$=\frac{8}{3}\int_0^2(1+0.1x)\,dx=\frac{8}{3}\left[x+0.1\cdot\frac{1}{2}x^2\right]_0^2=\frac{8}{3}(2.2)\approx 5.87$$

Similarly, the moment of inertia about the y –axis is given by

$$I_y=\iint_D x^2\rho(\text{x,y})\,dA=\int_0^2\int_0^2 x^2(1+0.1x)\,dy\,dx=\int_0^2 x^2(1+0.1x)\,[\,y\,]_{y=0}^{y=2}dx$$

$$=2\int_0^2(x^2+0.1x^3)\,dx=2\left[\frac{1}{3}x^3+0.1\cdot\frac{1}{4}x^4\right]_0^2=2\left(\frac{8}{3}+0.4\right)\approx 6.13$$

Since

$I_y > I_x$, more force is required to rotate the fan blade about the y –axis.

19. Using a CAS, we find □ . Then $\bar{x} = \frac{1}{m} \iint_D x\rho(x,y)\,dA = \frac{8}{\pi^2} \int_0^{\pi} \int_0^{\sin x} x^2 y\,dy\,dx = \frac{2\pi}{3} - \frac{1}{\pi}$ and

$\bar{y} = \frac{1}{m} \iint_D y\rho(x,y)\,dA = \frac{8}{\pi^2} \int_0^{\pi} \int_0^{\sin x} xy^2\,dy\,dx = \frac{16}{9\pi}$, so $(\bar{x},\bar{y}) = \left(\frac{2\pi}{3} - \frac{1}{\pi}, \frac{16}{9\pi} \right)$.

The moments of inertia are $I_x = \iint_D y^2 \rho(x,y)\,dA = \int_0^{\pi} \int_0^{\sin x} xy^3\,dy\,dx = \frac{3\pi^2}{64}$,

$I_y = \iint_D x^2 \rho(x,y)\,dA = \int_0^{\pi} \int_0^{\sin x} x^3 y\,dy\,dx = \frac{\pi^2}{16}(\pi^2 - 3)$, and $I_0 = I_x + I_y = \frac{\pi^2}{64}(4\pi^2 - 9)$.

20. Using a CAS, we find $m = \iint_D \sqrt{x^2+y^2}\,dA = \int_0^{2\pi} \int_0^{1+\cos\theta} r^2\,dr\,d\theta = \frac{5}{3}\pi$,

$\bar{x} = \frac{1}{m} \iint_D x\sqrt{x^2+y^2}\,dA = \frac{3}{5\pi} \int_0^{2\pi} \int_0^{1+\cos\theta} r^3 \cos\theta\,dr\,d\theta = \frac{21}{20}$ and

$\bar{y} = \frac{1}{m} \iint_D y\sqrt{x^2+y^2}\,dA = \frac{3}{5\pi} \int_0^{2\pi} \int_0^{1+\cos\theta} r^3 \sin\theta\,dr\,d\theta = 0$, so $(\bar{x},\bar{y}) = \left(\frac{21}{20}, 0 \right)$.

The moments of inertia are $I_x = \iint_D y^2\sqrt{x^2+y^2}\,dA = \int_0^{2\pi} \int_0^{1+\cos\theta} r^4 \sin^2\theta\,dr\,d\theta = \frac{33}{40}\pi$,

$I_y = \iint_D x^2\sqrt{x^2+y^2}\,dA = \int_0^{2\pi} \int_0^{1+\cos\theta} r^4 \cos^2\theta\,dr\,d\theta = \frac{93}{40}\pi$, and $I_0 = I_x + I_y = \frac{63}{20}\pi$.

21. $I_x = \int_0^a \int_0^a \rho y^2\,dx\,dy = \rho \int_0^a dx \int_0^a y^2\,dy = \rho\,[x]_0^a \left[\frac{1}{3} y^3 \right]_0^a = \rho a \left(\frac{1}{3} a^3 \right) = \frac{1}{3} \rho a^4 = I_y$ by symmetry, and $m = \rho a^2$

since the lamina is homogeneous. Hence $\bar{\bar{x}}^2 = \frac{I_y}{m} \Rightarrow \bar{\bar{x}} = \left[\left(\frac{1}{3} \rho a^4 \right) / (\rho a^2) \right]^{1/2} = \frac{1}{\sqrt{3}}\,a$ and

$\bar{\bar{y}}^2 = \frac{I_x}{m} \Rightarrow \bar{\bar{y}} = \frac{1}{\sqrt{3}}\,a$.

22.

$$m=\int_0^{\pi}\int_0^{\sin x}\rho\,dy\,dx=\rho\int_0^{\pi}\sin x\,dx=\rho\left[-\cos x\right]_0^{\pi}=2\rho\ ,$$

$$I_x=\int_0^{\pi}\int_0^{\sin x}\rho y^2\,dy\,dx=\frac{1}{3}\rho\int_0^{\pi}\sin^3 x\,dx=\frac{1}{3}\rho\int_0^{\pi}(1-\cos^2 x)\sin x\,dx$$

$$=\frac{1}{3}\rho\left[-\cos\theta+\frac{1}{3}\cos^3\theta\right]_0^{\pi}=\frac{4}{9}\rho\ ,$$

$$I_y=\int_0^{\pi}\int_0^{\sin x}\rho x^2\,dy\,dx=\rho\int_0^{\pi}x^2\sin x\,dx$$

$$=\rho\left[-x^2\cos x+2x\sin x+2\cos x\right]_0^{\pi}$$

$$=\rho(\pi^2-4)\ .$$

Then $\bar{\bar{y}}^2=\frac{I_x}{m}=\frac{2}{9}$, so $\bar{\bar{y}}=\frac{\sqrt{2}}{3}$ and $\bar{\bar{x}}^2=\frac{I_y}{m}=\frac{\pi^2-4}{2}$, so $\bar{\bar{x}}=\sqrt{\frac{\pi^2-4}{2}}$.

23. **(a)** $f(x,y)$ is a joint density function, so we know $\iint_{R^2} f(x,y)\,dA=1$. Since $f(x,y)=0$ outside the rectangle $[0,1]\times[0,2]$, we can say

$$\iint_{R^2} f(x,y)\,dA=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dy\,dx=\int_0^1\int_0^2 Cx(1+y)\,dy\,dx$$

$$=C\int_0^1 x\left[y+\frac{1}{2}y^2\right]_{y=0}^{y=2}dx=C\int_0^1 4x\,dx=C\left[2x^2\right]_0^1=2C$$

Then $2C=1\Rightarrow C=\frac{1}{2}$.

(b)

$$P(X\leq 1,Y\leq 1)=\int_{-\infty}^{1}\int_{-\infty}^{1}f(x,y)\,dy\,dx=\int_0^1\int_0^1\frac{1}{2}x(1+y)\,dy\,dx$$

$$=\int_0^1\frac{1}{2}x\left[y+\frac{1}{2}y^2\right]_{y=0}^{y=1}dx=\int_0^1\frac{1}{2}x\left(\frac{3}{2}\right)dx=\frac{3}{4}\left[\frac{1}{2}x^2\right]_0^1=\frac{3}{8}\text{ or }0.375$$

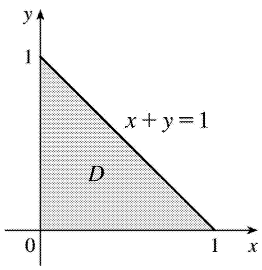
(c) $P(X+Y\leq 1)=P((X,Y)\in D)$ where D is the triangular region shown in the figure. Thus

$$P(X+Y\leq 1)=\iint_D f(x,y)\,dA=\int_0^1\int_0^{1-x}\frac{1}{2}x(1+y)\,dy\,dx$$

$$=\int_0^1\frac{1}{2}x\left[y+\frac{1}{2}y^2\right]_{y=0}^{y=1-x}dx=\int_0^1\frac{1}{2}x\left(\frac{1}{2}x^2-2x+\frac{3}{2}\right)dx$$

$$=\frac{1}{4}\int_0^1\left(x^3-4x^2+3x\right)dx=\frac{1}{4}\left[\frac{x^4}{4}-4\frac{x^3}{3}+3\frac{x^2}{2}\right]_0^1$$

$$=\frac{5}{48}\approx 0.1042$$



24. **(a)** $f(x,y)\geq 0$, so f is a joint density function if $\iint_{R^2} f(x,y)\,dA=1$. Here, $f(x,y)=0$ outside the square $[0,1]\times[0,1]$, so $\iint_{R^2} f(x,y)\,dA=\int_0^1\int_0^1 4xy\,dy\,dx=\int_0^1\left[2xy^2\right]_{y=0}^{y=1}dx=\int_0^1 2x\,dx=\left.x^2\right]_0^1=1$.

Thus, $f(x,y)$ is a joint density function.

(b)

(a) No restriction is placed on Y , so

$$P\left(X\geq\frac{1}{2}\right)=\int_{1/2}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dy\,dx=\int_{1/2}^{1}\int_0^1 4xy\,dy\,dx$$

$$=\int_{1/2}^{1}\left[2xy^2\right]_{y=0}^{y=1}dx=\int_{1/2}^{1}2x\,dx=\left.x^2\right]_{1/2}^{1}=\frac{3}{4}$$

(b) No restriction is placed on Y , so

$$P\left(X\geq\frac{1}{2}\right)=\int_{1/2}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dy\,dx=\int_{1/2}^{1}\int_0^1 4xy\,dy\,dx$$

$$=\int_{1/2}^{1}\left[2xy^2\right]_{y=0}^{y=1}dx=\int_{1/2}^{1}2x\,dx=\left.x^2\right]_{1/2}^{1}=\frac{3}{4}$$

(c)

$$P\left(X\geq \frac{1}{2},Y\leq \frac{1}{2}\right)=\int_{1/2}^{\infty}\int_{-\infty}^{1/2} f(x,y)\,dy\,dx=\int_{1/2}^{1}\int_{0}^{1/2} 4xy\,dy\,dx$$

$$=\int_{1/2}^{1}\left[2xy^2\right]_{y=0}^{y=1/2}dx=\int_{1/2}^{1}\frac{1}{2}x\,dx=\left.\frac{1}{2}\cdot\frac{1}{2}x^2\right]_{1/2}^{1}=\frac{3}{16}$$

(d)

$$P\left(X\geq \frac{1}{2},Y\leq \frac{1}{2}\right)=\int_{1/2}^{\infty}\int_{-\infty}^{1/2} f(x,y)\,dy\,dx=\int_{1/2}^{1}\int_{0}^{1/2} 4xy\,dy\,dx$$

$$=\int_{1/2}^{1}\left[2xy^2\right]_{y=0}^{y=1/2}dx=\int_{1/2}^{1}\frac{1}{2}x\,dx=\left.\frac{1}{2}\cdot\frac{1}{2}x^2\right]_{1/2}^{1}=\frac{3}{16}$$

(c) The expected value of X is given by $\mu_1=\iint_{R^2} x f(x,y)\,dA=\int_0^1\int_0^1 x(4xy)\,dy\,dx=\int_0^1 2x^2\left[y^2\right]_{y=0}^{y=1}dx$

$=2\int_0^1 x^2\,dx=2\left[\frac{1}{3}x^3\right]_0^1=\frac{2}{3}$

The expected value of Y is $\mu_2=\iint_{R^2} y f(x,y)\,dA=\int_0^1\int_0^1 y(4xy)\,dy\,dx=\int_0^1 4x\left[\frac{1}{3}y^3\right]_{y=0}^{y=1}dx$

$=\frac{4}{3}\int_0^1 x\,dx=\frac{4}{3}\left[\frac{1}{2}x^2\right]_0^1=\frac{2}{3}$

25. **(a)** $f(x,y)\geq 0$, so f is a joint density function if $\iint_{R^2} f(x,y)\,dA=1$. Here, $f(x,y)=0$ outside the first quadrant, so

$$\iint_{R^2} f(x,y)\,dA = \int_0^{\infty}\int_0^{\infty} 0.1e^{-(0.5x+0.2y)}\,dy\,dx=0.1\int_0^{\infty}\int_0^{\infty} e^{-0.5x}e^{-0.2y}\,dy\,dx$$

$$= 0.1\int_0^{\infty} e^{-0.5x}dx\int_0^{\infty} e^{-0.2y}dy=0.1\lim_{t\to\infty}\int_0^{t} e^{-0.5x}dx\lim_{t\to\infty}\int_0^{t} e^{-0.2y}dy$$

$$= 0.1\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t\lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t$$

$$= 0.1\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right]\lim_{t\to\infty}\left[-5(e^{-0.2t}-1)\right]$$

$$= (0.1)\cdot(-2)(0-1)\cdot(-5)(0-1)=1$$

Thus $f(x,y)$ is a joint density function.

(b)

(a) No restriction is placed on X , so

$$P(Y \geq 1)=\int_{-\infty}^{\infty}\int_{1}^{\infty} f(x,y)\,dy\,dx=\int_{0}^{\infty}\int_{1}^{\infty} 0.1e^{-(0.5x+0.2y)}\,dy\,dx$$

$$= 0.1\int_{0}^{\infty} e^{-0.5x}dx\int_{1}^{\infty} e^{-0.2y}dy=0.1\lim_{t\to\infty}\int_{0}^{t} e^{-0.5x}dx\lim_{t\to\infty}\int_{1}^{t} e^{-0.2y}dy$$

$$= 0.1\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_{0}^{t}\lim_{t\to\infty}\left[-5e^{-0.2y}\right]_{1}^{t}$$

$$= 0.1\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right]\lim_{t\to\infty}\left[-5(e^{-0.2t}-e^{-0.2})\right]$$

$$= (0.1)\cdot(-2)(0-1)\cdot(-5)(0-e^{-0.2})=e^{-0.2}\approx 0.8187$$

(b) No restriction is placed on X , so

$$P(Y \geq 1)=\int_{-\infty}^{\infty}\int_{1}^{\infty} f(x,y)\,dy\,dx=\int_{0}^{\infty}\int_{1}^{\infty} 0.1e^{-(0.5x+0.2y)}\,dy\,dx$$

$$= 0.1\int_{0}^{\infty} e^{-0.5x}dx\int_{1}^{\infty} e^{-0.2y}dy=0.1\lim_{t\to\infty}\int_{0}^{t} e^{-0.5x}dx\lim_{t\to\infty}\int_{1}^{t} e^{-0.2y}dy$$

$$= 0.1\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_{0}^{t}\lim_{t\to\infty}\left[-5e^{-0.2y}\right]_{1}^{t}$$

$$= 0.1\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right]\lim_{t\to\infty}\left[-5(e^{-0.2t}-e^{-0.2})\right]$$

$$= (0.1)\cdot(-2)(0-1)\cdot(-5)(0-e^{-0.2})=e^{-0.2}\approx 0.8187$$

(c)

$$P(X\leq 2, Y\leq 4) =\int_{-\infty}^{2}\int_{-\infty}^{4} f(x,y)\,dy\,dx=\int_{0}^{2}\int_{0}^{4} 0.1e^{-(0.5x+0.2y)}\,dy\,dx$$

$$=0.1\int_{0}^{2} e^{-0.5x}dx\int_{0}^{4} e^{-0.2y}dy=0.1\left[-2e^{-0.5x}\right]_{0}^{2}\left[-5e^{-0.2y}\right]_{0}^{4}$$

$$=(0.1)\cdot(-2)(e^{-1}-1)\cdot(-5)(e^{-0.8}-1)$$

$$=(e^{-1}-1)(e^{-0.8}-1)=1+e^{-1.8}-e^{-0.8}-e^{-1}\approx 0.3481$$

(d)

$$P(X\le 2, Y\le 4) = \int_{-\infty}^{2}\int_{-\infty}^{4} f(x,y)\,dy\,dx = \int_0^2\int_0^4 0.1e^{-(0.5x+0.2y)}\,dy\,dx$$

$$=0.1\int_0^2 e^{-0.5x}dx\int_0^4 e^{-0.2y}dy = 0.1\left[-2e^{-0.5x}\right]_0^2\left[-5e^{-0.2y}\right]_0^4$$

$$=(0.1)\cdot(-2)(e^{-1}-1)\cdot(-5)(e^{-0.8}-1)$$

$$=(e^{-1}-1)(e^{-0.8}-1)=1+e^{-1.8}-e^{-0.8}-e^{-1}\approx 0.3481$$

(c) The expected value of X is given by

$$\mu_1 = \iint_{R^2} x f(x,y)\,dA = \int_0^{\infty}\int_0^{\infty} x\left[0.1e^{-(0.5x+0.2y)}\right]dy\,dx$$

$$= 0.1\int_0^{\infty} xe^{-0.5x}dx\int_0^{\infty} e^{-0.2y}dy = 0.1\lim_{t\to\infty}\int_0^t xe^{-0.5x}dx\lim_{t\to\infty}\int_0^t e^{-0.2y}dy$$

To evaluate the first integral, we integrate by parts with $u=x$ and $dv=e^{-0.5x}dx$ (or we can use Formula 96
in the Table of Integrals):
$\int xe^{-0.5x}dx=-2xe^{-0.5x}-\int -2e^{-0.5x}dx=-2xe^{-0.5x}-4e^{-0.5x}=-2(x+2)e^{-0.5x}$. Thus

$$\mu_1=0.1\lim_{t\to\infty}\left[-2(x+2)e^{-0.5x}\right]_0^t\lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t$$

$$=0.1\lim_{t\to\infty}(-2)\left[(t+2)e^{-0.5t}-2\right]\lim_{t\to\infty}(-5)\left[e^{-0.2t}-1\right]$$

$$=0.1(-2)\left(\lim_{t\to\infty}\frac{t+2}{e^{0.5t}}-2\right)(-5)(-1)=2 \text{ [by l'Hospital's Rule]}$$

The expected value of Y is given by

$$\mu_2 \&=\& \iint_{R^2} y f(x,y)\,dA=\int_0^{\infty}\int_0^{\infty} y\left[0.1e^{-(0.5+0.2y)}\right]dy\,dx$$

$$=0.1\int_0^{\infty} e^{-0.5x}dx\int_0^{\infty} ye^{-0.2y}dy=0.1\lim_{t\to\infty}\int_0^t e^{-0.5x}dx\lim_{t\to\infty}\int_0^t ye^{-0.2y}dy$$

To evaluate the second integral, we integrate by parts with $u=y$ and $dv=e^{-0.2y}dy$ (or again we can use Formula 96 in the Table of Integrals) which gives $\int ye^{-0.2y}dy=-5ye^{-0.2y}+\int 5e^{-0.2y}dy=-5(y+5)e^{-0.2y}$. Then

$$\mu_2 = 0.1 \lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty}\left[-5(y+5)e^{-0.2y}\right]_0^t$$
$$= 0.1 \lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right] \lim_{t\to\infty}\left(-5\left[(t+5)e^{-0.2t}-5\right]\right)$$
$$= 0.1(-2)(-1)\cdot(-5)\left(\lim_{t\to\infty}\frac{t+5}{e^{0.2t}}-5\right) = 5 \text{ [by l'Hospital's Rule]}$$

26. **(a)** Each lamp has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t<0 \\ \dfrac{1}{1000}e^{-t/1000} & \text{if } t\geq 0 \end{cases}$$

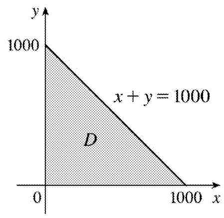
If X and Y are the lifetimes of the individual bulbs, then X and Y are independent, so the joint density function is the product of the individual density functions:

$$f(x,y) = \begin{cases} 10^{-6}e^{-(x+y)/1000} & \text{if } x\geq 0, y\geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that both of the bulbs fail within 1000 hours is

$$P(X\leq 1000, Y\leq 1000) = \int_{-\infty}^{1000}\int_{-\infty}^{1000} f(x,y)\,dy\,dx$$
$$= \int_0^{1000}\int_0^{1000} 10^{-6}e^{-(x+y)/1000}\,dy\,dx$$
$$= 10^{-6}\int_0^{1000} e^{-x/1000}\,dx \int_0^{1000} e^{-y/1000}\,dy$$
$$= 10^{-6}\left[-1000e^{-x/1000}\right]_0^{1000}\left[-1000e^{-y/1000}\right]_0^{1000}$$
$$= \left(e^{-1}-1\right)^2 \approx 0.3996$$

(b) Now we are asked for the probability that the combined lifetimes of both bulbs is 1000 hours or less. Thus we want to find $P(X+Y\leq 1000)$, or equivalently $P\left((X,Y)\in D\right)$ where D is the triangular region shown in the figure.
Then



$$P(X+Y\leq 1000) = \iint_D f(\mathrm{x,y})\,dA = \int_0^{1000}\int_0^{1000-x} 10^{-6} e^{-(x+y)/1000}\,dy\,dx$$

$$= 10^{-6}\int_0^{1000}\left[-1000e^{-(x+y)/1000}\right]_{y=0}^{y=1000-x} dx$$

$$= -10^{-3}\int_0^{1000}\left(e^{-1}-e^{-x/1000}\right)dx$$

$$= -10^{-3}\left[e^{-1}x+1000e^{-x/1000}\right]_0^{1000} = 1-2e^{-1}\approx 0.2642$$

27. **(a)** The random variables X and Y are normally distributed with $\mu_1=45$, $\mu_2=20$, $\sigma_1=0.5$, and $\sigma_2=0.1$.

The individual density functions for X and Y , then, are $f_1(x)=\dfrac{1}{0.5\sqrt{2\pi}}\,e^{-(x-45)^2/0.5}$ and $f_2(y)=\dfrac{1}{0.1\sqrt{2\pi}}\,e^{-(y-20)^2/0.02}$. Since X and Y are independent, the joint density function is the product $f(\mathrm{x,y})$ & = & $f_1(x)f_2(y)=\dfrac{1}{0.5\sqrt{2\pi}}\,e^{-(x-45)^2/0.5}\,\dfrac{1}{0.1\sqrt{2\pi}}\,e^{-(y-20)^2/0.02}$

& = & $\dfrac{10}{\pi}\,e^{-2(x-45)^2-50(y-20)^2}$ Then

$P(40$ & $\leq$ & $X\leq 50, 20\leq Y\leq 25)=\int_{40}^{50}\int_{20}^{25} f(\mathrm{x,y})\,dy\,dx$

& = & $\dfrac{10}{\pi}\int_{40}^{50}\int_{20}^{25} e^{-2(x-45)^2-50(y-20)^2}\,dy\,dx$ Using a CAS or calculator to evaluate the integral, we get $P(40\leq X\leq 50\ ,\ 20\leq Y\leq 25)\approx 0.500$.

(b) $P(4(X-45)^2+100(Y-20)^2\leq 2)=\iint_D \dfrac{10}{\pi}\,e^{-2(x-45)^2-50(y-20)^2}\,dA$, where D is the region enclosed by the ellipse

$4(x-45)^2+100(y-20)^2=2$. Solving for y gives $y=20\pm\frac{1}{10}\sqrt{2-4(x-45)^2}$, the upper and lower halves of the ellipse, and these two halves meet where $y=20\Rightarrow 4(x-45)^2=2\Rightarrow x=45\pm\frac{1}{\sqrt{2}}$. Thus

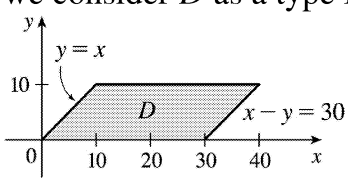
$$\iint_D \frac{10}{\pi}e^{-2(x-45)^2-50(y-20)^2}dA=\frac{10}{\pi}\int_{45-1/\sqrt{2}}^{45+1/\sqrt{2}}\int_{20-\frac{1}{10}\sqrt{2-4(x-45)^2}}^{20+\frac{1}{10}\sqrt{2-4(x-45)^2}} e^{-2(x-45)^2-50(y-20)^2}dy\,dx .$$

Using a CAS or calculator to evaluate the integral, we get $P(4(X-45)^2+100(Y-20)^2\leq 2)\approx 0.632$.

28. Because X and Y are independent, the joint density function for Xavier's and Yolanda's arrival times is the product of the individual density functions:

$$f(x,y)=f_1(x)f_2(y)=\begin{cases}\frac{1}{50}e^{-x}y & \text{if } x\geq 0, 0\leq y\leq 10\\ 0 & \text{otherwise}\end{cases}$$

Since Xavier won't wait for Yolanda, they won't meet unless $X\geq Y$. Additionally, Yolanda will wait up to half an
hour but no longer, so they won't meet unless $X-Y\leq 30$. Thus the probability that they meet is $P((X,Y)\in D)$ where D is the parallelogram shown in the figure. The integral is simpler to evaluate if we consider D as a type II region, so



$$P((X,Y)\in D)=\iint_D f(x,y)\,dx\,dy=\int_0^{10}\int_y^{y+30}\frac{1}{50}e^{-x}y\,dx\,dy=\frac{1}{50}\int_0^{10}y\left[-e^{-x}\right]_{x=y}^{x=y+30}dy$$

$$=\frac{1}{50}\int_0^{10}y(-e^{-(y+30)}+e^{-y})dy=\frac{1}{50}(1-e^{-30})\int_0^{10}ye^{-y}dy$$

By integration by parts (or Formula 96 in the Table of Integrals), this is
$\frac{1}{50}(1-e^{-30})\left[-(y+1)e^{-y}\right]_0^{10}=\frac{1}{50}(1-e^{-30})(1-11e^{-10})\approx 0.020$. Thus there is only about a 2% chance they will meet. Such is student life!

29. **(a)** If $f(P,A)$ is the probability that an individual at A will be infected by an individual at P , and $k\,dA$ is the number of infected individuals in an element of area dA , then $f(P,A)k\,dA$ is the number of

infections that should result from exposure of the individual at A to infected people in the element of area dA . Integration over D gives the number of infections of the person at A due to all the infected people in D . In rectangular coordinates (with the origin at the city's center), the exposure of a person at A is

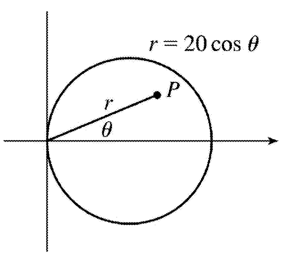
$$E=\iint_D kf(\mathrm{P,A})\,dA=k\iint_D \frac{20-d(\mathrm{P,A})}{20}\,dA=k\iint_D \left[1-\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{20}\right]dx\,dy\,.$$

(b) If $A=(0,0)$, then

$$E = k\iint_D \left[1-\frac{1}{20}\sqrt{x^2+y^2}\right]dx\,dy$$

$$= k\int_0^{2\pi}\int_0^{10}\left(1-\frac{r}{20}\right)r\,dr\,d\theta=2\pi k\left[\frac{r^2}{2}-\frac{r^3}{60}\right]_0^{10}$$

$$= 2\pi k\left(50-\frac{50}{3}\right)=\frac{200}{3}\pi k\approx 209k$$



For A at the edge of the city, it is convenient to use a polar coordinate system centered at A . Then the polar equation for the circular boundary of the city becomes $r=20\cos\theta$ instead of $r=10$, and the distance from A to a point P in the city is again r (see the figure). So

$$E = k\int_{-\pi/2}^{\pi/2}\int_0^{20\cos\theta}\left(1-\frac{r}{20}\right)r\,dr\,d\theta=k\int_{-\pi/2}^{\pi/2}\left[\frac{r^2}{2}-\frac{r^3}{60}\right]_{r=0}^{r=20\cos\theta}d\theta$$

$$= k\int_{-\pi/2}^{\pi/2}\left(200\cos^2\theta-\frac{400}{3}\cos^3\theta\right)d\theta=200k\int_{-\pi/2}^{\pi/2}\left[\frac{1}{2}+\frac{1}{2}\cos 2\theta-\frac{2}{3}\left(1-\sin^2\theta\right)\cos\theta\right]d\theta$$

$$= 200k\left[\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta-\frac{2}{3}\sin\theta+\frac{2}{3}\cdot\frac{1}{3}\sin^3\theta\right]_{-\pi/2}^{\pi/2}=200k\left[\frac{\pi}{4}+0-\frac{2}{3}+\frac{2}{9}+\frac{\pi}{4}+0-\frac{2}{3}+\frac{2}{9}\right]$$

$$= 200k\left(\frac{\pi}{2}-\frac{8}{9}\right)\approx 136k$$

Therefore the risk of infection is much lower at the edge of the city than in the middle, so it is better to live at the edge.

1. Here $z=f(x,y)=2+3x+4y$ and D is the rectangle $[0,5]\times[1,4]$, so by Formula 2 the area of the surface is

$$A(S) = \iint_D \sqrt{[f_x(x,y)]^2+[f_y(x,y)]^2+1}\,dA=\iint_D \sqrt{3^2+4^2+1}\,dA=\sqrt{26}\iint_D dA$$
$$= \sqrt{26}\,A(D)=\sqrt{26}\,(5)(3)=15\sqrt{26}$$

2. $z=f(x,y)=10-2x-5y$ and D is the disk $x^2+y^2\le 9$, so by Formula 2

$$A(S) = \iint_D \sqrt{(-2)^2+(-5)^2+1}\,dA=\sqrt{30}\iint_D dA=\sqrt{30}\,A(D)$$
$$= \sqrt{30}\,(\pi\cdot 3^2)=9\sqrt{30}\,\pi$$

3. $z=f(x,y)=6-3x-2y$ which intersects the xy –plane in the line $3x+2y=6$, so D is the triangular region given by $\left\{ (x,y) \;\middle|\; 0\le x\le 2, 0\le y\le 3-\frac{3}{2}x \right\}$. Thus

$$A(S) = \iint_D \sqrt{(-3)^2+(-2)^2+1}\,dA=\sqrt{14}\iint_D dA=\sqrt{14}\,A(D)$$
$$= \sqrt{14}\left(\frac{1}{2}\cdot 2\cdot 3\right)=3\sqrt{14}$$

4. $z=f(x,y)=1+3x+2y^2$ with $0\le x\le 2y$, $0\le y\le 1$. Thus by Formula 2,

$$A(S) =\iint_D \sqrt{(3)^2+(4y)^2+1}\,dA=\int_0^1\int_0^{2y} \sqrt{10+16y^2}\,dx\,dy=\int_0^1 \sqrt{10+16y^2}\,[x]_{x=0}^{x=2y}\,dy$$
$$=\int_0^1 2y\sqrt{10+16y^2}\,dy= \left. 2\cdot\frac{1}{32}\cdot\frac{2}{3}(10+16y^2)^{3/2}\right]_0^1=\frac{1}{24}(26^{3/2}-10^{3/2})$$

5. $y^2+z^2=9\Rightarrow z=\sqrt{9-y^2}$. $f_x=0$, $f_y=-y(9-y^2)^{-1/2}\Rightarrow$

$$A(S) =\int_0^4\int_0^2 \sqrt{0^2+[-y(9-y^2)^{-1/2}]^2+1}\,dy\,dx=\int_0^4\int_0^2 \sqrt{\frac{y^2}{9-y^2}+1}\,dy\,dx$$
$$=\int_0^4\int_0^2 \frac{3}{\sqrt{9-y^2}}\,dy\,dx=3\int_0^4\left[\sin^{-1}\frac{y}{3}\right]_{y=0}^{y=2}dx=3\left[\left(\sin^{-1}\left(\frac{2}{3}\right)\right)x\right]_0^4=12\sin^{-1}\left(\frac{2}{3}\right)$$

6. $z=f(x,y)=4-x^2-y^2$ and D is the projection of the paraboloid $z=4-x^2-y^2$ onto the xy –plane, that is, $D=\left\{(x,y)|x^2+y^2\le 4\right\}$. So $f_x=-2x$, $f_y=-2y\Rightarrow$

$$A(S)=\iint_D\sqrt{(-2x)^2+(-2y)^2+1}\,dA=\iint_D\sqrt{4(x^2+y^2)+1}\,dA=\int_0^{2\pi}\int_0^2\sqrt{4r^2+1}\,r\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[\frac{1}{12}(4r^2+1)^{3/2}\right]_{r=0}^{r=2}d\theta=\int_0^{2\pi}\frac{1}{12}\left(17\sqrt{17}-1\right)d\theta=\frac{\pi}{6}\left(17\sqrt{17}-1\right)$$

7. $z=f(x,y)=y^2-x^2$ with $1\le x^2+y^2\le 4$. Then

$$A(S)=\iint_D\sqrt{1+4x^2+4y^2}\,dA=\int_0^{2\pi}\int_1^2\sqrt{1+4r^2}\,r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_1^2 r\sqrt{1+4r^2}\,dr$$

$$=\left[\theta\right]_0^{2\pi}\left[\frac{1}{12}(1+4r^2)^{3/2}\right]_1^2=\frac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right)$$

8. $z=f(x,y)=\frac{2}{3}(x^{3/2}+y^{3/2})$ and $D=\{(x,y)|\,0\le x\le 1,0\le y\le 1\}$. Then $f_x=x^{1/2}$, $f_y=y^{1/2}$ and

$$A(S)=\iint_D\sqrt{\left(\sqrt{x}\right)^2+\left(\sqrt{y}\right)^2+1}\,dA=\int_0^1\int_0^1\sqrt{x+y+1}\,dy\,dx$$

$$=\int_0^1\left[\frac{2}{3}(x+y+1)^{3/2}\right]_{y=0}^{y=1}dx=\frac{2}{3}\int_0^1\left[(x+2)^{3/2}-(x+1)^{3/2}\right]dx$$

$$=\frac{2}{3}\left[\frac{2}{5}(x+2)^{5/2}-\frac{2}{5}(x+1)^{5/2}\right]_0^1=\frac{4}{15}(3^{5/2}-2^{5/2}-2^{5/2}+1)$$

$$=\frac{4}{15}(3^{5/2}-2^{7/2}+1)$$

9. $z=f(x,y)=xy$ with $0\le x^2+y^2\le 1$, so $f_x=y$, $f_y=x\Rightarrow$

$$A(S)=\iint_D\sqrt{y^2+x^2+1}\,dA=\int_0^{2\pi}\int_0^1\sqrt{r^2+1}\,r\,dr\,d\theta=\int_0^{2\pi}\left[\frac{1}{3}(r^2+1)^{3/2}\right]_{r=0}^{r=1}d\theta$$

$$=\int_0^{2\pi}\frac{1}{3}\left(2\sqrt{2}-1\right)d\theta=\frac{2\pi}{3}\left(2\sqrt{2}-1\right)$$

10. Given the sphere $x^2+y^2+z^2=4$, when $z=1$, we get $x^2+y^2=3$ so $D=\left\{ (\mathrm{x,y})|x^2+y^2\le 3\right\}$ and $z=f(\mathrm{x,y})=\sqrt{4-x^2-y^2}$. Thus

$$A(S)=\iint_D \sqrt{[(-x)(4-x^2-y^2)^{-1/2}]^2+[(-y)(4-x^2-y^2)^{-1/2}]^2+1}\;dA$$

$$=\int_0^{2\pi}\int_0^{\sqrt{3}}\sqrt{\frac{r^2}{4-r^2}+1}\;r\,dr\,d\theta=\int_0^{2\pi}\int_0^{\sqrt{3}}\frac{2r}{\sqrt{4-r^2}}\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[-2(4-r^2)^{1/2}\right]_{r=0}^{r=\sqrt{3}}d\theta=\int_0^{2\pi}(-2+4)\,d\theta=2\theta\,\Big]_0^{2\pi}=4\pi$$

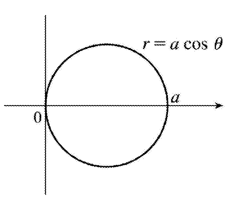
11. $z=\sqrt{a^2-x^2-y^2}$, $z_x=-x(a^2-x^2-y^2)^{-1/2}$, $z_y=-y(a^2-x^2-y^2)^{-1/2}$,

$$A(S)=\iint_D \sqrt{\frac{x^2+y^2}{a^2-x^2-y^2}+1}\;dA$$

$$=\int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta}\sqrt{\frac{r^2}{a^2-r^2}+1}\;r\,dr\,d\theta$$

$$=\int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta}\frac{ar}{\sqrt{a^2-r^2}}\,dr\,d\theta$$

$$=\int_{-\pi/2}^{\pi/2}\left[-a\sqrt{a^2-r^2}\,\right]_{r=0}^{r=a\cos\theta}d\theta$$



$$=\int_{-\pi/2}^{\pi/2}-a\left(\sqrt{a^2-a^2\cos^2\theta}-a\right)d\theta=2a^2\int_0^{\pi/2}\left(1-\sqrt{1-\cos^2\theta}\,\right)d\theta$$

$$=2a^2\int_0^{\pi/2} d\theta - 2a^2\int_0^{\pi/2}\sqrt{\sin^2\theta}\, d\theta = a^2\pi - 2a^2\int_0^{\pi/2}\sin\theta\, d\theta = a^2(\pi-2)$$

12. To find the region D : $z=x^2+y^2$ implies $z+z^2=4z$ or $z^2-3z=0$. Thus $z=0$ or $z=3$ are the planes where the surfaces intersect. But $x^2+y^2+z^2=4z$ implies $x^2+y^2+(z-2)^2=4$, so $z=3$ intersects the upper hemisphere. Thus $(z-2)^2=4-x^2-y^2$ or $z=2+\sqrt{4-x^2-y^2}$. Therefore D is the region inside the circle $x^2+y^2+(3-2)^2=4$, that is, $D=\left\{ (x,y) \mid x^2+y^2\le 3\right\}$.

$$A(S) = \iint_D \sqrt{1+[(-x)(4-x^2-y^2)^{-1/2}]^2+[(-y)(4-x^2-y^2)^{-1/2}]^2}\, dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}}\sqrt{1+\frac{r^2}{4-r^2}}\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}}\frac{2r\,dr}{\sqrt{4-r^2}}\, d\theta = \int_0^{2\pi}\Big[-2(4-r^2)^{1/2}\Big]_{r=0}^{r=\sqrt{3}} d\theta$$

$$= \int_0^{2\pi}(-2+4)\,d\theta = 2\theta\Big]_0^{2\pi} = 4\pi$$

13. $z=f(x,y)=e^{-x^2-y^2}$, $f_x=-2xe^{-x^2-y^2}$, $f_y=-2ye^{-x^2-y^2}$. Then

$$A(S)= \iint_{x^2+y^2\le 4}\sqrt{(-2xe^{-x^2-y^2})^2+(-2ye^{-x^2-y^2})^2+1}\, dA= \iint_{x^2+y^2\le 4}\sqrt{4(x^2+y^2)e^{-2(x^2+y^2)}+1}\, dA .$$

Converting to polar coordinates we have

$$A(S) = \int_0^{2\pi}\int_0^{2}\sqrt{4r^2e^{-2r^2}+1}\, r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^{2} r\sqrt{4r^2e^{-2r^2}+1}\, dr$$

$$=2\pi\int_0^{2} r\sqrt{4r^2e^{-2r^2}+1}\, dr \approx 13.9783 \text{ using a calculator.}$$

14. $z=f(x,y)=\cos(x^2+y^2)$, $f_x=-2x\sin(x^2+y^2)$, $f_y=-2y\sin(x^2+y^2)$.

$$A(S) = \iint_{x^2+y^2\le 1}\sqrt{4x^2\sin^2(x^2+y^2)+4y^2\sin^2(x^2+y^2)+1}\, dA$$

$$= \iint_{x^2+y^2\le 1}\sqrt{4(x^2+y^2)\sin^2(x^2+y^2)+1}\, dA$$

Converting to polar coordinates gives

$$A(S)=\int_0^{2\pi}\int_0^1\sqrt{4r^2\sin^2(r^2)+1}\; r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_0^1 r\sqrt{4r^2\sin^2(r^2)+1}\;dr$$

$$=2\pi\int_0^1 r\sqrt{4r^2\sin^2(r^2)+1}\;dr\approx 4.1073 \text{ using a calculator.}$$

15. **(a)** The midpoints of the four squares are $\left(\frac{1}{4},\frac{1}{4}\right)$, $\left(\frac{1}{4},\frac{3}{4}\right)$, $\left(\frac{3}{4},\frac{1}{4}\right)$, and $\left(\frac{3}{4},\frac{3}{4}\right)$.
Here $f(x,y)=x^2+y^2$, so the Midpoint Rule gives

$$A(S)=\iint_D\sqrt{[f_x(x,y)]^2+[f_y(x,y)]^2+1}\;dA=\iint_D\sqrt{(2x)^2+(2y)^2+1}\;dA$$

$$\approx\frac{1}{4}\left(\sqrt{\left[2\left(\frac{1}{4}\right)\right]^2+\left[2\left(\frac{1}{4}\right)\right]^2+1}+\sqrt{\left[2\left(\frac{1}{4}\right)\right]^2+\left[2\left(\frac{3}{4}\right)\right]^2+1}\right.$$

$$\left.+\sqrt{\left[2\left(\frac{3}{4}\right)\right]^2+\left[2\left(\frac{1}{4}\right)\right]^2+1}+\sqrt{\left[2\left(\frac{3}{4}\right)\right]^2+\left[2\left(\frac{3}{4}\right)\right]^2+1}\right)$$

$$=\frac{1}{4}\left(\sqrt{\frac{3}{2}}+2\sqrt{\frac{7}{2}}+\sqrt{\frac{11}{2}}\right)\approx 1.8279$$

(b) A CAS estimates the integral to be $A(S)=\iint_D\sqrt{1+(2x)^2+(2y)^2}\;dA=\int_0^1\int_0^1\sqrt{1+4x^2+4y^2}\;dy\,dx\approx 1.8616$.
This agrees with the Midpoint estimate only in the first decimal place.

16. **(a)** With $m=n=2$ we have four squares with midpoints $\left(\frac{1}{2},\frac{1}{2}\right)$, $\left(\frac{1}{2},\frac{3}{2}\right)$, $\left(\frac{3}{2},\frac{1}{2}\right)$, and $\left(\frac{3}{2},\frac{3}{2}\right)$. Since $z=xy+x^2+y^2$, the Midpoint Rule gives

$$A(S)=\iint_D\sqrt{[f_x(x,y)]^2+[f_y(x,y)]^2+1}\;dA=\iint_D\sqrt{(y+2x)^2+(x+2y)^2+1}\;dA$$

$$\approx 1\left(\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^2+1}+\sqrt{\left(\frac{5}{2}\right)^2+\left(\frac{7}{2}\right)^2+1}+\sqrt{\left(\frac{7}{2}\right)^2+\left(\frac{5}{2}\right)^2+1}+\sqrt{\left(\frac{9}{2}\right)^2+\left(\frac{9}{2}\right.}\right.$$

$$=\frac{\sqrt{22}}{2}+\frac{\sqrt{78}}{2}+\frac{\sqrt{78}}{2}+\frac{\sqrt{166}}{2}\approx 17.619$$

(b) Using a CAS, we have

$A(S)=\iint_D \sqrt{(y+2x)^2+(x+2y)^2+1}\, dA=\int_0^2\int_0^2 \sqrt{1+(y+2x)^2+(x+2y)^2}\, dy dx \approx 17.7165$. This is within about 0.1 of the Midpoint Rule estimate.

17. $z=1+2x+3y+4y^2$, so

$$A(S)=\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA=\int_1^4\int_0^1 \sqrt{1+4+(3+8y)^2}\, dy dx$$

$$=\int_1^4\int_0^1 \sqrt{14+48y+64y^2}\, dy dx .$$

Using a CAS, we have

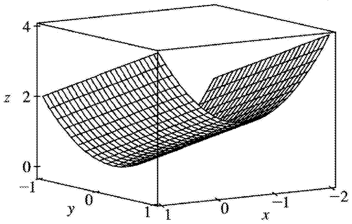
$$\int_1^4\int_0^1 \sqrt{14+48y+64y^2}\, dy dx=\frac{45}{8}\sqrt{14}+\frac{15}{16}\ln\left(11\sqrt{5}+3\sqrt{14}\sqrt{5}\right)-\frac{15}{16}\ln\left(3\sqrt{5}+\sqrt{14}\sqrt{5}\right)$$

or $\frac{45}{8}\sqrt{14}+\frac{15}{16}\ln\frac{11\sqrt{5}+3\sqrt{70}}{3\sqrt{5}+\sqrt{70}}$.

18. $f(x,y)=1+x+y+x^2 \Rightarrow f_x=1+2x$, $f_y=1$. We use a CAS to calculate the integral

$A(S)=\int_{-2}^{1}\int_{-1}^{1}\sqrt{f_x^2+f_y^2+1}\, dy dx=\int_{-2}^{1}\int_{-1}^{1}\sqrt{(1+2x)^2+2}\, dy dx=2\int_{-2}^{1}\sqrt{4x^2+4x+3}\, dx$ and find that

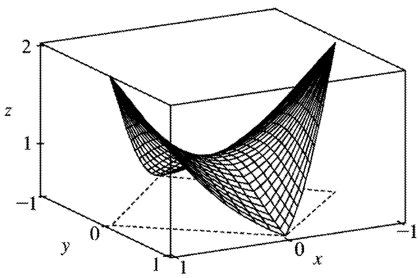
$A(S)=3\sqrt{11}+2\sinh^{-1}\left(\frac{3\sqrt{2}}{2}\right)$ or $A(S)=3\sqrt{11}+\ln\left(10+3\sqrt{11}\right)$.



19. $f(x,y)=1+x^2y^2 \Rightarrow f_x=2xy^2$, $f_y=2x^2y$. We use a CAS (with precision reduced to five significant digits, to speed up the calculation) to estimate the integral

$A(S)=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{f_x^2+f_y^2+1}\, dy dx=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{4x^2y^4+4x^4y^2+1}\, dy dx$, and find that $A(S)\approx 3.3213$.

20. Let $f(x,y)=\frac{1+x^2}{1+y^2}$. Then $f_x=\frac{2x}{1+y^2}$, $f_y=\left(1+x^2\right)\left[-\frac{2y}{\left(1+y^2\right)^2}\right]=-\frac{2y\left(1+x^2\right)}{\left(1+y^2\right)^2}$. We use a CAS to estimate $\int_{-1}^{1}\int_{-(1-|x|)}^{1-|x|}\sqrt{f_x^2+f_y^2+1}\,dy\,dx\approx 2.6959$. In order to graph only the part of the surface above the square, we use $-(1-|x|)\le y\le 1-|x|$ as the y –range in our plot command.



21. Here $z=f(x,y)=ax+by+c$, $f_x(x,y)=a$, $f_y(x,y)=b$, so
$A(S)=\iint_D\sqrt{a^2+b^2+1}\,dA=\sqrt{a^2+b^2+1}\iint_D dA=\sqrt{a^2+b^2+1}\,A(D)$.

22. Let S be the upper hemisphere. Then $z=f(x,y)=\sqrt{a^2-x^2-y^2}$, so

$$A(S)=\iint_D\sqrt{[-x(a^2-x^2-y^2)^{-1/2}]^2+[-y(a^2-x^2-y^2)^{-1/2}]^2+1}\,dA$$

$$=\iint_D\sqrt{\frac{x^2+y^2}{a^2-x^2-y^2}+1}\,dA=\int_0^{2\pi}\int_0^t\sqrt{\frac{r^2}{a^2-r^2}+1}\,r\,dr\,d\theta$$

$$=\lim_{t\to a^-}\int_0^{2\pi}\int_0^t\frac{ar}{\sqrt{a^2-r^2}}\,dr\,d\theta=2\pi\lim_{t\to a^-}\left[-a\sqrt{a^2-r^2}\right]_0^t=2\pi\lim_{t\to a^-}-a\left[\sqrt{a^2-t^2}-a\right]$$

$=2\pi(-a)(-a)=2\pi a^2$. Thus the surface area of the entire sphere is $4\pi a^2$.

23. If we project the surface onto the xz –plane, then the surface lies ‘‘above’’ the disk $x^2+z^2\le 25$ in the xz –plane.

We have $y=f(x,z)=x^2+z^2$ and, adapting Formula 2, the area of the surface is

$$A(S)=\iint_{x^2+z^2\le 25}\sqrt{[f_x(x,z)]^2+[f_z(x,z)]^2+1}\,dA=\iint_{x^2+z^2\le 25}\sqrt{4x^2+4z^2+1}\,dA$$

Converting to polar coordinates $x = r\cos\theta$, $z = r\sin\theta$ we have

$$A(S) = \int_0^{2\pi}\int_0^5 \sqrt{4r^2+1}\, r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^5 r(4r^2+1)^{1/2} dr = [\theta]_0^{2\pi}\left[\frac{1}{12}(4r^2+1)^{3/2}\right]_0^5$$

$$= \frac{\pi}{6}\left(101\sqrt{101}-1\right)$$

24. First we find the area of the face of the surface that intersects the positive y –axis. As in Exercise 23 , we can project the face onto the xz –plane, so the surface lies ‘‘above’’ the disk $x^2+z^2 \le 1$. Then $z = f(\text{x,z}) = \sqrt{1-z^2}$ and the area is

$$A(S) = \iint_{x^2+z^2\le 1} \sqrt{[f_x(\text{x,z})]^2+[f_z(\text{x,z})]^2+1}\, dA = \iint_{x^2+z^2\le 1} \sqrt{0+\left(\frac{-z}{\sqrt{1-z^2}}\right)^2+1}\, dA$$

$$= \iint_{x^2+z^2\le 1} \sqrt{\frac{z^2}{1-z^2}+1}\, dA = \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}}\, dx\,dz$$

$$= 4\int_0^1\int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}}\, dx\,dz \text{ [by the symmetry of the surface]}$$

This integral is improper (when $z=1$), so

$$A(S) = \lim_{t\to 1^-} 4\int_0^t\int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}}\, dx\,dz = \lim_{t\to 1^-} 4\int_0^t \frac{\sqrt{1-z^2}}{\sqrt{1-z^2}}\, dz = \lim_{t\to 1^-} 4\int_0^t dz = \lim_{t\to 1^-} 4t = 4 .$$

Since the complete surface consists of four congruent faces, the total surface area is $4(4)=16$.

1.

$$\iiint_B xyz^2\,dV = \int_0^1\int_{-1}^2\int_0^3 xyz^2\,dz\,dx\,dy = \int_0^1\int_{-1}^2 xy\left[\frac{1}{3}z^3\right]_{z=0}^{z=3}dx\,dy = \int_0^1\int_{-1}^2 9xy\,dx\,dy$$

$$= \int_0^1\left[\frac{9}{2}x^2y\right]_{x=-1}^{x=2}dy = \int_0^1\frac{27}{2}\,y\,dy = \left.\frac{27}{4}y^2\right]_0^1 = \frac{27}{4}$$

2. There are six different possible orders of integration.

$$\iiint_E (xz-y^3)\,dV = \int_{-1}^1\int_0^2\int_0^1 (xz-y^3)\,dz\,dy\,dx = \int_{-1}^1\int_0^2\left[\frac{1}{2}xz^2-y^3z\right]_{z=0}^{z=1}dy\,dx$$

$$= \int_{-1}^1\int_0^2\left(\frac{1}{2}x-y^3\right)dy\,dx = \int_{-1}^1\left[\frac{1}{2}xy-\frac{1}{4}y^4\right]_{y=0}^{y=2}dx$$

$$= \int_{-1}^1 (x-4)\,dx = \left[\frac{1}{2}x^2-4x\right]_{-1}^1 = -8$$

$$\iiint_E (xz-y^3)\,dV = \int_0^2\int_{-1}^1\int_0^1 (xz-y^3)\,dz\,dx\,dy = \int_0^2\int_{-1}^1\left[\frac{1}{2}xz^2-y^3z\right]_{z=0}^{z=1}dx\,dy$$

$$= \int_0^2\int_{-1}^1\left(\frac{1}{2}x-y^3\right)dx\,dy = \int_0^2\left[\frac{1}{4}x^2-xy^3\right]_{x=-1}^{x=1}dy$$

$$= \int_0^2 -2y^3\,dy = \left.-\frac{1}{2}y^4\right]_0^2 = -8$$

$$\iiint_E (xz-y^3)\,dV = \int_{-1}^1\int_0^1\int_0^2 (xz-y^3)\,dy\,dz\,dx = \int_{-1}^1\int_0^1\left[xyz-\frac{1}{4}y^4\right]_{y=0}^{y=2}dz\,dx$$

$$= \int_{-1}^1\int_0^1 (2xz-4)\,dz\,dx = \int_{-1}^1\left[xz^2-4z\right]_{z=0}^{z=1}dx$$

$$= \int_{-1}^1 (x-4)\,dx = \left[\frac{1}{2}x^2-4x\right]_{-1}^1 = -8$$

$$\iiint_E (xz-y^3)\,dV = \int_0^1\int_{-1}^1\int_0^2 (xz-y^3)\,dy\,dx\,dz = \int_0^1\int_{-1}^1\left[xyz-\frac{1}{4}y^4\right]_{y=0}^{y=2}dx\,dz$$

$$=\int_0^1\int_{-1}^1 (2xz-4)\,dx\,dz=\int_0^1\left[x^2z-4x\right]_{x=-1}^{x=1}dz$$

$$=\int_0^1 -8\,dz=\left.-8z\right]_0^1=-8$$

$$\iiint_E (xz-y^3)\,dV=\int_0^2\int_0^1\int_{-1}^1 (xz-y^3)\,dx\,dz\,dy=\int_0^2\int_0^1\left[\frac{1}{2}x^2z-xy^3\right]_{x=-1}^{x=1}dz\,dy$$

$$=\int_0^2\int_0^1 -2y^3\,dz\,dy=\int_0^2\left[-2y^3z\right]_{z=0}^{z=1}dy=\int_0^2 -2y^3\,dy=\left.-\frac{1}{2}y^4\right]_0^2=-8$$

$$\iiint_E \left(xz-y^3\right)dV=\int_0^1\int_0^2\int_{-1}^1 (xz-y^3)\,dx\,dy\,dz=\int_0^1\int_0^2\left[\frac{1}{2}x^2z-xy^3\right]_{x=-1}^{x=1}dy\,dz$$

$$=\int_0^1\int_0^2 -2y^3\,dy\,dz=\int_0^1\left[-\frac{1}{2}y^4\right]_{y=0}^{y=2}dz=\int_0^1 -8\,dz=\left.-8z\right]_0^1=-8$$

3.

$$\int_0^1\int_0^z\int_0^{x+z} 6xz\,dy\,dx\,dz=\int_0^1\int_0^z\left[6xyz\right]_{y=0}^{y=x+z}dx\,dz=\int_0^1\int_0^z 6xz(x+z)\,dx\,dz$$

$$=\int_0^1\left[2x^3z+3x^2z^2\right]_{x=0}^{x=z}dz=\int_0^1(2z^4+3z^4)\,dz=\int_0^1 5z^4\,dz=\left.z^5\right]_0^1=1$$

4.

$$\int_0^1\int_x^{2x}\int_0^y 2xyz\,dz\,dy\,dx=\int_0^1\int_x^{2x}\left[xyz^2\right]_{z=0}^{z=y}dy\,dx=\int_0^1\int_x^{2x} xy^3\,dy\,dx$$

$$=\int_0^1\left[\frac{1}{4}xy^4\right]_{y=x}^{y=2x}dx=\int_0^1\frac{15}{4}x^5\,dx=\left.\frac{5}{8}x^6\right]_0^1=\frac{5}{8}$$

5.

$$\int_0^3\int_0^1\int_0^{\sqrt{1-z^2}} ze^y\,dx\,dz\,dy = \int_0^3\int_0^1 \Big[xze^y\Big]_{x=0}^{x=\sqrt{1-z^2}} dz\,dy = \int_0^3\int_0^1 ze^y\sqrt{1-z^2}\,dz\,dy$$

$$= \int_0^3 \left[-\frac{1}{3}(1-z^2)^{3/2}e^y\right]_{z=0}^{z=1} dy = \int_0^3 \frac{1}{3}e^y\,dy = \left.\frac{1}{3}e^y\right]_0^3 = \frac{1}{3}(e^3-1)$$

6.

$$\int_0^1\int_0^z\int_0^y ze^{-y^2}\,dx\,dy\,dz = \int_0^1\int_0^z \Big[xze^{-y^2}\Big]_{x=0}^{x=y} dy\,dz = \int_0^1\int_0^z yze^{-y^2}\,dy\,dz = \int_0^1 \left[-\frac{1}{2}ze^{-y^2}\right]_{y=0}^{y=z} dz$$

$$= \int_0^1 -\frac{1}{2}z\left(e^{-z^2}-1\right)dz = \frac{1}{2}\int_0^1 \left(z-ze^{-z^2}\right)dz$$

$$= \frac{1}{2}\left[\frac{1}{2}z^2+\frac{1}{2}e^{-z^2}\right]_0^1 = \frac{1}{4}(1+e^{-1}-0-1) = \frac{1}{4e}$$

7.

$$\iiint_E 2x\,dV = \int_0^2\int_0^{\sqrt{4-y^2}}\int_0^y 2x\,dz\,dx\,dy = \int_0^2\int_0^{\sqrt{4-y^2}} [2xz]_{z=0}^{z=y}\,dx\,dy = \int_0^2\int_0^{\sqrt{4-y^2}} 2xy\,dx\,dy$$

$$= \int_0^2 \Big[x^2y\Big]_{x=0}^{x=\sqrt{4-y^2}} dy = \int_0^2 (4-y^2)y\,dy = \left[2y^2-\frac{1}{4}y^4\right]_0^2 = 4$$

8.

$$\iiint_E yz\cos(x^5)\,dV = \int_0^1\int_0^x\int_x^{2x} yz\cos(x^5)\,dz\,dy\,dx = \int_0^1\int_0^x \left[\frac{1}{2}yz^2\cos(x^5)\right]_{z=x}^{z=2x} dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^x 3x^2y\cos(x^5)\,dy\,dx = \frac{1}{2}\int_0^1 \left[\frac{3}{2}x^2y^2\cos(x^5)\right]_{y=0}^{y=x} dx$$

$$= \frac{3}{4}\int_0^1 x^4\cos(x^5)\,dx = \frac{3}{4}\left[\frac{1}{5}\sin(x^5)\right]_0^1 = \frac{3}{20}(\sin 1-\sin 0) = \frac{3}{20}\sin 1$$

9. Here $E=\left\{(x,y,z)\,|\,0\le x\le 1, 0\le y\le \sqrt{x}, 0\le z\le 1+x+y\right\}$, so

$$\iiint_E 6xy\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{x}} \left[6xyz\right]_{z=0}^{z=1+x+y} dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{x}} 6xy(1+x+y)\,dy\,dx = \int_0^1 \left[3xy^2+3x^2y^2+2xy^3\right]_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^1 (3x^2+3x^3+2x^{5/2})\,dx = \left[x^3+\frac{3}{4}x^4+\frac{4}{7}x^{7/2}\right]_0^1 = \frac{65}{28}$$

10. Here E is the region in the first octant that lies below the plane $2x+2y+z=4$ (and above the region in the xy –plane bounded by the lines $x=0$, $y=0$, $x+y=2$). So

$$\iiint_E y\,dV = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} y\,dz\,dy\,dx = \int_0^2 \int_0^{2-x} y(4-2x-2y)\,dy\,dx$$

$$= \int_0^2 \int_0^{2-x} (4y-2xy-2y^2)\,dy\,dx = \int_0^2 \left[2y^2-xy^2-\frac{2}{3}y^3\right]_{y=0}^{y=2-x} dx$$

$$= \int_0^2 \left[2(2-x)^2-x(2-x)^2-\frac{2}{3}(2-x)^3\right] dx$$

$$= \int_0^2 \left[(2-x)(2-x)^2-\frac{2}{3}(2-x)^3\right] dx = \frac{1}{3}\int_0^2 (2-x)^3\,dx$$

$$= \frac{1}{3}\left[-\frac{1}{4}(2-x)^4\right]_0^2 = -\frac{1}{12}(0-16) = \frac{4}{3}$$

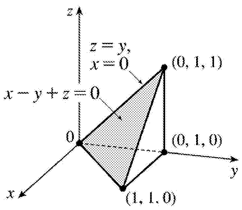
11. Here E is the region that lies below the plane with x –, y –, and z –intercepts 1 , 2 , and 3 respectively, that is, below the plane $2z+6x+3y=6$ and above the region in the xy –plane bounded by the lines $x=0$, $y=0$ and $6x+3y=6$. So

$$\iiint_E xy\,dV = \int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} xy\,dz\,dy\,dx = \int_0^1 \int_0^{2-2x} \left(3xy-3x^2y-\frac{3}{2}xy^2\right) dy\,dx$$

$$= \int_0^1 \left[\frac{3}{2}xy^2-\frac{3}{2}x^2y^2-\frac{1}{2}xy^3\right]_{y=0}^{y=2-2x} dx = \int_0^1 (2x-6x^2+6x^3-2x^4)\,dx$$

$$= \left[x^2-2x^3+\frac{3}{2}x^4-\frac{2}{5}x^5\right]_0^1 = \frac{1}{10}\ .$$

12.

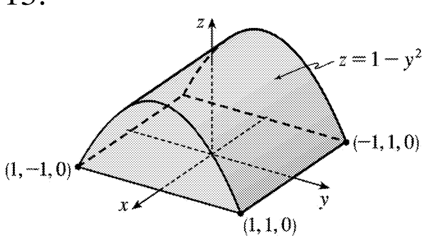


$$\int_0^1\int_0^y\int_0^{y-z} xz\,dx\,dz\,dy = \int_0^1\int_0^y \frac{1}{2}(y-z)^2 z\,dz\,dy$$

$$= \frac{1}{2}\int_0^1 \left[\frac{1}{2}y^2z^2 - \frac{2}{3}yz^3 + \frac{1}{4}z^4\right]_{z=0}^{z=y} dy$$

$$= \frac{1}{24}\int_0^1 y^4\,dy = \frac{1}{24}\left[\frac{1}{5}y^5\right]_0^1 = \frac{1}{120}$$

13.



E is the region below the parabolic cylinder $z=1-y^2$ and above the square $[-1,1]\times[-1,1]$ in the xy – plane.

$$\iiint_E x^2e^y\,dV = \int_{-1}^1\int_{-1}^1\int_0^{1-y^2} x^2e^y\,dz\,dy\,dx$$

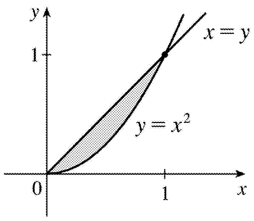
$$= \int_{-1}^1\int_{-1}^1 x^2e^y(1-y^2)\,dy\,dx$$

$$= \int_{-1}^1 x^2\,dx \int_{-1}^1 (e^y - y^2e^y)\,dy$$

$$= \left[\frac{1}{3}x^3\right]_{-1}^1 \left[e^y - (y^2-2y+2)e^y\right]_{-1}^1$$

$$= \frac{1}{3}(2)[e-e-e^{-1}+5e^{-1}] = \frac{8}{3e}$$

14.



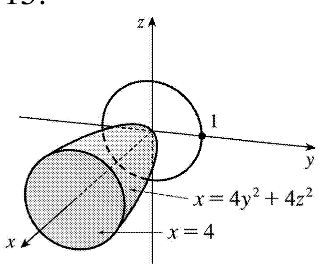
E is the solid above the region shown in the xy –plane and below the plane $z=x$. Thus,

$$\iiint_E (x+2y)\,dV = \int_0^1 \int_{x^2}^{x} \int_0^x (x+2y)\,dz\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} (x^2+2yx)\,dy\,dx = \int_0^1 \left[x^2y+xy^2\right]_{y=x^2}^{y=x} dx$$

$$= \int_0^1 (2x^3-x^4-x^5)\,dx = \left[\frac{1}{2}x^4-\frac{1}{5}x^5-\frac{1}{6}x^6\right]_0^1 = \frac{2}{15}$$

15.



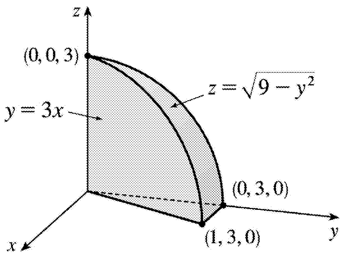
The projection E on the yz –plane is the disk $y^2+z^2 \le 1$. Using polar coordinates $y=r\cos\theta$ and $z=r\sin\theta$, we get

$$\iiint_E x\,dV = \iint_D \left[\int_{4y^2+4z^2}^{4} x\,dx\right] dA$$

$$= \frac{1}{2}\iint_D \left[4^2-(4y^2+4z^2)^2\right] dA = 8\int_0^{2\pi}\int_0^1 (1-r^4)\,r\,dr\,d\theta$$

$$= 8\int_0^{2\pi} d\theta \int_0^1 (r-r^5)\,dr = 8(2\pi)\left[\frac{1}{2}r^2-\frac{1}{6}r^6\right]_0^1 = \frac{16\pi}{3}$$

16.



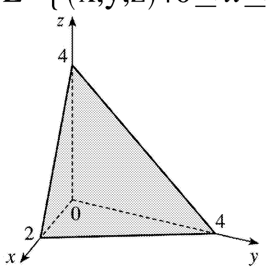
$$\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx=\int_0^1\int_{3x}^3 \frac{1}{2}(9-y^2)\,dy\,dx$$

$$=\int_0^1\left[\frac{9}{2}y-\frac{1}{6}y^3\right]_{y=3x}^{y=3}dx$$

$$=\int_0^1\left[9-\frac{27}{2}x+\frac{9}{2}x^3\right]dx$$

$$=\left[9x-\frac{27}{4}x^2+\frac{9}{8}x^4\right]_0^1=\frac{27}{8}$$

17. The plane $2x+y+z=4$ intersects the xy –plane when $2x+y+0=4\Rightarrow y=4-2x$, so $E=\{(x,y,z)\,|\,0\le x\le 2, 0\le y\le 4-2x, 0\le z\le 4-2x-y\}$ and



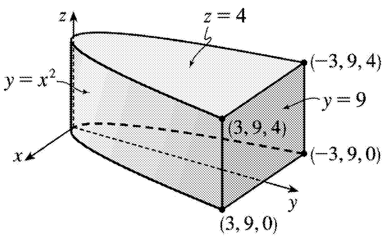
$$V=\int_0^2\int_0^{4-2x}\int_0^{4-2x-y} dz\,dy\,dx=\int_0^2\int_0^{4-2x}(4-2x-y)\,dy\,dx$$

$$=\int_0^2\left[4y-2xy-\frac{1}{2}y^2\right]_{y=0}^{y=4-2x}dx$$

$$=\int_0^2\left[4(4-2x)-2x(4-2x)-\frac{1}{2}(4-2x)^2\right]dx$$

$$=\int_0^2 (2x^2-8x+8)\,dx=\left[\frac{2}{3}x^3-4x^2+8x\right]_0^2=\frac{16}{3}$$

18.



$$V=\iiint_E dV=\int_{-3}^{3}\int_{x^2}^{9}\int_0^4 dz\,dy\,dx$$

$$=4\int_{-3}^{3}\int_{x^2}^{9} dy\,dx=4\int_{-3}^{3}\left(9-x^2\right)dx$$

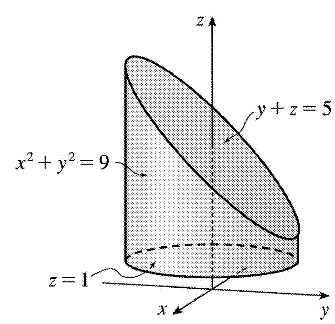
$$=4\left[9x-\frac{1}{3}x^3\right]_{-3}^{3}=4(27-9+27-9)=144$$

19.

$$V=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_1^{5-y} dz\,dy\,dx=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}(5-y-1)\,dy\,dx=\int_{-3}^{3}\left[4y-\frac{1}{2}y^2\right]_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}}dx$$

$$=\int_{-3}^{3} 8\sqrt{9-x^2}\,dx=8\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_{-3}^{3}$$

[using trigonometric substitutio
or Formula 30 in the Table of



$$=8\left[\frac{9}{2}\sin^{-1}(1)-\frac{9}{2}\sin^{-1}(-1)\right]=36\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=36\pi$$

Alternatively, use polar coordinates to evaluate the double integral:

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}(4-y)\,dy\,dx=\int_0^{2\pi}\int_0^3(4-r\sin\theta)\,r\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[2r^2-\frac{1}{3}r^3\sin\theta\right]_{r=0}^{r=3}d\theta$$

$$=\int_0^{2\pi}(18-9\sin\theta)\,d\theta$$

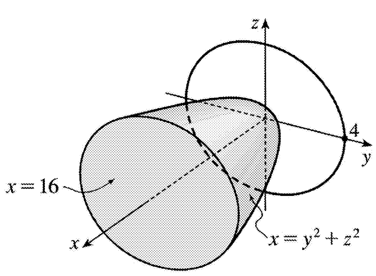
$$=18\theta+9\cos\theta\Big]_0^{2\pi}=36\pi$$

20. The paraboloid $x=y^2+z^2$ intersects the plane $x=16$ in the circle $y^2+z^2=16$, $x=16$.
Thus, $E=\left\{(x,y,z)\,|\,y^2+z^2\le x\le 16, y^2+z^2\le 16\right\}$.
Let $D=\left\{(y,z)\,|\,y^2+z^2\le 16\right\}$. Then using polar coordinates $y=r\cos\theta$ and $z=r\sin\theta$, we have

$$V=\iint_D\left(\int_{y^2+z^2}^{16}dx\right)dA=\iint_D(16-(y^2+z^2))\,dA$$

$$=\int_0^{2\pi}\int_0^4(16-r^2)\,r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_0^4(16r-r^3)\,dr$$

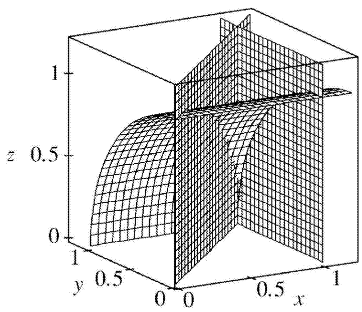
$$=[\theta]_0^{2\pi}\left[8r^2-\frac{1}{4}r^4\right]_0^4=2\pi(128-64)=128\pi$$



21. **(a)** The wedge can be described as the region
$D=\left\{(x,y,z)\,|\,y^2+z^2\le 1, 0\le x\le 1, 0\le y\le x\right\}=\left\{(x,y,z)\,|\,0\le x\le 1, 0\le y\le x, 0\le z\le\sqrt{1-y^2}\right\}$

So the integral expressing the volume of the wedge is $\iiint_D dV=\int_0^1\int_0^x\int_0^{\sqrt{1-y^2}}dz\,dy\,dx$.

(b) A CAS gives $\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\,dy\,dx = \frac{\pi}{4} - \frac{1}{3}$.
(Or use Formulas 30 and 87 from the Table of Integrals.)



22. **(a)** Note that $\Delta V = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$, so the Midpoint Rule gives

$$\iiint_B f(x,y,z)\,dV \approx \frac{1}{8}\left[f\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)+f\left(\frac{1}{4},\frac{1}{4},\frac{3}{4}\right)+f\left(\frac{1}{4},\frac{3}{4},\frac{1}{4}\right)+f\left(\frac{3}{4},\frac{1}{4},\frac{1}{4}\right)\right.$$
$$\left.+\ f\left(\frac{1}{4},\frac{3}{4},\frac{3}{4}\right)+f\left(\frac{3}{4},\frac{1}{4},\frac{3}{4}\right)+f\left(\frac{3}{4},\frac{3}{4},\frac{1}{4}\right)+f\left(\frac{3}{4},\frac{3}{4},\frac{3}{4}\right)\right]$$
$$= \frac{1}{8}\left[e^{-3(1/4)^2}+3e^{-2(1/4)^2-(3/4)^2}+3e^{-(1/4)^2-2(3/4)^2}+e^{-3(3/4)^2}\right] \approx 0.42968$$

(b) A CAS estimates the integral to be $\iiint_B e^{-x^2-y^2-z^2}\,dV \approx 0.42$. The estimate in part (a) is correct to one decimal place, and is larger than the actual value of the integral.

23. Here $f(x,y,z) = \frac{1}{\ln(1+x+y+z)}$ and $\Delta V = 2\cdot 4\cdot 2 = 16$, so the Midpoint Rule gives

$$\begin{aligned}\iiint_B f(x,y,z)\,dV &\approx \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f\left(\bar{x}_i,\bar{y}_j,\bar{z}_k\right)\Delta V\\ &= 16[f(1,2,1)+f(1,2,3)+f(1,6,1)+f(1,6,3)\\ &\quad + f(3,2,1)+f(3,2,3)+f(3,6,1)+f(3,6,3)]\\ &= 16\left[\frac{1}{\ln 5}+\frac{1}{\ln 7}+\frac{1}{\ln 9}+\frac{1}{\ln 11}+\frac{1}{\ln 7}+\frac{1}{\ln 9}+\frac{1}{\ln 11}+\frac{1}{\ln 13}\right] \approx 60.533\end{aligned}$$

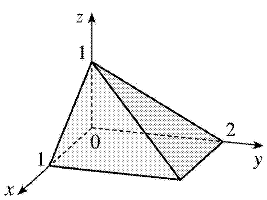
24. Here $f(x,y,z)=\sin(xy^2z^3)$ and $\Delta V=2\cdot 1\cdot \frac{1}{2}=1$, so the Midpoint Rule gives

$$\iiint_B f(x,y,z)\,dV \approx \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f\left(\bar{x}_i,\bar{y}_j,\bar{z}_k\right)\Delta V$$

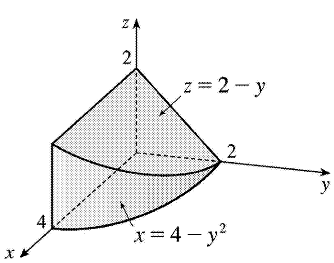
$$= 1\left[f\left(1,\frac{1}{2},\frac{1}{4}\right)+f\left(1,\frac{1}{2},\frac{3}{4}\right)+f\left(1,\frac{3}{2},\frac{1}{4}\right)+f\left(1,\frac{3}{2},\frac{3}{4}\right)\right.$$
$$\left.+f\left(3,\frac{1}{2},\frac{1}{4}\right)+f\left(3,\frac{1}{2},\frac{3}{4}\right)+f\left(3,\frac{3}{2},\frac{1}{4}\right)+f\left(3,\frac{3}{2},\frac{3}{4}\right)\right]$$

$$= \sin\frac{1}{256}+\sin\frac{27}{256}+\sin\frac{9}{256}+\sin\frac{243}{256}+\sin\frac{3}{256}+\sin\frac{81}{256}+\sin\frac{27}{256}+\sin\frac{729}{256}\approx 1.675$$

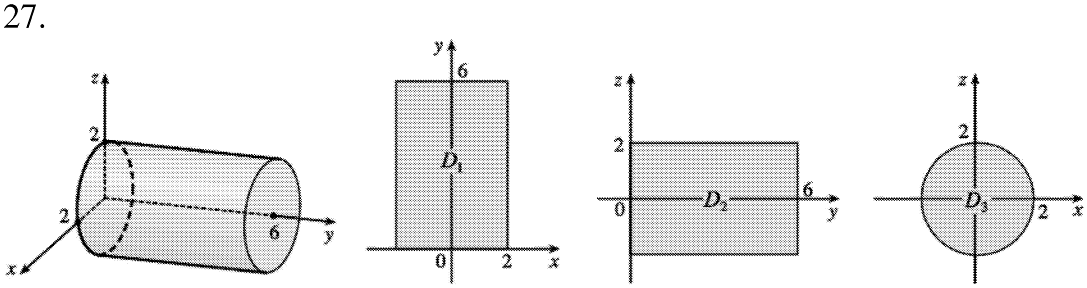
25. $E=\{(x,y,z)\,|\,0\leq x\leq 1, 0\leq z\leq 1-x, 0\leq y\leq 2-2z\}$,
the solid bounded by the three coordinate planes and the planes $z=1-x$, $y=2-2z$.



26. $E=\left\{(x,y,z)\,|\,0\leq y\leq 2, 0\leq z\leq 2-y, 0\leq x\leq 4-y^2\right\}$,
the solid bounded by the three coordinate planes, the plane $z=2-y$, and the cylindrical surface $x=4-y^2$
.



27.



If D_1 , D_2 , D_3 are the projections of E on the xy –, yz –, and xz –planes, then

$D_1 = \{(x,y) \mid -2 \le x \le 2, 0 \le y \le 6\}$

$D_2 = \{(y,z) \mid -2 \le z \le 2, 0 \le y \le 6\}$

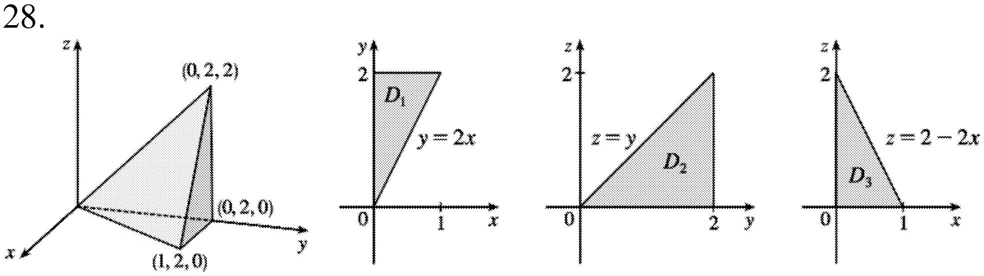
$D_3 = \left\{ (x,z) \mid x^2 + z^2 \le 4 \right\}$

Therefore

$$E = \left\{ (x,y,z) \mid -\sqrt{4-x^2} \le z \le \sqrt{4-x^2}, -2 \le x \le 2, 0 \le y \le 6 \right\}$$
$$= \left\{ (x,y,z) \mid -\sqrt{4-z^2} \le x \le \sqrt{4-z^2}, -2 \le z \le 2, 0 \le y \le 6 \right\}$$

$$\iiint_E f(x,y,z)\,dV = \int_{-2}^{2}\int_{0}^{6}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y,z)\,dz\,dy\,dx = \int_{0}^{6}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y,z)\,dz\,dx\,dy$$
$$= \int_{0}^{6}\int_{-2}^{2}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} f(x,y,z)\,dx\,dz\,dy = \int_{-2}^{2}\int_{0}^{6}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} f(x,y,z)\,dx\,dy\,dz$$
$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{6} f(x,y,z)\,dy\,dz\,dx = \int_{-2}^{2}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}}\int_{0}^{6} f(x,y,z)\,dy\,dx\,dz$$

28.



If D_1, D_2, and D_3 are the projections of E on the xy–, yz–, and xz–planes, then

$$D_1 = \{(x,y) \mid 0 \le x \le 1, 2x \le y \le 2\} = \{(x,y) \mid 0 \le y \le 2, 0 \le x \le y/2\},$$

$D_2 = \{(y,z) \mid 0 \le y \le 2, 0 \le z \le y\} = \{(y,z) \mid 0 \le z \le 2, z \le y \le 2\}$, and

$D_3 = \{(x,z) \mid 0 \le x \le 1, 0 \le z \le 2-2x\} = \{(x,z) \mid 0 \le z \le 2, 0 \le x \le (2-z)/2\}$

Therefore

$$E = \{(x,y,z) \mid 0 \le x \le 1, 2x \le y \le 2, 0 \le z \le y-2x\}$$
$$= \{(x,y,z) \mid 0 \le y \le 2, 0 \le x \le y/2, 0 \le z \le y-2x\}$$

$= \{(x,y,z) \mid 0 \le y \le 2, 0 \le z \le y, 0 \le x \le (y-z)/2\}$
$= \{(x,y,z) \mid 0 \le z \le 2, z \le y \le 2, 0 \le x \le (y-z)/2\}$
$= \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le 2-2x, z+2x \le y \le 2\}$
$= \{(x,y,z) \mid 0 \le z \le 2, 0 \le x \le (2-z)/2, z+2x \le y \le 2\}$

Then

$$\iiint_E f(x,y,z)\,dV = \int_0^1 \int_{2x}^2 \int_0^{y-2x} f(x,y,z)\,dz\,dy\,dx$$

$$= \int_0^2 \int_0^{y/2} \int_0^{y-2x} f(x,y,z)\,dz\,dx\,dy$$

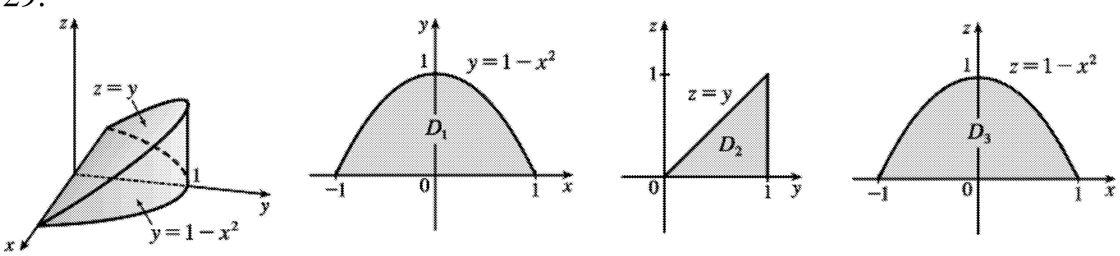
$$= \int_0^2 \int_0^y \int_0^{(y-z)/2} f(x,y,z)\,dx\,dz\,dy$$

$$= \int_0^2 \int_z^2 \int_0^{(y-z)/2} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^{2-2x} \int_{z+2x}^2 f(x,y,z)\,dy\,dz\,dx$$

$$= \int_0^2 \int_0^{(2-z)/2} \int_{z+2x}^2 f(x,y,z)\,dy\,dx\,dz$$

29.



If D_1, D_2, and D_3 are the projections of E on the xy–, yz–, and xz–planes, then

$D_1 = \{(x,y) \mid -1 \le x \le 1, 0 \le y \le 1-x^2\} = \{(x,y) \mid 0 \le y \le 1, -\sqrt{1-y} \le x \le \sqrt{1-y}\}$,

$D_2 = \{(y,z) \mid 0 \le y \le 1, 0 \le z \le y\} = \{(y,z) \mid 0 \le z \le 1, z \le y \le 1\}$, and

$D_3 = \{(x,z) \mid -1 \le x \le 1, 0 \le z \le 1-x^2\} = \{(x,z) \mid 0 \le z \le 1, -\sqrt{1-z} \le x \le \sqrt{1-z}\}$

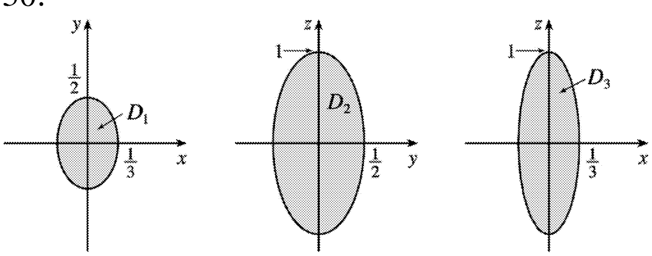
Therefore

$$\begin{aligned}E&=\left\{(x,y,z)\mid -1\le x\le 1, 0\le y\le 1-x^2, 0\le z\le y\right\}\\ &=\left\{(x,y,z)\mid 0\le y\le 1, -\sqrt{1-y}\le x\le \sqrt{1-y}, 0\le z\le y\right\}\\ &=\left\{(x,y,z)\mid 0\le y\le 1, 0\le z\le y, -\sqrt{1-y}\le x\le \sqrt{1-y}\right\}\\ &=\left\{(x,y,z)\mid 0\le z\le 1, z\le y\le 1, -\sqrt{1-y}\le x\le \sqrt{1-y}\right\}\\ &=\left\{(x,y,z)\mid -1\le x\le 1, 0\le z\le 1-x^2, z\le y\le 1-x^2\right\}\\ &=\left\{(x,y,z)\mid 0\le z\le 1, -\sqrt{1-z}\le x\le \sqrt{1-z}, z\le y\le 1-x^2\right\}\end{aligned}$$

Then

$$\begin{aligned}\iiint_E f(x,y,z)\,dV &= \int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{y} f(x,y,z)\,dz\,dy\,dx=\int_{0}^{1}\int_{-\sqrt{1-y}}^{\sqrt{1-y}}\int_{0}^{y} f(x,y,z)\,dz\,dx\,dy\\ &= \int_{0}^{1}\int_{0}^{y}\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y,z)\,dx\,dz\,dy=\int_{0}^{1}\int_{z}^{1}\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y,z)\,dx\,dy\,dz\\ &= \int_{-1}^{1}\int_{0}^{1-x^2}\int_{z}^{1-x^2} f(x,y,z)\,dy\,dz\,dx=\int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{z}^{1-x^2} f(x,y,z)\,dy\,dx\,dz\end{aligned}$$

30.



If D_1, D_2 and D_3 are the projections of E on the xy–, yz–, and xz–planes, then

$D_1=\left\{(x,y)\mid 9x^2+4y^2\le 1\right\}$, $D_2=\left\{(y,z)\mid 4y^2+z^2\le 1\right\}$, $D_3=\left\{(x,z)\mid 9x^2+z^2\le 1\right\}$. Therefore

$$\begin{aligned}\iiint_E f(x,y,z)\,dV &= \int_{-1/3}^{1/3}\int_{-\sqrt{1-9x^2}/2}^{\sqrt{1-9x^2}/2}\int_{-\sqrt{1-9x^2-4y^2}}^{\sqrt{1-9x^2-4y^2}} f(x,y,z)\,dz\,dy\,dx\\ &= \int_{-1/2}^{1/2}\int_{-\sqrt{1-4y^2}/3}^{\sqrt{1-4y^2}/3}\int_{-\sqrt{1-9x^2-4y^2}}^{\sqrt{1-9x^2-4y^2}} f(x,y,z)\,dz\,dx\,dy\end{aligned}$$

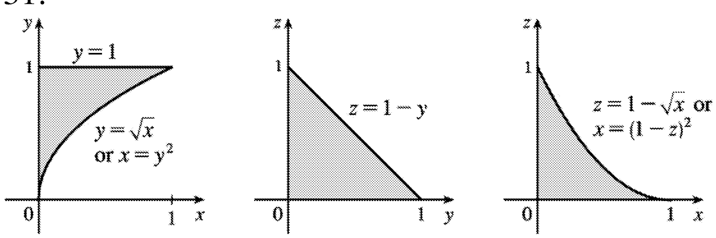
$$= \int_{-1/2}^{1/2} \int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} \int_{-\sqrt{1-4y^2-z^2}/3}^{\sqrt{1-4y^2-z^2}/3} f(\mathrm{x,y,z})\,dx\,dz\,dy$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-z^2}/2}^{\sqrt{1-z^2}/2} \int_{-\sqrt{1-4y^2-z^2}/3}^{\sqrt{1-4y^2-z^2}/3} f(\mathrm{x,y,z})\,dx\,dy\,dz$$

$$= \int_{-1/3}^{1/3} \int_{-\sqrt{1-9x^2}}^{\sqrt{1-9x^2}} \int_{-\sqrt{1-9x^2-z^2}/2}^{\sqrt{1-9x^2-z^2}/2} f(\mathrm{x,y,z})\,dy\,dz\,dx$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-z^2}/3}^{\sqrt{1-z^2}/3} \int_{-\sqrt{1-9x^2-z^2}/2}^{\sqrt{1-9x^2-z^2}/2} f(\mathrm{x,y,z})\,dy\,dx\,dz$$

31.



The diagrams show the projections of E on the xy –, yz –, and xz –planes. Therefore

$$\int_0^1 \int_{\sqrt{x}}^{1} \int_0^{1-y} f(\mathrm{x,y,z})\,dz\,dy\,dx = \int_0^1 \int_0^{y^2} \int_0^{1-y} f(\mathrm{x,y,z})\,dz\,dx\,dy$$

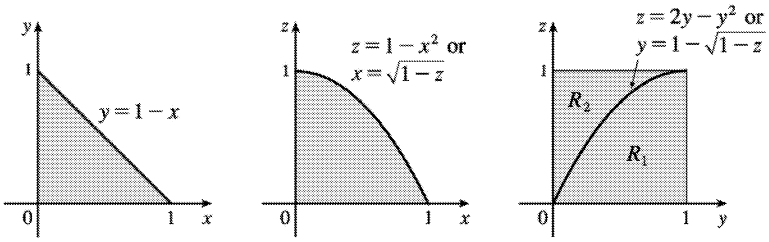
$$= \int_0^1 \int_0^{1-z} \int_0^{y^2} f(\mathrm{x,y,z})\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^{1-y} \int_0^{y^2} f(\mathrm{x,y,z})\,dx\,dz\,dy$$

$$= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(\mathrm{x,y,z})\,dy\,dz\,dx$$

$$= \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(\mathrm{x,y,z})\,dy\,dx\,dz$$

32.



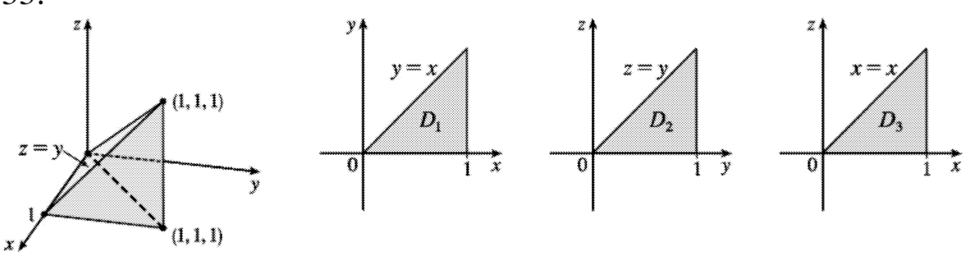
The projections of E onto the xy – and xz –planes are as in the first two diagrams and so

$$\int_0^1\int_0^{1-x^2}\int_0^{1-x} f(x,y,z)\,dy\,dz\,dx=\int_0^1\int_0^{\sqrt{1-z}}\int_0^{1-x} f(x,y,z)\,dy\,dx\,dz=\int_0^1\int_0^{1-y}\int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy=\int_0^1\int_0^{1-x}\int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dx$$

Now the surface $z=1-x^2$ intersects the plane $y=1-x$ in a curve whose projection in the yz –plane is $z=1-(1-y)^2$ or $z=2y-y^2$. So we must split up the projection of E on the yz –plane into two regions as in the third diagram. For (y,z) in R_1 , $0\le x\le 1-y$ and for (y,z) in R_2 , $0\le x\le \sqrt{1-z}$, and so the given integral is also equal to

$$\int_0^1\int_0^{1-\sqrt{1-z}}\int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dy\,dz+\int_0^1\int_{1-\sqrt{1-z}}^{1}\int_0^{1-y} f(x,y,z)\,dx\,dy\,dz=\int_0^1\int_0^{2y-y^2}\int_0^{1-y} f(x,y,z)\,dx\,dz\,dy+\int_0^1\int_{2y-y^2}^{1}\int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dz$$

33.



$$\int_0^1\int_y^1\int_0^y f(x,y,z)\,dz\,dx\,dy=\iiint_E f(x,y,z)\,dV$$ where $E=\{(x,y,z)|0\le z\le y, y\le x\le 1, 0\le y\le 1\}$.

If D_1 , D_2 , and D_3 are the projections of E on the xy –, yz – and xz –planes then

$D_1=\{(x,y)|0\le y\le 1, y\le x\le 1\}=\{(x,y)|0\le x\le 1, 0\le y\le x\}$,

$D_2=\{(y,z)|0\le y\le 1, 0\le z\le y\}=\{(y,z)|0\le z\le 1, z\le y\le 1\}$, and

$D_3=\{(x,z)|0\le x\le 1, 0\le z\le x\}=\{(x,z)|0\le z\le 1, z\le x\le 1\}$.

Thus we also have

$$\begin{aligned} E &=\{(x,y,z)|0\le x\le 1, 0\le y\le x, 0\le z\le y\}=\{(x,y,z)|0\le y\le 1, 0\le z\le y, y\le x\le 1\}\\ &=\{(x,y,z)|0\le z\le 1, z\le y\le 1, y\le x\le 1\}=\{(x,y,z)|0\le x\le 1, 0\le z\le x, z\le y\le x\}\\ &=\{(x,y,z)|0\le z\le 1, z\le x\le 1, z\le y\le x\}. \end{aligned}$$

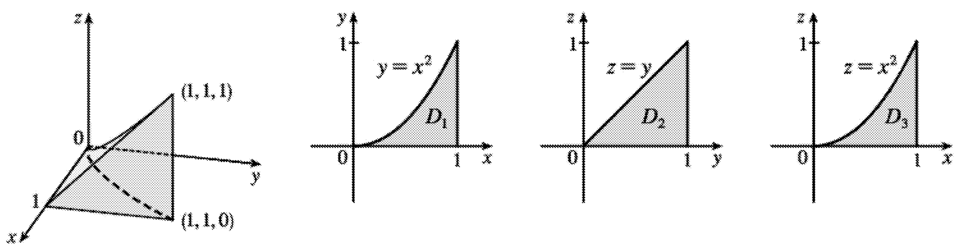
Then

$$\int_0^1\int_y^1\int_0^y f(x,y,z)\,dz\,dx\,dy = \int_0^1\int_0^x\int_0^y f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^y\int_y^1 f(x,y,z)\,dx\,dz\,dy$$

$$= \int_0^1\int_z^1\int_y^1 f(x,y,z)\,dx\,dy\,dz = \int_0^1\int_0^x\int_z^x f(x,y,z)\,dy\,dz\,dx$$

$$= \int_0^1\int_z^1\int_z^x f(x,y,z)\,dy\,dx\,dz$$

34.



$\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx = \iiint_E f(x,y,z)\,dV$ where $E = \left\{ (x,y,z) \mid 0 \le x \le 1, 0 \le y \le x^2, 0 \le z \le y \right\}$. If D_1 , D_2 , D_3 are the projections of E on the

xy –, yz –, and xz –planes, then $D_1 = \left\{ (x,y) \mid 0 \le x \le 1, 0 \le y \le x^2 \right\} = \left\{ (x,y) \mid 0 \le y \le 1, \sqrt{y} \le x \le 1 \right\}$,

$D_2 = \{ (y,z) \mid 0 \le y \le 1, 0 \le z \le y \} = \{ (y,z) \mid 0 \le z \le 1, z \le y \le 1 \}$,

$D_3 = \left\{ (x,z) \mid 0 \le x \le 1, 0 \le z \le x^2 \right\} = \left\{ (x,z) \mid 0 \le z \le 1, \sqrt{z} \le x \le 1 \right\}$. Thus we also have

$$\begin{aligned} E &= \left\{ (x,y,z) \mid 0 \le y \le 1, \sqrt{y} \le x \le 1, 0 \le z \le y \right\} \\ &= \left\{ (x,y,z) \mid 0 \le y \le 1, 0 \le z \le y, \sqrt{y} \le x \le 1 \right\} \\ &= \left\{ (x,y,z) \mid 0 \le z \le 1, z \le y \le 1, \sqrt{y} \le x \le 1 \right\} \\ &= \left\{ (x,y,z) \mid 0 \le x \le 1, 0 \le z \le x^2, z \le y \le x^2 \right\} \\ &= \left\{ (x,y,z) \mid 0 \le z \le 1, \sqrt{z} \le x \le 1, z \le y \le x^2 \right\} \end{aligned}$$

Then

$$\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_{\sqrt{y}}^1\int_0^y f(x,y,z)\,dz\,dx\,dy$$

$$= \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy$$

$$= \int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^{x^2} \int_z^{x^2} f(x,y,z)\,dy\,dz\,dx$$

$$= \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f(x,y,z)\,dy\,dx\,dz$$

35.

$$m = \iiint_E \rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{x}} 2(1+x+y)\,dy\,dx = \int_0^1 \left[2y+2xy+y^2\right]_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \left(2\sqrt{x}+2x^{3/2}+x\right)dx = \left[\frac{4}{3}x^{3/2}+\frac{4}{5}x^{5/2}+\frac{1}{2}x^2\right]_0^1 = \frac{79}{30}$$

$$M_{yz} = \iiint_E x\rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2x\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{x}} 2x(1+x+y)\,dy\,dx = \int_0^1 \left[2xy+2x^2y+xy^2\right]_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^1 (2x^{3/2}+2x^{5/2}+x^2)\,dx = \left[\frac{4}{5}x^{5/2}+\frac{4}{7}x^{7/2}+\frac{1}{3}x^3\right]_0^1 = \frac{179}{105}$$

$$M_{xz} = \iiint_E y\rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2y\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{x}} 2y(1+x+y)\,dy\,dx = \int_0^1 \left[y^2+xy^2+\frac{2}{3}y^3\right]_{y=0}^{y=\sqrt{x}} dx$$

$$=\int_0^1 \left(x+x^2+\frac{2}{3}x^{3/2} \right) dx=\left[\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{4}{15}x^{5/2} \right]_0^1=\frac{11}{10}$$

$$M_{xy}=\iiint_E z\rho(\mathrm{x,y,z})\,dV=\int_0^1\int_0^{\sqrt{x}}\int_0^{1+x+y} 2z\,dz\,dy\,dx$$

$$=\int_0^1\int_0^{\sqrt{x}} \left[z^2\right]_{z=0}^{z=1+x+y} dy\,dx=\int_0^1\int_0^{\sqrt{x}} (1+x+y)^2\,dy\,dx$$

$$=\int_0^1\int_0^{\sqrt{x}} (1+2x+2y+2xy+x^2+y^2)\,dy\,dx$$

$$=\int_0^1 \left[y+2xy+y^2+xy^2+x^2y+\frac{1}{3}y^3 \right]_{y=0}^{y=\sqrt{x}} dx=\int_0^1 \left(\sqrt{x}+\frac{7}{3}x^{3/2}+x+x^2+x^{5/2} \right) dx$$

$$=\left[\frac{2}{3}x^{3/2}+\frac{14}{15}x^{5/2}+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{2}{7}x^{7/2} \right]_0^1=\frac{571}{210}$$

Thus the mass is $\frac{79}{30}$ and the center of mass is $(\overline{\mathrm{x}},\overline{\mathrm{y}},\overline{\mathrm{z}})=\left(\frac{M_{yz}}{m},\frac{M_{xz}}{m},\frac{M_{xy}}{m} \right)=\left(\frac{358}{553},\frac{33}{79},\frac{571}{553} \right)$.

36.

$$m=\int_{-1}^1\int_0^{1-y^2}\int_0^{1-z} 4\,dx\,dz\,dy=4\int_{-1}^1\int_0^{1-y^2} (1-z)\,dz\,dy$$

$$=4\int_{-1}^1 \left[z-\frac{1}{2}z^2 \right]_{z=0}^{z=1-y^2} dy=2\int_{-1}^1 (1-y^4)\,dy=\frac{16}{5}\ ,$$

$$M_{yz}=\int_{-1}^1\int_0^{1-y^2}\int_0^{1-z} 4x\,dx\,dz\,dy=2\int_{-1}^1\int_0^{1-y^2} (1-z)^2\,dz\,dy=2\int_{-1}^1 \left[-\frac{1}{3}(1-z)^3 \right]_{z=0}^{z=1-y^2} dy$$

$$=\frac{2}{3}\int_{-1}^1 \left(1-y^6\right) dy=\left(\frac{4}{3} \right)\left(\frac{6}{7} \right)=\frac{24}{21}$$

$$M_{xz}=\int_{-1}^1\int_0^{1-y^2}\int_0^{1-z} 4y\,dx\,dz\,dy=\int_{-1}^1\int_0^{1-y^2} 4y(1-z)\,dz\,dy$$

$$=\int_{-1}^{1}\left[4y(1-y^2)-2y(1-y^2)^2\right]dy=\int_{-1}^{1}(2y-2y^5)\,dy=0 \text{ [the integrand is odd]}$$

$$M_{xy}=\int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z}4z\,dx\,dz\,dy=\int_{-1}^{1}\int_{0}^{1-y^2}(4z-4z^2)\,dz\,dy=2\int_{-1}^{1}\left[(1-y^2)^2-\frac{2}{3}(1-y^2)^3\right]dy$$

$$=2\int_{-1}^{1}\left[\frac{1}{3}-y^4+\frac{2}{3}y^6\right]dy=\left[\frac{4}{3}y-\frac{4}{5}y^5+\frac{8}{21}y^7\right]_0^1=\frac{96}{105}=\frac{32}{35}$$

Thus, $(\bar{x},\bar{y},\bar{z})=\left(\frac{5}{14},0,\frac{2}{7}\right)$

37.

$$m=\int_0^a\int_0^a\int_0^a(x^2+y^2+z^2)\,dx\,dy\,dz=\int_0^a\int_0^a\left[\frac{1}{3}x^3+xy^2+xz^2\right]_{x=0}^{x=a}dy\,dz$$

$$=\int_0^a\int_0^a\left(\frac{1}{3}a^3+ay^2+az^2\right)dy\,dz=\int_0^a\left[\frac{1}{3}a^3y+\frac{1}{3}ay^3+ayz^2\right]_{y=0}^{y=a}dz$$

$$=\int_0^a\left(\frac{2}{3}a^4+a^2z^2\right)dz=\left[\frac{2}{3}a^4z+\frac{1}{3}a^2z^3\right]_0^a=\frac{2}{3}a^5+\frac{1}{3}a^5=a^5$$

$$M_{yz}=\int_0^a\int_0^a\int_0^a\left[x^3+x(y^2+z^2)\right]dx\,dy\,dz=\int_0^a\int_0^a\left[\frac{1}{4}a^4+\frac{1}{2}a^2(y^2+z^2)\right]dy\,dz$$

$$=\int_0^a\left(\frac{1}{4}a^5+\frac{1}{6}a^5+\frac{1}{2}a^3z^2\right)dz=\frac{1}{4}a^6+\frac{1}{3}a^6=\frac{7}{12}a^6$$

$=M_{xz}=M_{xy}$ by symmetry of E and $\rho(x,y,z)$

Hence $(\bar{x},\bar{y},\bar{z})=\left(\frac{7}{12}a,\frac{7}{12}a,\frac{7}{12}a\right)$.

38.

$$m=\int_0^1\int_0^{1-x}\int_0^{1-x-y}y\,dz\,dy\,dx=\int_0^1\int_0^{1-x}\left[(1-x)y-y^2\right]dy\,dx$$

$$=\int_0^1 \left[\frac{1}{2}(1-x)^3 - \frac{1}{3}(1-x)^3 \right] dx = \frac{1}{6}\int_0^1 (1-x)^3 dx = \frac{1}{24}$$

$$M_{yz} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} xy\,dz\,dy\,dx = \int_0^1 \int_0^{1-x} \left[(x-x^2)y - xy^2 \right] dy\,dx$$

$$=\int_0^1 \left[\frac{1}{2}x(1-x)^3 - \frac{1}{3}x(1-x)^3 \right] dx = \frac{1}{6}\int_0^1 \left(x-3x^2+3x^3-x^4 \right) dx$$

$$=\frac{1}{6}\left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5} \right) = \frac{1}{120}$$

$$M_{xz} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y^2\,dz\,dy\,dx = \int_0^1 \int_0^{1-x} \left[(1-x)y^2 - y^3 \right] dy\,dx$$

$$=\int_0^1 \left[\frac{1}{3}(1-x)^4 - \frac{1}{4}(1-x)^4 \right] dx = \frac{1}{12}\left[-\frac{1}{5}(1-x)^5 \right]_0^1 = \frac{1}{60}$$

$$M_{xy} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} yz\,dz\,dy\,dx = \int_0^1 \int_0^{1-x} \left[\frac{1}{2}y(1-x-y)^2 \right] dy\,dx$$

$$=\frac{1}{2}\int_0^1 \int_0^{1-x} \left[(1-x)^2 y - 2(1-x)y^2 + y^3 \right] dy\,dx$$

$$=\frac{1}{2}\int_0^1 \left[\frac{1}{2}(1-x)^4 - \frac{2}{3}(1-x)^4 + \frac{1}{4}(1-x)^4 \right] dx$$

$$=\frac{1}{24}\int_0^1 (1-x)^4 dx = -\frac{1}{24}\left[\frac{1}{5}(1-x)^5 \right]_0^1 = \frac{1}{120}$$

Hence $(\overline{x},\overline{y},\overline{z}) = \left(\frac{1}{5}, \frac{2}{5}, \frac{1}{5} \right)$.

39. **(a)** $m = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} \sqrt{x^2+y^2}\,dz\,dy\,dx$

(b) $(\overline{x},\overline{y},\overline{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right)$ where

$M_{yz}=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} x\sqrt{x^2+y^2}\,dz\,dy\,dx$, $M_{xz}=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} y\sqrt{x^2+y^2}\,dz\,dy\,dx$, and

$M_{xy}=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} z\sqrt{x^2+y^2}\,dz\,dy\,dx$.

(c) $I_z=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} (x^2+y^2)\sqrt{x^2+y^2}\,dz\,dy\,dx=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} (x^2+y^2)^{3/2}\,dz\,dy\,dx$

40. **(a)** $m=\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{0}^{\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$

(b) $(\overline{x},\overline{y},\overline{z})$ where $\overline{x}=m^{-1}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{0}^{\sqrt{1-x^2-y^2}} x\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$,

$\overline{y}=m^{-1}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{0}^{\sqrt{1-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$, $\overline{z}=m^{-1}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{0}^{\sqrt{1-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$

(c) $I_z=\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{0}^{\sqrt{1-x^2-y^2}} (x^2+y^2)(1+x+y+z)\,dz\,dx\,dy$

41. **(a)** $m=\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{y}(1+x+y+z)\,dz\,dy\,dx=\frac{3\pi}{32}+\frac{11}{24}$

(b)

$$(\overline{x},\overline{y},\overline{z})=\Bigg(m^{-1}\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{y} x(1+x+y+z)\,dz\,dy\,\mathrm{dx},$$

$$m^{-1}\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{y} y(1+x+y+z)\,dz\,dy\,\mathrm{dx},$$

$$m^{-1}\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y z(1+x+y+z)\,dz\,dy\,dx\Bigg)$$
$$=\left(\frac{28}{9\pi+44},\frac{30\pi+128}{45\pi+220},\frac{45\pi+208}{135\pi+660}\right)$$

(c) $I_z=\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y (x^2+y^2)(1+x+y+z)\,dz\,dy\,dx=\frac{68+15\pi}{240}$

42. **(a)** $m=\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}}(x^2+y^2)\,dz\,dy\,dx=\frac{56}{5}=11.2$

(b) $(\bar{x},\bar{y},\bar{z})$ where $\bar{x}=m^{-1}\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}}x(x^2+y^2)\,dz\,dy\,dx\approx 0.375$,

$\bar{y}=m^{-1}\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}}y(x^2+y^2)\,dz\,dy\,dx=\frac{45\pi}{64}\approx 2.209$, $\bar{z}=m^{-1}\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}}z(x^2+y^2)\,dz\,dy\,dx=\frac{15}{16}=0.9375$.

(c) $I_z=\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}}(x^2+y^2)^2\,dz\,dy\,dx=\frac{10{,}464}{175}\approx 59.79$

43.

$$I_x=\int_0^L\int_0^L\int_0^L k(y^2+z^2)\,dz\,dy\,dx=k\int_0^L\int_0^L\left(Ly^2+\frac{1}{3}L^3\right)dy\,dx=k\int_0^L\frac{2}{3}L^4\,dx=\frac{2}{3}kL^5.$$

By symmetry, $I_x=I_y=I_z=\frac{2}{3}kL^5$.

44. Let k be the density. Then

$$I_x=\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}k(y^2+z^2)\,dx\,dy\,dz=ka\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}(y^2+z^2)\,dy\,dz$$
$$=ak\int_{-c/2}^{c/2}\left[\frac{1}{3}y^3+z^2y\right]_{y=-b/2}^{y=b/2}dz=ak\int_{-c/2}^{c/2}\left(\frac{1}{12}b^3+bz^2\right)dz=ak\left[\frac{1}{12}b^3z+\frac{1}{3}bz^3\right]_{-c/2}^{c/2}$$

$$= ak\left(\frac{1}{12}b^3c+\frac{1}{12}bc^3\right)=\frac{1}{12}kabc(b^2+c^2)$$

By symmetry, $I_y=\frac{1}{12}kabc(a^2+c^2)$ and $I_z=\frac{1}{12}kabc(a^2+b^2)$.

45. **(a)** $f(x,y,z)$ is a joint density function, so we know $\iiint_{R^3} f(x,y,z)dV=1$. Here we have

$$\iiint_{R^3} f(x,y,z)dV = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)dzdydx=\int_0^2\int_0^2\int_0^2 Cxyzdzdydx$$

$$= C\int_0^2 xdx\int_0^2 ydy\int_0^2 zdz=C\left[\frac{x^2}{2}\right]_0^2\left[\frac{y^2}{2}\right]_0^2\left[\frac{z^2}{2}\right]_0^2$$

$$= 8C$$

Then we must have $8C=1\Rightarrow C=\frac{1}{8}$.

(b)

$$P(X\le 1,Y\le 1,Z\le 1) =\int_{-\infty}^{1}\int_{-\infty}^{1}\int_{-\infty}^{1} f(x,y,z)dzdydx$$

$$=\int_0^1\int_0^1\int_0^1 \frac{1}{8}xyzdzdydx=\frac{1}{8}\int_0^1 xdx\int_0^1 ydy\int_0^1 zdz$$

$$=\frac{1}{8}\left[\frac{x^2}{2}\right]_0^1\left[\frac{y^2}{2}\right]_0^1\left[\frac{z^2}{2}\right]_0^1=\frac{1}{8}\left(\frac{1}{2}\right)^3=\frac{1}{64}$$

(c) $P(X+Y+Z\le 1)=P((X,Y,Z)\in E)$ where E is the solid region in the first octant bounded by the coordinate planes and the plane $x+y+z=1$. The plane $x+y+z=1$ meets the xy –plane in the line $x+y=1$, so we have

$$P(X+Y+Z\le 1) = \iiint_E f(x,y,z)dV=\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{8}xyzdzdydx$$

$$= \frac{1}{8}\int_0^1\int_0^{1-x} xy\left[\frac{1}{2}z^2\right]_{z=0}^{z=1-x-y}dydx$$

$$= \frac{1}{16}\int_0^1\int_0^{1-x} xy(1-x-y)^2dydx$$

$$= \frac{1}{16}\int_0^1\int_0^{1-x}\left[(x^3-2x^2+x)y+(2x^2-2x)y^2+xy^3\right]dy\,dx$$

$$= \frac{1}{16}\int_0^1\left[(x^3-2x^2+x)\frac{1}{2}y^2+(2x^2-2x)\frac{1}{3}y^3+x\left(\frac{1}{4}y^4\right)\right]_{y=0}^{y=1-x}dx$$

$$= \frac{1}{192}\int_0^1(x-4x^2+6x^3-4x^4+x^5)dx=\frac{1}{192}\left(\frac{1}{30}\right)=\frac{1}{5760}$$

46. **(a)** $f(x,y,z)$ is a joint density function, so we know $\iiint_{R^3} f(x,y,z)dV=1$. Here we have

$$\iiint_{R^3} f(x,y,z)dV = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)dz\,dy\,dx$$

$$= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} Ce^{-(0.5x+0.2y+0.1z)}dz\,dy\,dx$$

$$= C\int_0^{\infty} e^{-0.5x}dx\int_0^{\infty} e^{-0.2y}dy\int_0^{\infty} e^{-0.1z}dz$$

$$= C\lim_{t\to\infty}\int_0^t e^{-0.5x}dx\lim_{t\to\infty}\int_0^t e^{-0.2y}dy\lim_{t\to\infty}\int_0^t e^{-0.1z}dz$$

$$= C\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t\lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t\lim_{t\to\infty}\left[-10e^{-0.1z}\right]_0^t$$

$$= C\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right]\lim_{t\to\infty}\left[-5(e^{-0.2t}-1)\right]\lim_{t\to\infty}\left[-10(e^{-0.1t}-1)\right]$$

$$= C\cdot(-2)(0-1)\cdot(-5)(0-1)\cdot(-10)(0-1)=100C$$

So we must have $100C=1\Rightarrow C=\frac{1}{100}$.

(b) We have no restriction on Z , so

$$P(X\leq 1,Y\leq 1) = \int_{-\infty}^{1}\int_{-\infty}^{1}\int_{-\infty}^{\infty} f(x,y,z)dz\,dy\,dx$$

$$=\int_0^1\int_0^1\int_0^{\infty}\frac{1}{100}e^{-(0.5x+0.2y+0.1z)}dz\,dy\,dx$$

$$= \frac{1}{100} \int_0^1 e^{-0.5x} dx \int_0^1 e^{-0.2y} dy \int_0^\infty e^{-0.1z} dz$$

$$= \frac{1}{100} \left[-2e^{-0.5x}\right]_0^1 \left[-5e^{-0.2y}\right]_0^1 \lim_{t\to\infty} \left[-10e^{-0.1z}\right]_0^t \text{ [by part (a)]}$$

$$= \frac{1}{100} (2-2e^{-0.5})(5-5e^{-0.2})(10) = (1-e^{-0.5})(1-e^{-0.2}) \approx 0.07132$$

(c)

$$P(X\le 1, Y\le 1, Z\le 1) = \int_{-\infty}^1 \int_{-\infty}^1 \int_{-\infty}^1 f(x,y,z)\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^1 \int_0^1 \frac{1}{100} e^{-(0.5x+0.2y+0.1z)} dz\, dy\, dx$$

$$= \frac{1}{100} \int_0^1 e^{-0.5x} dx \int_0^1 e^{-0.2y} dy \int_0^1 e^{-0.1z} dz$$

$$= \frac{1}{100} \left[-2e^{-0.5x}\right]_0^1 \left[-5e^{-0.2y}\right]_0^1 \left[-10e^{-0.1z}\right]_0^1$$

$$= (1-e^{-0.5})(1-e^{-0.2})(1-e^{-0.1}) \approx 0.006787$$

47. $V(E) = L^3$,

$$f_{\text{ave}} = \frac{1}{L^3} \int_0^L \int_0^L \int_0^L xyz\, dx\, dy\, dz = \frac{1}{L^3} \int_0^L x\, dx \int_0^L y\, dy \int_0^L z\, dz$$

$$= \frac{1}{L^3} \left[\frac{x^2}{2}\right]_0^L \left[\frac{y^2}{2}\right]_0^L \left[\frac{z^2}{2}\right]_0^L = \frac{1}{L^3} \frac{L^2}{2} \frac{L^2}{2} \frac{L^2}{2} = \frac{L^3}{8}$$

48.

$$V(E) = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} dz\, dy\, dx = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2)\, dy\, dx$$

$$= \int_0^{2\pi} \int_0^1 (1-r^2) r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^1 (r-r^3)\, dr = 2\pi \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg]_0^1 = \frac{\pi}{2} \ .$$

Then

$$f_{\text{ave}} = \frac{1}{\pi/2}\iiint_E (x^2z+y^2z)\,dV = \frac{2}{\pi}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{1-x^2-y^2}(x^2+y^2)\,z\,dz\,dy\,dx$$

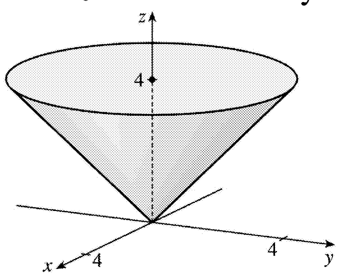
$$= \frac{2}{\pi}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(x^2+y^2)\cdot\frac{1}{2}(1-x^2-y^2)^2\,dy\,dx = \frac{1}{\pi}\int_{0}^{2\pi}\int_{0}^{1} r^2(1-r^2)^2\,r\,dr\,d\theta$$

$$= \frac{1}{\pi}\int_{0}^{2\pi}d\theta\int_{0}^{1}(r^3-2r^5+r^7)\,dr = \frac{1}{\pi}(2\pi)\left[\frac{1}{4}r^4-\frac{1}{3}r^6+\frac{1}{8}r^8\right]_0^1$$

$$=2\left(\frac{1}{24}\right)=\frac{1}{12}$$

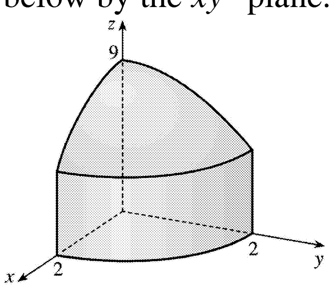
49. The triple integral will attain its maximum when the integrand $1-x^2-2y^2-3z^2$ is positive in the region E and negative everywhere else. For if E contains some region F where the integrand is negative, the integral could be increased by excluding F from E , and if E fails to contain some part G of the region where the integrand is positive, the integral could be increased by including G in E . So we require that $x^2+2y^2+3z^2\le 1$. This describes the region bounded by the ellipsoid $x^2+2y^2+3z^2=1$.

1. The region of integration is given in cylindrical coordinates by $E=\{(r,\theta,z)|0\le\theta\le 2\pi, 0\le r\le 4, r\le z\le 4\}$. This represents the solid region bounded below by the cone $z=r$ and above by the horizontal plane $z=4$.



$$\int_0^4\int_0^{2\pi}\int_r^4 r\,dz\,d\theta\,dr = \int_0^4\int_0^{2\pi}[rz]_{z=r}^{z=4}\,d\theta\,dr$$
$$=\int_0^4\int_0^{2\pi} r(4-r)\,d\theta\,dr$$
$$=\int_0^4 (4r-r^2)\,dr\int_0^{2\pi} d\theta$$
$$=\left[2r^2-\frac{1}{3}r^3\right]_0^4[\theta]_0^{2\pi}$$
$$=\left(32-\frac{64}{3}\right)(2\pi)=\frac{64\pi}{3}$$

2. The region of integration is given in cylindrical coordinates by $E=\left\{(r,\theta,z)|0\le\theta\le\pi/2, 0\le r\le 2, 0\le z\le 9-r^2\right\}$. This represents the solid region in the first octant enclosed by the circular cylinder $r=2$, bounded above by $z=9-r^2$, a circular paraboloid, and bounded below by the xy –plane.



$$\int_0^{\pi/2}\int_0^2\int_0^{9-r^2} r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^2[rz]_{z=0}^{z=9-r^2}\,dr\,d\theta$$

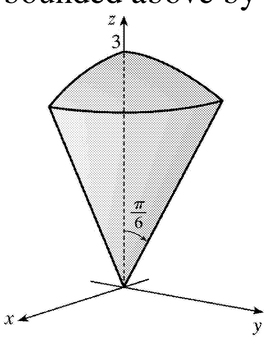
$$=\int_0^{\pi/2}\int_0^2 r(9-r^2)\,dr\,d\theta$$

$$=\int_0^{\pi/2} d\theta \int_0^2 (9r-r^3)\,dr$$

$$=[\theta]_0^{\pi/2}\left[\frac{9}{2}r^2-\frac{1}{4}r^4\right]_0^2$$

$$=\frac{\pi}{2}(18-4)=7\pi$$

3. The region of integration is given in spherical coordinates by $E=\{(\rho,\theta,\phi)|0\le\rho\le 3, 0\le\theta\le\pi/2, 0\le\phi\le\pi/6\}$. This represents the solid region in the first octant bounded above by the sphere $\rho=3$ and below by the cone $\phi=\pi/6$.



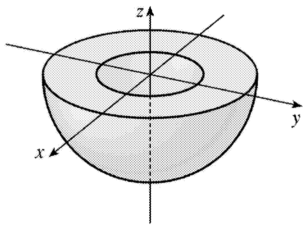
$$\int_0^{\pi/6}\int_0^{\pi/2}\int_0^3 \rho^2\sin\phi\,d\rho\,d\theta\,d\phi=\int_0^{\pi/6}\sin\phi\,d\phi\int_0^{\pi/2}d\theta\int_0^3\rho^2\,d\rho$$

$$=[-\cos\phi]_0^{\pi/6}[\theta]_0^{\pi/2}\left[\frac{1}{3}\rho^3\right]_0^3$$

$$=\left(1-\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{2}\right)(9)$$

$$=\frac{9\pi}{4}\left(2-\sqrt{3}\right)$$

4. The region of integration is given in spherical coordinates by $E=\{(\rho,\theta,\phi)|1\le\rho\le 2, 0\le\theta\le 2\pi, \pi/2\le\phi\le\pi\}$. This represents the solid region between the spheres $\rho=1$ and $\rho=2$ and below the xy –plane.



$$\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_1^2 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \int_0^{2\pi} d\theta \int_{\pi/2}^{\pi} \sin\phi\, d\phi \int_1^2 \rho^2 d\rho$$

$$= [\theta]_0^{2\pi} [-\cos\phi]_{\pi/2}^{\pi} \left[\frac{1}{3}\rho^3 \right]_1^2$$

$$= 2\pi(1)\left(\frac{7}{3}\right) = \frac{14\pi}{3}$$

5. The solid E is most conveniently described if we use cylindrical coordinates:
$E = \left\{ (r,\theta,z) | 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le 3, 0 \le z \le 2 \right\}$. Then

$$\iiint_E f(x,y,z)\, dV = \int_0^{\pi/2}\int_0^3\int_0^2 f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta \ .$$

6. The solid E is most conveniently described if we use spherical coordinates:
$E = \left\{ (\rho,\theta,\phi) | 1 \le \rho \le 2, \frac{\pi}{2} \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2} \right\}$. Then □ .

7. In cylindrical coordinates, E is given by $\{(r,\theta,z) | 0 \le \theta \le 2\pi, 0 \le r \le 4, -5 \le z \le 4\}$. So

$$\iiint_E \sqrt{x^2+y^2}\, dV = \int_0^{2\pi}\int_0^4\int_{-5}^4 \sqrt{r^2}\, r\, dz\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^4 r^2 dr \int_{-5}^4 dz$$

$$= [\theta]_0^{2\pi} \left[\frac{1}{3} r^3 \right]_0^4 [z]_{-5}^4 = (2\pi)\left(\frac{64}{3}\right)(9) = 384\pi$$

8. The paraboloid $z = 1 - x^2 - y^2$ intersects the xy –plane in the circle $x^2 + y^2 = r^2 = 1$ or $r = 1$, so in cylindrical coordinates, E is given by $\left\{ (r,\theta,z) \,\middle|\, 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le 1, 0 \le z \le 1 - r^2 \right\}$. Thus

$$\iiint_E (x^3 + xy^2)\, dV = \int_0^{\pi/2}\int_0^1\int_0^{1-r^2} (r^3\cos^3\theta + r^3\cos\theta\sin^2\theta)\, r\, dz\, dr\, d\theta$$

$$=\int_0^{\pi/2}\int_0^1\int_0^{1-r^2} r^4\cos\theta\,dz\,dr\,d\theta=\int_0^{\pi/2}\int_0^1 r^4\cos\theta\,[z]_{z=0}^{z=1-r^2}\,dr\,d\theta$$

$$=\int_0^{\pi/2}\int_0^1 r^4(1-r^2)\cos\theta\,dr\,d\theta=\int_0^{\pi/2}\cos\theta\left[\frac{1}{5}r^5-\frac{1}{7}r^7\right]_{r=0}^{r=1}d\theta$$

$$=\int_0^{\pi/2}\frac{2}{35}\cos\theta\,d\theta=\frac{2}{35}\,[\sin\theta]_0^{\pi/2}=\frac{2}{35}$$

9. In cylindrical coordinates E is bounded by the paraboloid $z=1+r^2$, the cylinder $r^2=5$ or $r=\sqrt{5}$, and the xy –plane, so E is given by $\left\{(r,\theta,z)\,|\,0\le\theta\le 2\pi, 0\le r\le\sqrt{5}, 0\le z\le 1+r^2\right\}$. Thus

$$\iiint_E e^z\,dV=\int_0^{2\pi}\int_0^{\sqrt{5}}\int_0^{1+r^2} e^z\,r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^{\sqrt{5}} r\left[e^z\right]_{z=0}^{z=1+r^2}dr\,d\theta=\int_0^{2\pi}\int_0^{\sqrt{5}} r(e^{1+r^2}-1)\,dr\,d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^{\sqrt{5}}\left(re^{1+r^2}-r\right)dr=2\pi\left[\frac{1}{2}e^{1+r^2}-\frac{1}{2}r^2\right]_0^{\sqrt{5}}=\pi(e^6-e-5)$$

10. In cylindrical coordinates E is bounded by the planes $z=0$, $z=r\cos\theta+r\sin\theta+3$ and the cylinders $r=2$ and $r=3$, so E is given by $\{(r,\theta,z)\,|\,0\le\theta\le 2\pi, 2\le r\le 3, 0\le z\le r\cos\theta+r\sin\theta+3\}$. Thus

$$\iiint_E x\,dV=\int_0^{2\pi}\int_2^3\int_0^{r\cos\theta+r\sin\theta+3}(r\cos\theta)\,r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_2^3 (r^2\cos\theta)\,]_{z=0}^{z=r\cos\theta+r\sin\theta+3}\,dr\,d\theta=\int_0^{2\pi}\int_2^3 (r^2\cos\theta)(r\cos\theta+r\sin\theta+3)\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_2^3 (r^3(\cos^2\theta+\cos\theta\sin\theta)+3r^2\cos\theta)\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[\frac{1}{4}r^4(\cos^2\theta+\cos\theta\sin\theta)+r^3\cos\theta\right]_{r=2}^{r=3}d\theta$$

$$=\int_0^{2\pi}\left[\left(\frac{81}{4}-\frac{16}{4}\right)(\cos^2\theta+\cos\theta\sin\theta)+(27-8)\cos\theta\right]d\theta$$

$$=\int_0^{2\pi}\left(\frac{65}{4}\left(\frac{1}{2}(1+\cos 2\theta)+\cos\theta\sin\theta\right)+19\cos\theta\right)d\theta$$

$$=\left[\frac{65}{8}\theta+\frac{65}{16}\sin 2\theta+\frac{65}{8}\sin^2\theta+19\sin\theta\right]_0^{2\pi}=\frac{65}{4}\pi$$

11. In cylindrical coordinates, E is bounded by the cylinder $r=1$, the plane $z=0$, and the cone $z=2r$. So $E=\{(r,\theta,z)|0\le\theta\le 2\pi, 0\le r\le 1, 0\le z\le 2r\}$ and

$$\iiint_E x^2\,dV=\int_0^{2\pi}\int_0^1\int_0^{2r} r^2\cos^2\theta\, r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^1\left[r^3\cos^2\theta\, z\right]_{z=0}^{z=2r}dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1 2r^4\cos^2\theta\,dr\,d\theta=\int_0^{2\pi}\left[\frac{2}{5}r^5\cos^2\theta\right]_{r=0}^{r=1}d\theta=\frac{2}{5}\int_0^{2\pi}\cos^2\theta\,d\theta$$

$$=\frac{2}{5}\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta=\frac{1}{5}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{2\pi}=\frac{2\pi}{5}$$

12. In cylindrical coordinates E is the solid region within the cylinder $r=1$ bounded above and below by the sphere $r^2+z^2=4$, so $E=\left\{(r,\theta,z)|0\le\theta\le 2\pi, 0\le r\le 1, -\sqrt{4-r^2}\le z\le\sqrt{4-r^2}\right\}$. Thus the volume is

$$\iiint_E dV=\int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^1 2r\sqrt{4-r^2}\,dr\,d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^1 2r\sqrt{4-r^2}\,dr=2\pi\left[-\frac{2}{3}(4-r^2)^{3/2}\right]_0^1=\frac{4}{3}\pi(8-3^{3/2})$$

13. **(a)** The paraboloids intersect when $x^2+y^2=36-3x^2-3y^2\Rightarrow x^2+y^2=9$, so the region of integration is $D=\left\{(x,y)\,|\,x^2+y^2\le 9\right\}$. Then, in cylindrical coordinates, $E=\left\{(r,\theta,z)\,|\,r^2\le z\le 36-3r^2, 0\le r\le 3, 0\le\theta\le 2\pi\right\}$ and

$$V=\int_0^{2\pi}\int_0^3\int_{r^2}^{36-3r^2} r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^3\left(36r-4r^3\right)dr\,d\theta$$

$$=\int_0^{2\pi}\left[18r^2-r^4\right]_{r=0}^{r=3}d\theta=\int_0^{2\pi}81\,d\theta=162\pi$$

(b) For constant density K , $m=KV=162\pi K$ from part (a). Since the region is homogeneous and symmetric,

$M_{yz}=M_{xz}=0$ and

$$M_{xy}=\int_0^{2\pi}\int_0^3\int_{r^2}^{36-3r^2}(zK)\,r\,dz\,dr\,d\theta=K\int_0^{2\pi}\int_0^3 r\left[\frac{1}{2}z^2\right]_{z=r^2}^{z=36-3r^2}dr\,d\theta$$

$$=\frac{K}{2}\int_0^{2\pi}\int_0^3 r((36-3r^2)^2-r^4)\,dr\,d\theta=\frac{K}{2}\int_0^{2\pi}d\theta\int_0^3(8r^5-216r^3+1296r)\,dr$$

$$=\frac{K}{2}(2\pi)\left[\frac{8}{6}r^6-\frac{216}{4}r^4+\frac{1296}{2}r^2\right]_0^3=\pi K(2430)=2430\pi K$$

Thus $\left(\overline{x},\overline{y},\overline{z}\right)=\left(\frac{M_{yz}}{m},\frac{M_{xz}}{m},\frac{M_{xy}}{m}\right)=\left(0,0,\frac{2430\pi K}{162\pi K}\right)=(0,0,15)$.

14. **(a)** $V=\int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}}r\,dz\,dr\,d\theta$

$$=4\int_0^{\pi/2}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}}r\,dz\,dr\,d\theta$$

$$=4\int_0^{\pi/2}\int_0^{a\cos\theta}r\sqrt{a^2-r^2}\,dr\,d\theta$$

$$=-\frac{4}{3}\int_0^{\pi/2}\left[\left(a^2-r^2\right)^{3/2}\right]_{r=0}^{r=a\cos\theta}d\theta$$

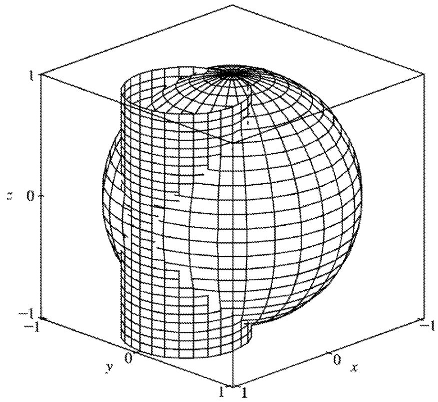
$$=-\frac{4}{3}\int_0^{\pi/2}\left[\left(a^2-a^2\cos^2\theta\right)^{3/2}-a^3\right]d\theta$$

$$=-\frac{4}{3}\int_0^{\pi/2}\left[\left(a^2\sin^2\theta\right)^{3/2}-a^3\right]d\theta$$

$$=-\frac{4}{3}\int_0^{\pi/2}\left(a^3\sin^3\theta-a^3\right)d\theta$$

$$=-\frac{4a^3}{3}\int_0^{\pi/2}\left[\sin\theta\left(1-\cos^2\theta\right)-1\right]d\theta$$

(b)



$$=-\frac{4a^3}{3}\left[-\cos\theta+\frac{1}{3}\cos^3\theta-\theta\right]_0^{\pi/2}=-\frac{4a^3}{3}\left(-\frac{\pi}{2}+\frac{2}{3}\right)=\frac{2}{9}a^3(3\pi-4)$$

To plot the cylinder and the sphere on the same screen in Maple, we can use the sequence of commands

```
sphere:=plot3d(1,theta=0..2*Pi,phi=0..Pi,coords=spherical):
cylinder:=plot3d(,
theta=0..2*Pi,z=–1..1,coords=cylindrical):
with(plots): display3d(sphere,cylinder});
```

In Mathematica, we can use

```
sphere==SphericalPlot3d[1,{theta,0,2Pi},{phi,0,Pi}],
cylinder=ParametricPlot3d[{Sin[theta],Cos[theta],z},
{theta,0,2Pi},{z,–1,1}]
Show[{sphere, cylinder}]
```

15. The paraboloid $z=4x^2+4y^2$ intersects the plane $z=a$ when $a=4x^2+4y^2$ or $x^2+y^2=\frac{1}{4}a$. So, in cylindrical coordinates, $E=\left\{(r,\theta,z)|0\le r\le\frac{1}{2}\sqrt{a},0\le\theta\le 2\pi,4r^2\le z\le a\right\}$. Thus

$$m=\int_0^{2\pi}\int_0^{\sqrt{a}/2}\int_{4r^2}^{a}Kr\,dz\,dr\,d\theta=K\int_0^{2\pi}\int_0^{\sqrt{a}/2}(ar-4r^3)\,dr\,d\theta$$

$$=K\int_0^{2\pi}\left[\frac{1}{2}ar^2-r^4\right]_{r=0}^{r=\sqrt{a}/2}d\theta=K\int_0^{2\pi}\frac{1}{16}a^2\,d\theta=\frac{1}{8}a^2\pi K$$

Since the region is homogeneous and symmetric, $M_{yz}=M_{xz}=0$ and

$$M_{xy} = \int_0^{2\pi}\int_0^{\sqrt{a}/2}\int_{4r^2}^{a} Krz\,dz\,dr\,d\theta = K\int_0^{2\pi}\int_0^{\sqrt{a}/2}\left(\frac{1}{2}a^2r - 8r^5\right)dr\,d\theta$$

$$= K\int_0^{2\pi}\left[\frac{1}{4}a^2r^2 - \frac{4}{3}r^6\right]_{r=0}^{r=\sqrt{a}/2} d\theta = K\int_0^{2\pi}\frac{1}{24}a^3\,d\theta = \frac{1}{12}a^3\pi K$$

Hence $(\bar{x},\bar{y},\bar{z}) = \left(0,0,\frac{2}{3}a\right)$.

16. Since density is proportional to the distance from the z –axis, we can say $\rho(x,y,z) = K\sqrt{x^2+y^2}$. Then

$$m = 2\int_0^{2\pi}\int_0^{a}\int_0^{\sqrt{a^2-r^2}} Kr^2\,dz\,dr\,d\theta = 2K\int_0^{2\pi}\int_0^{a} r^2\sqrt{a^2-r^2}\,dr\,d\theta$$

$$= 2K\int_0^{2\pi}\left[\frac{1}{8}r(2r^2-a^2)\sqrt{a^2-r^2} + \frac{1}{8}a^4\sin^{-1}(r/a)\right]_{r=0}^{r=a} d\theta$$

$$= 2K\int_0^{2\pi}\left[\left(\frac{1}{8}a^4\right)\left(\frac{\pi}{2}\right)\right]d\theta = \frac{1}{4}a^4\pi^2K .$$

17. In spherical coordinates, B is represented by $\{(\rho,\theta,\phi) \mid 0\le\rho\le 1, 0\le\theta\le 2\pi, 0\le\phi\le\pi\}$. Thus

$$\iiint_B (x^2+y^2+z^2)\,dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{1}(\rho^2)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^{1}\rho^4\,d\rho$$

$$= [-\cos\phi]_0^{\pi}[\theta]_0^{2\pi}\left[\frac{1}{5}\rho^5\right]_0^1 = (2)(2\pi)\left(\frac{1}{5}\right) = \frac{4\pi}{5}$$

18. In spherical coordinates, H is represented by $\left\{(\rho,\theta,\phi) \;\middle|\; 0\le\rho\le 1, 0\le\theta\le 2\pi, 0\le\phi\le\frac{\pi}{2}\right\}$. Thus

$$\iiint_H (x^2+y^2)\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{1}(\rho^2\sin^2\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}d\theta\int_0^{\pi/2}\sin^3\phi\,d\phi\int_0^{1}\rho^4\,d\rho$$

$$= [\theta]_0^{2\pi}\left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^{\pi/2}\left[\frac{1}{5}\rho^5\right]_0^1 = \frac{4\pi}{15}$$

19. In spherical coordinates, E is represented by

$\left\{ (\rho,\theta,\phi) \,\middle|\, 1 \le \rho \le 2, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{2} \right\}$. Thus

$$\iiint_E z\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 (\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\int_0^{\pi/2}d\theta\int_1^2\rho^3\,d\rho = \left[\frac{1}{2}\sin^2\phi\right]_0^{\pi/2}\left[\theta\right]_0^{\pi/2}\left[\frac{1}{4}\rho^4\right]_1^2$$

$$= \left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)\left(\frac{15}{4}\right) = \frac{15\pi}{16}$$

20.

$$\iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 e^{\sqrt{\rho^2}}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \rho^2 e^{\rho}\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{\pi/2}d\theta\int_0^{\pi/2}\sin\phi\,d\phi\int_0^3\rho^2 e^{\rho}\,d\rho = \left[\theta\right]_0^{\pi/2}\left[-\cos\phi\right]_0^{\pi/2}\left[(\rho^2-2\rho+2)e^{\rho}\right]_0^3$$

[integrate by parts twice]

$$= \frac{\pi}{2}(0+1)(5e^3-2) = \frac{\pi}{2}(5e^3-2)$$

21.

$$\iiint_E x^2\,dV = \int_0^{\pi}\int_0^{\pi}\int_3^4 (\rho\sin\phi\cos\theta)^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{\pi}\cos^2\theta\,d\theta\int_0^{\pi}\sin^3\phi\,d\phi\int_3^4\rho^4\,d\rho$$

$$= \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi}\left[-\frac{1}{3}(2+\sin^2\phi)\cos\phi\right]_0^{\pi}\left[\frac{1}{5}\rho^5\right]_3^4$$

$$= \left(\frac{\pi}{2}\right)\left(\frac{2}{3}+\frac{2}{3}\right)\frac{1}{5}(4^5-3^5) = \frac{1562}{15}\pi$$

22.

$$\iiint_E xyz\,dV = \int_0^{\pi/3}\int_0^{2\pi}\int_2^4 (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$=\int_0^{\pi/3}\sin^3\phi\cos\phi\,d\phi\int_0^{2\pi}\sin\theta\cos\theta\,d\theta\int_2^4\rho^5\,d\rho=\left[\frac{1}{4}\sin^4\phi\right]_0^{\pi/3}\left[\frac{1}{2}\sin^2\theta\right]_0^{2\pi}\left[\frac{1}{6}\rho^6\right]_2^4=0$$

23. Since $\rho=4\cos\phi$ implies $\rho^2=4\rho\cos\phi$, the equation is that of a sphere of radius 2 with center at $(0,0,2)$. Thus

$$V=\int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\int_0^{2\pi}\int_0^{\pi/3}\left[\frac{1}{3}\rho^3\right]_{\rho=0}^{\rho=4\cos\phi}\sin\phi\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi/3}\left(\frac{64}{3}\cos^3\phi\right)\sin\phi\,d\phi\,d\theta=\int_0^{2\pi}\left[-\frac{16}{3}\cos^4\phi\right]_{\phi=0}^{\phi=\pi/3}d\theta$$

$$=\int_0^{2\pi}-\frac{16}{3}\left(\frac{1}{16}-1\right)d\theta=5\theta\Big]_0^{2\pi}=10\pi$$

24. In spherical coordinates, the sphere $x^2+y^2+z^2=4$ is equivalent to $\rho=2$ and the cone $z=\sqrt{x^2+y^2}$ is represented by $\phi=\frac{\pi}{4}$. Thus, the solid is given by $\left\{(\rho,\theta,\phi)\,\middle|\,0\le\rho\le 2, 0\le\theta\le 2\pi, \frac{\pi}{4}\le\phi\le\frac{\pi}{2}\right\}$ and

$$V=\int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^2\rho^2\sin\phi\,d\rho\,d\theta\,d\phi=\int_{\pi/4}^{\pi/2}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^2\rho^2\,d\rho$$

$$=[-\cos\phi]_{\pi/4}^{\pi/2}[\theta]_0^{2\pi}\left[\frac{1}{3}\rho^3\right]_0^2=\left(\frac{\sqrt{2}}{2}\right)(2\pi)\left(\frac{8}{3}\right)=\frac{8\sqrt{2}\,\pi}{3}$$

25. By the symmetry of the region, $M_{xy}=0$ and $M_{yz}=0$. Assuming constant density K,

$$m=\iiint_E KV=K\int_0^{\pi}\int_0^{\pi}\int_3^4\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=K\int_0^{\pi}d\theta\int_0^{\pi}\sin\phi\,d\phi\int_3^4\rho^2\,d\rho$$

$$=K\pi[-\cos\phi]_0^{\pi}\left[\frac{1}{3}\rho^3\right]_3^4=2K\pi\cdot\frac{37}{3}=\frac{74}{3}\pi K$$

and

$$M_{xz}=\iiint_E yK\,dV=K\int_0^{\pi}\int_0^{\pi}\int_3^4(\rho\sin\phi\sin\theta)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=K\int_0^{\pi} \sin\theta\, d\theta \int_0^{\pi} \sin^2\phi\, d\phi \int_3^4 \rho^3 d\rho$$

$$=K\left[-\cos\theta\right]_0^{\pi}\left[\frac{1}{2}\phi-\frac{1}{4}\sin 2\phi\right]_0^{\pi}\left[\frac{1}{4}\rho^4\right]_3^4$$

$$=K(2)\left(\frac{\pi}{2}\right)\frac{1}{4}(256-81)=\frac{175}{4}\pi K$$

Thus the centroid is $(\overline{x},\overline{y},\overline{z})=\left(\frac{M_{yz}}{m},\frac{M_{xz}}{m},\frac{M_{xy}}{m}\right)=\left(0,\frac{175\pi K/4}{74\pi K/3},0\right)=\left(0,\frac{525}{296},0\right)$.

26. **(a)** Placing the center of the base at $(0,0,0)$, $\rho(x,y,z)=K\sqrt{x^2+y^2+z^2}$ is the density function. So

$$m=\int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^3\sin\phi\, d\rho\, d\phi\, d\theta=K\int_0^{2\pi} d\theta\int_0^{\pi/2}\sin\phi\, d\phi\int_0^a\rho^3 d\rho$$

$$=K\left[\theta\right]_0^{2\pi}\left[-\cos\phi\right]_0^{\pi/2}\left[\frac{1}{4}\rho^4\right]_0^a=K(2\pi)(1)\left(\frac{1}{4}a^4\right)=\frac{1}{2}\pi Ka^4$$

(b) By the symmetry of the problem $M_{yz}=M_{xz}=0$. Then

$$M_{xy}=\int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^4\sin\phi\cos\phi\, d\rho\, d\phi\, d\theta=K\int_0^{2\pi} d\theta\int_0^{\pi/2}\sin\phi\cos\phi\, d\phi\int_0^a\rho^4 d\rho$$

$$=K\left[\theta\right]_0^{2\pi}\left[\frac{1}{2}\sin^2\phi\right]_0^{\pi/2}\left[\frac{1}{5}\rho^5\right]_0^a=K(2\pi)\left(\frac{1}{2}\right)\left(\frac{1}{5}a^5\right)=\frac{1}{5}\pi Ka^5$$

Hence $(\overline{x},\overline{y},\overline{z})=\left(0,0,\frac{2}{5}a\right)$.

(c)

$$I_z=\int_0^{2\pi}\int_0^{\pi/2}\int_0^a(K\rho^3\sin\phi)(\rho^2\sin^2\phi)\, d\rho\, d\phi\, d\theta=K\int_0^{2\pi} d\theta\int_0^{\pi/2}\sin^3\phi\, d\phi\int_0^a\rho^5 d\rho$$

$$=K\left[\theta\right]_0^{2\pi}\left[-\cos\phi+\frac{1}{3}\cos^3\phi\right]_0^{\pi/2}\left[\frac{1}{6}\rho^6\right]_0^a=K(2\pi)\left(\frac{2}{3}\right)\left(\frac{1}{6}a^6\right)=\frac{2}{9}\pi Ka^6$$

27. **(a)** The density function is $\rho(x,y,z)=K$, a constant, and by the symmetry of the problem $M_{xz}=M_{yz}=0$. Then $M_{xy}=\int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^3\sin\phi\cos\phi\, d\rho\, d\phi\, d\theta=\frac{1}{2}\pi Ka^4\int_0^{\pi/2}\sin\phi\cos\phi\, d\phi=\frac{1}{8}\pi Ka^4$. But

the mass is K (volume of the hemisphere)$=\frac{2}{3}\pi Ka^3$, so the centroid is $\left(0,0,\frac{3}{8}a\right)$.

(b) Place the center of the base at $(0,0,0)$; the density function is $\rho(x,y,z)=K$. By symmetry, the moments of inertia about any two such diameters will be equal, so we just need to find I_x :

$$\begin{aligned}
I_x &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a}\left(K\rho^2\sin\phi\right)\rho^2\left(\sin^2\phi\sin^2\theta+\cos^2\phi\right)d\rho\,d\phi\,d\theta\\
&= K\int_0^{2\pi}\int_0^{\pi/2}\left(\sin^3\phi\sin^2\theta+\sin\phi\cos^2\phi\right)\left(\frac{1}{5}a^5\right)d\phi\,d\theta\\
&= \frac{1}{5}Ka^5\int_0^{2\pi}\left[\sin^2\theta\left(-\cos\phi+\frac{1}{3}\cos^3\phi\right)+\left(-\frac{1}{3}\cos^3\phi\right)\right]_{\phi=0}^{\phi=\pi/2}d\theta\\
&= \frac{1}{5}Ka^5\int_0^{2\pi}\left[\frac{2}{3}\sin^2\theta+\frac{1}{3}\right]d\theta=\frac{1}{5}Ka^5\left[\frac{2}{3}\left(\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right)+\frac{1}{3}\theta\right]_0^{2\pi}\\
&= \frac{1}{5}Ka^5\left[\frac{2}{3}(\pi-0)+\frac{1}{3}(2\pi-0)\right]=\frac{4}{15}Ka^5\pi
\end{aligned}$$

28. Place the center of the base at $(0,0,0)$, then the density is $\rho(x,y,z)=Kz$, K a constant. Then

$$\begin{aligned}
m &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a}(K\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\\
&= 2\pi K\int_0^{\pi/2}\cos\phi\sin\phi\cdot\frac{1}{4}a^4\,d\phi\\
&= \frac{1}{2}\pi Ka^4\left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/2}=\frac{\pi}{4}Ka^4
\end{aligned}$$

By the symmetry of the problem $M_{xz}=M_{yz}=0$, and

$$\begin{aligned}
M_{xy} &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a}K\rho^4\cos^2\phi\sin\phi\,d\rho\,d\phi\,d\theta\\
&= \frac{2}{5}\pi Ka^5\int_0^{\pi/2}\cos^2\phi\sin\phi\,d\phi\\
&= \frac{2}{5}\pi Ka^5\left[-\frac{1}{3}\cos^3\theta\right]_0^{\pi/2}=\frac{2}{15}\pi Ka^5
\end{aligned}$$

Hence $(\overline{x},\overline{y},\overline{z})=\left(0,0,\frac{8}{15}a\right)$.

29. In spherical coordinates $z=\sqrt{x^2+y^2}$ becomes $\cos\phi=\sin\phi$ or $\phi=\frac{\pi}{4}$. Then

$$V=\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\int_0^{2\pi}d\theta\int_0^{\pi/4}\sin\phi\,d\phi\int_0^1\rho^2\,d\rho=\frac{1}{3}\pi\left(2-\sqrt{2}\right),$$

$$M_{xy}=\int_0^{2\pi}\int_0^{\pi/4}\int_0^1\rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta=2\pi\left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/4}\left(\frac{1}{4}\right)=\frac{\pi}{8}$$ and by symmetry $M_{yz}=M_{xz}=0$

. Hence $\left(\bar{x},\bar{y},\bar{z}\right)=\left(0,0,\frac{3}{8\left(2-\sqrt{2}\right)}\right)$.

30. Place the center of the sphere at $(0,0,0)$, let the diameter of intersection be along the z –axis, one of the planes be the xz –plane and the other be the plane whose angle with the xz –plane is $\theta=\frac{\pi}{6}$. Then in spherical coordinates the volume is given by

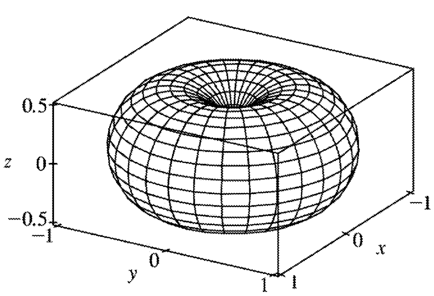
$$V=\int_0^{\pi/6}\int_0^{\pi}\int_0^a\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\int_0^{\pi/6}d\theta\int_0^{\pi}\sin\phi\,d\phi\int_0^a\rho^2\,d\rho=\frac{\pi}{6}(2)\left(\frac{1}{3}a^3\right)=\frac{1}{9}\pi a^3 .$$

31. In cylindrical coordinates the paraboloid is given by $z=r^2$ and the plane by $z=2r\sin\theta$ and they intersect in the circle $r=2\sin\theta$. Then $\iiint_E z\,dV=\int_0^{\pi}\int_0^{2\sin\theta}\int_{r^2}^{2r\sin\theta} rz\,dz\,dr\,d\theta=\frac{5\pi}{6}$ [using a CAS].

32. **(a)** The region enclosed by the torus is $\{(\rho,\theta,\phi)\,|\,0\le\theta\le 2\pi, 0\le\phi\le\pi, 0\le\rho\le\sin\phi\}$, so its volume is

$$V=\int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=2\pi\int_0^{\pi}\frac{1}{3}\sin^4\phi\,d\phi=\frac{2}{3}\pi\left[\frac{3}{8}\phi-\frac{1}{4}\sin 2\phi+\frac{1}{16}\sin 4\phi\right]_0^{\pi}=\frac{1}{4}\pi^2 .$$

(b) In Maple, we can plot the torus using the plots[sphereplot] command, or with the coords=spherical option in a regular plot command. In Mathematica, use ParametricPlot3d.



33. The region E of integration is the region above the paraboloid $z=x^2+y^2$, or $z=r^2$, and below the

paraboloid $z=2-x^2-y^2$, or $z=2-r^2$. Also, we have $-1\leq x\leq 1$ with $-\sqrt{1-x^2}\leq y\leq\sqrt{1-x^2}$ which describes the unit circle in the xy –plane. Thus,

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{2-x^2-y^2}(x^2+y^2)^{3/2}\,dz\,dy\,dx=\int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2}(r^2)^{3/2}\,r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1\left[r^4z\right]_{z=r^2}^{z=2-r^2}dr\,d\theta=\int_0^{2\pi}\int_0^1(2r^4-r^6-r^6)\,dr\,d\theta=\int_0^{2\pi}\left(\frac{2}{5}-\frac{2}{7}\right)d\theta=\frac{8\pi}{35}$$

34. The region E of integration is the region above the paraboloid $z=x^2+y^2=r^2$ and below the cone $z=\sqrt{x^2+y^2}=r$. Also, we have $0\leq y\leq 1$, $0\leq x\leq\sqrt{1-y^2}$ which is equivalent to $0\leq\theta\leq\frac{\pi}{2}$, $0\leq r\leq 1$. Thus

$$\int_0^1\int_0^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}}xyz\,dz\,dx\,dy=\int_0^{\pi/2}\int_0^1\int_{r^2}^{r}r^2\cos\theta\sin\theta\,zr\,dz\,dr\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}\int_0^1 r^3\cos\theta\sin\theta\left[z^2\right]_{z=r^2}^{z=r}dr\,d\theta=\frac{1}{2}\int_0^{\pi/2}\int_0^1(r^5-r^7)\cos\theta\sin\theta\,dr\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}\left[\frac{1}{6}r^6-\frac{1}{8}r^8\right]_{r=0}^{r=1}\cos\theta\sin\theta\,d\theta=\frac{1}{2}\int_0^{\pi/2}\frac{1}{24}\cos\theta\sin\theta\,d\theta$$

$$=\frac{1}{48}\int_0^{\pi/2}\frac{1}{2}\sin 2\theta\,d\theta=\frac{1}{96}\left[-\frac{1}{2}\cos 2\theta\right]_0^{\pi/2}=\frac{1}{96}$$

35. The region of integration E is the top half of the sphere $x^2+y^2+z^2=9$. So

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2-y^2}}z\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx=\iiint_E z\sqrt{x^2+y^2+z^2}\,dV$$

$$=\int_0^{2\pi}\int_0^{\pi/2}\int_0^3\left(\rho^2\cos\phi\right)\left(\rho^2\sin\phi\right)d\rho\,d\phi\,d\theta=\int_0^{2\pi}d\theta\int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\int_0^3\rho^4\,d\rho$$

$$=[\theta]_0^{2\pi}\left[\frac{1}{2}\sin^2\phi\right]_0^{\pi/2}\left[\frac{1}{5}\rho^5\right]_0^3=(2\pi)\left(\frac{1}{2}\right)\left(\frac{243}{5}\right)=\frac{243}{5}\pi$$

36. The region of integration E is the region above the cone $z=\sqrt{x^2+y^2}$ and below the sphere

$x^2+y^2+z^2=18$ in the first octant. Because E is in the first octant we have $0\le\theta\le\frac{\pi}{2}$. The cone has equation $\phi=\frac{\pi}{4}$ (as in Example 4) and so $0\le\phi\le\frac{\pi}{4}$. Also $0\le\rho\le\sqrt{18}=3\sqrt{2}$. So the integral becomes

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{3\sqrt{2}}\rho^4\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/2}d\theta\int_0^{\pi/4}\sin\phi\,d\phi\int_0^{3\sqrt{2}}\rho^4\,d\rho$$
$$=[\theta]_0^{\pi/2}[-\cos\phi]_0^{\pi/4}\left[\frac{1}{5}\rho^5\right]_0^{3\sqrt{2}}$$
$$=\left(\frac{\pi}{2}\right)\left(1-\frac{\sqrt{2}}{2}\right)\left(\frac{972\sqrt{2}}{5}\right)=486\pi\left(\frac{\sqrt{2}-1}{5}\right)$$

37. If E is the solid enclosed by the surface $\rho=1+\frac{1}{5}\sin 6\theta\sin 5\phi$, it can be described in spherical coordinates as $E=\left\{(\rho,\theta,\phi)\,|\,0\le\rho\le 1+\frac{1}{5}\sin 6\theta\sin 5\phi, 0\le\theta\le 2\pi, 0\le\phi\le\pi\right\}$. Its volume is given by $V(E)=\iiint_E dV=\int_0^{\pi}\int_0^{2\pi}\int_0^{1+(\sin 6\theta\sin 5\phi)/5}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi=\frac{136\pi}{99}$.

38. The given integral is equal to

$\lim_{R\to\infty}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\rho e^{-\rho^2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\lim_{R\to\infty}\left(\int_0^{2\pi}d\theta\right)\left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_0^{R}\rho^3 e^{-\rho^2}d\rho\right)$. Now use integration

by parts with $u=\rho^2$, $dv=\rho e^{-\rho^2}d\rho$ to get

$$\lim_{R\to\infty}2\pi(2)\left(\rho^2\left(-\frac{1}{2}\right)e^{-\rho^2}\Big]_0^R-\int_0^R 2\rho\left(-\frac{1}{2}\right)e^{-\rho^2}d\rho\right)=\lim_{R\to\infty}4\pi\left(-\frac{1}{2}R^2e^{-R^2}+\left[-\frac{1}{2}e^{-\rho^2}\right]_0^R\right)$$
$$=4\pi\lim_{R\to\infty}\left[-\frac{1}{2}R^2e^{-R^2}-\frac{1}{2}e^{-R^2}+\frac{1}{2}\right]$$
$$=4\pi\left(\frac{1}{2}\right)=2\pi$$

(Note that $R^2e^{-R^2}\to 0$ as $R\to\infty$ by l'Hospital's Rule.)

39. **(a)** From the diagram, $z=r\cot\phi_0$ to

$z=\sqrt{a^2-r^2}$, $r=0$ to $r=a\sin\phi_0$ (or use $a^2-r^2=r^2\cot^2\phi_0$). Thus

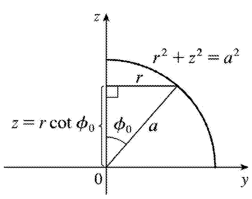
$$V=\int_0^{2\pi}\int_0^{a\sin\phi_0}\int_{r\cot\phi_0}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$$

$$=2\pi\int_0^{a\sin\phi_0}\left(r\sqrt{a^2-r^2}-r^2\cot\phi_0\right)dr$$

$$=\frac{2\pi}{3}\left[-(a^2-r^2)^{3/2}-r^3\cot\phi_0\right]_0^{a\sin\phi_0}$$

$$=\frac{2\pi}{3}\left[-\left(a^2-a^2\sin^2\phi_0\right)^{3/2}-a^3\sin^3\phi_0\cot\phi_0+a^3\right]$$

$$=\frac{2}{3}\pi a^3\left[1-\left(\cos^3\phi_0+\sin^2\phi_0\cos\phi_0\right)\right]=\frac{2}{3}\pi a^3(1-\cos\phi_0)$$

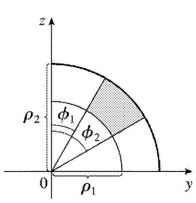


(b) The wedge in question is the shaded area rotated from $\theta=\theta_1$ to $\theta=\theta_2$. Letting

$V_{ij}=$ volume of the region bounded by the sphere of radius ρ_i

and the cone with angle ϕ_j ($\theta=\theta_1$ to θ_2)

and letting V be the volume of the wedge, we have



$$V=\left(V_{22}-V_{21}\right)-\left(V_{12}-V_{11}\right)$$

$$=\frac{1}{3}\left(\theta_2-\theta_1\right)\left[\rho_2^3\left(1-\cos\phi_2\right)-\rho_2^3\left(1-\cos\phi_1\right)-\rho_1^3\left(1-\cos\phi_2\right)+\rho_1^3\left(1-\cos\phi_1\right)\right]$$
$$=\frac{1}{3}\left(\theta_2-\theta_1\right)\left[\left(\rho_2^3-\rho_1^3\right)\left(1-\cos\phi_2\right)-\left(\rho_2^3-\rho_1^3\right)\left(1-\cos\phi_1\right)\right]$$
$$=\frac{1}{3}\left(\theta_2-\theta_1\right)\left[\left(\rho_2^3-\rho_1^3\right)\left(\cos\phi_1-\cos\phi_2\right)\right]$$

Or: Show that $V=\int_{\theta_1}^{\theta_2}\int_{\rho_1\sin\phi_1}^{\rho_2\sin\phi_2}\int_{r\cot\phi_2}^{r\cot\phi_1} r\,dz\,dr\,d\theta$.

(c) By the Mean Value Theorem with $f(\rho)=\rho^3$ there exists some $\tilde{\rho}$ with $\rho_1\le\tilde{\rho}\le\rho_2$ such that $f\left(\rho_2\right)-f\left(\rho_1\right)=f'\left(\tilde{\rho}\right)\left(\rho_2-\rho_1\right)$ or $\rho_1^3-\rho_2^3=3\tilde{\rho}^2\,\Delta\rho$. Similarly there exists $\tilde{\phi}$ with $\phi_1\le\tilde{\phi}\le\phi_2$ such that $\cos\phi_2-\cos\phi_1=\left(-\sin\tilde{\phi}\right)\Delta\phi$. Substituting into the result from (b) gives $\Delta V=(\tilde{\rho}^2\,\Delta\rho)(\theta_2-\theta_1)(\sin\tilde{\phi})\,\Delta\phi=\tilde{\rho}^2\,\sin\tilde{\phi}\,\Delta\rho\,\Delta\phi\,\Delta\theta$.

40. **(a)** The mountain comprises a solid conical region C . The work done in lifting a small volume of material ΔV with density $g(P)$ to a height $h(P)$ above sea level is $h(P)g(P)\,\Delta V$. Summing over the whole mountain we get $W=\iiint_C h(P)g(P)\,\Delta V$.

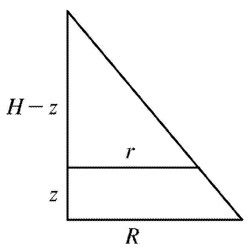
(b) Here C is a solid right circular cone with radius $R=62,000$ ft, height $H=12,400$ ft, and density $g(P)=200\text{ lb}/\text{ft}^3$ at all points P in C . We use cylindrical coordinates:

$$W=\int_0^{2\pi}\int_0^H\int_0^{R(1-z/H)} z\cdot 200r\,dr\,dz\,d\theta$$
$$=2\pi\int_0^H 200z\left[\frac{1}{2}r^2\right]_{r=0}^{r=R(1-z/H)}dz$$
$$=400\pi\int_0^H z\,\frac{R^2}{2}\left(1-\frac{z}{H}\right)^2dz$$
$$=200\pi R^2\int_0^H\left(z-\frac{2z^2}{H}+\frac{z^3}{H^2}\right)dz$$

$$= 200\pi R^2 \left[\frac{z^2}{2} - \frac{2z^3}{3H} + \frac{z^4}{4H^2} \right]_0^H$$

$$= 200\pi R^2 \left(\frac{H^2}{2} - \frac{2H^2}{3} + \frac{H^2}{4} \right) = \frac{50}{3}\pi R^2 H^2$$

$$= \frac{50}{3}\pi (62{,}000)^2 (12{,}400)^2 \approx 3.1 \times 10^{19} \text{ ft–lb}$$



$$\frac{r}{R} = \frac{H-z}{H} = 1 - \frac{z}{H}$$

1. $x=u+4v$, $y=3u-2v$.

The Jacobian is $\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}=\begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix}=1(-2)-4(3)=-14$.

2. $\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}=\begin{vmatrix} 2u & -2v \\ 2u & 2v \end{vmatrix}=4uv-(-4uv)=8uv$

3.

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}=\begin{vmatrix} \dfrac{v}{(u+v)^2} & -\dfrac{u}{(u+v)^2} \\ -\dfrac{v}{(u-v)^2} & \dfrac{u}{(u-v)^2} \end{vmatrix}=\frac{uv}{(u+v)^2(u-v)^2}-\frac{uv}{(u+v)^2(u-v)^2}=0$$

4. $\dfrac{\partial(x,y)}{\partial(\alpha,\beta)}=\begin{vmatrix} \partial x/\partial\alpha & \partial x/\partial\beta \\ \partial y/\partial\alpha & \partial y/\partial\beta \end{vmatrix}=\begin{vmatrix} \sin\beta & \alpha\cos\beta \\ \cos\beta & -\alpha\sin\beta \end{vmatrix}=-\alpha\sin^2\beta-\alpha\cos^2\beta=-\alpha$

5.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}=\begin{vmatrix} v & u & 0 \\ 0 & w & v \\ w & 0 & u \end{vmatrix}$$

$$=v\begin{vmatrix} w & v \\ 0 & u \end{vmatrix}-u\begin{vmatrix} 0 & v \\ w & u \end{vmatrix}+0\begin{vmatrix} 0 & w \\ w & 0 \end{vmatrix}=v(uw-0)-u(0-vw)=2uvw$$

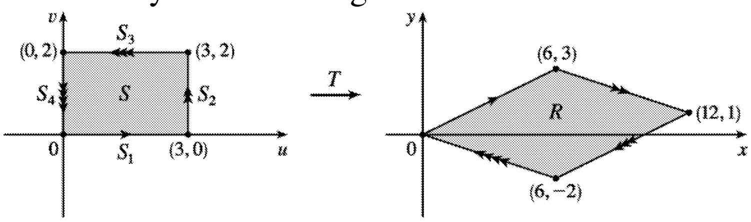
6.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} e^{u-v} & -e^{u-v} & 0 \\ e^{u+v} & e^{u+v} & 0 \\ e^{u+v+w} & e^{u+v+w} & e^{u+v+w} \end{vmatrix}=e^{u+v+w}\begin{vmatrix} e^{u-v} & -e^{u-v} \\ e^{u+v} & e^{u+v} \end{vmatrix}$$

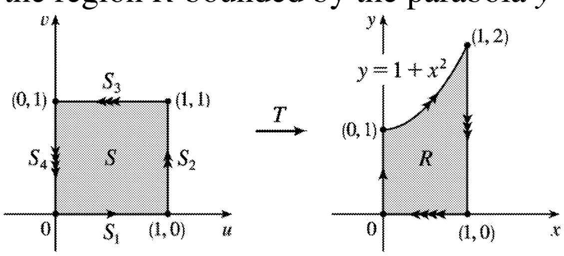
$$=e^{u+v+w}\left(e^{u-v}e^{u+v}+e^{u-v}e^{u+v}\right)=e^{u+v+w}\left(2e^{2u}\right)=2e^{3u+v+w}$$

7. The transformation maps the boundary of S to the boundary of the image R , so we first look at side S_1 in the uv –plane. S_1 is described by $v=0$ ($0\le u\le 3$), so $x=2u+3v=2u$ and $y=u-v=u$. Eliminating u , we have $x=2y$, $0\le x\le 6$. S_2 is the line segment $u=3$, $0\le v\le 2$, so $x=6+3v$ and $y=3-v$. Then $v=3-y \Rightarrow x=6+3(3-y)=15-3y$, $6\le x\le 12$. S_3 is the line segment $v=2$, $0\le u\le 3$, so $x=2u+6$ and $y=u-2$, giving $u=y+2\Rightarrow x=2y+10$, $6\le x\le 12$. Finally, S_4 is the segment $u=0$, $0\le v\le 2$, so $x=3v$ and $y=-v\Rightarrow x=-3y$, $0\le x\le 6$. The image of set S is the region R shown in the xy –plane, a parallelogram

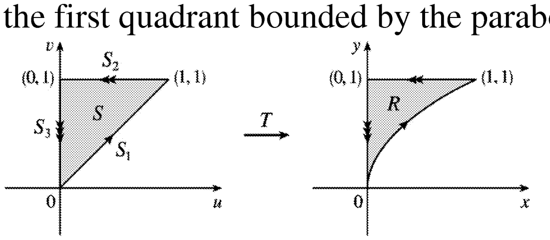
bounded by these four segments.



8. S_1 is the line segment $v=0$, $0\leq u\leq 1$, so $x=v=0$ and $y=u(1+v^2)=u$. Since $0\leq u\leq 1$, the image is the line segment $x=0$, $0\leq y\leq 1$. S_2 is the segment $u=1$, $0\leq v\leq 1$, so $x=v$ and $y=u(1+v^2)=1+x^2$. Thus the image is the portion of the parabola $y=1+x^2$ for $0\leq x\leq 1$. S_3 is the segment $v=1$, $0\leq u\leq 1$, so $x=1$ and $y=2u$. The image is the segment $x=1$, $0\leq y\leq 2$. S_4 is described by $u=0$, $0\leq v\leq 1$, so $0\leq x=v\leq 1$ and $y=u(1+v^2)=0$. The image is the line segment $y=0$, $0\leq x\leq 1$. Thus, the image of S is the region R bounded by the parabola $y=1+x^2$, the x –axis, and the lines $x=0$, $x=1$.

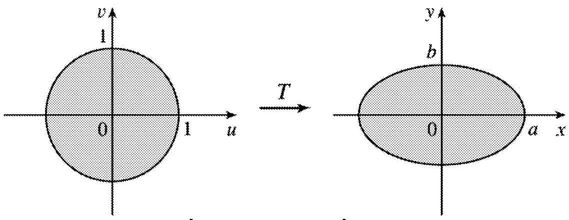


9. S_1 is the line segment $u=v$, $0\leq u\leq 1$, so $y=v=u$ and $x=u^2=y^2$. Since $0\leq u\leq 1$, the image is the portion of the parabola $x=y^2$, $0\leq y\leq 1$. S_2 is the segment $v=1$, $0\leq u\leq 1$, thus $y=v=1$ and $x=u^2$, so $0\leq x\leq 1$. The image is the line segment $y=1$, $0\leq x\leq 1$. S_3 is the segment $u=0$, $0\leq v\leq 1$, so $x=u^2=0$ and $y=v\Rightarrow 0\leq y\leq 1$. The image is the segment $x=0$, $0\leq y\leq 1$. Thus, the image of S is the region R in the first quadrant bounded by the parabola $x=y^2$, the y –axis, and the line $y=1$.



10. Substituting $u=\frac{x}{a}$, $v=\frac{y}{b}$ into $u^2+v^2\leq 1$ gives $\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1$, so the image of $u^2+v^2\leq 1$ is the elliptical region

$\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$.



11. $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}=3$ and $x-3y=(2u+v)-3(u+2v)=-u-5v$. To find the region S in the uv –plane that corresponds to R we first find the corresponding boundary under the given transformation. The line through $(0,0)$ and $(2,1)$ is $y=\frac{1}{2}x$ which is the image of $u+2v=\frac{1}{2}(2u+v)\Rightarrow v=0$; the line through $(2,1)$ and $(1,2)$ is $x+y=3$ which is the image of $(2u+v)+(u+2v)=3\Rightarrow u+v=1$; the line through $(0,0)$ and $(1,2)$ is $y=2x$ which is the image of $u+2v=2(2u+v)\Rightarrow u=0$. Thus S is the triangle $0\le v\le 1-u$, $0\le u\le 1$ in the uv –plane and

$$\iint_R (x-3y)\,dA=\int_0^1\int_0^{1-u}(-u-5v)|3|\,dv\,du$$
$$=-3\int_0^1\left[uv+\frac{5}{2}v^2\right]_{v=0}^{v=1-u}du=-3\int_0^1\left(u-u^2+\frac{5}{2}(1-u)^2\right)du$$
$$=-3\left[\frac{1}{2}u^2-\frac{1}{3}u^3-\frac{5}{6}(1-u)^3\right]_0^1=-3\left(\frac{1}{2}-\frac{1}{3}+\frac{5}{6}\right)=-3$$

12. $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{vmatrix}=\frac{1}{4}$, $4x+8y=4\cdot\frac{1}{4}(u+v)+8\cdot\frac{1}{4}(v-3u)=3v-5u$. R is a parallelogram bounded by the lines $x-y=-4$, $x-y=4$, $3x+y=0$, $3x+y=8$. Since $u=x-y$ and $v=3x+y$, R is the image of the rectangle enclosed by the lines $u=-4$, $u=4$, $v=0$, and $v=8$. Thus

$$\iint_R(4x+8y)\,dA=\int_{-4}^4\int_0^8(3v-5u)\left|\frac{1}{4}\right|dv\,du=\frac{1}{4}\int_{-4}^4\left[\frac{3}{2}v^2-5uv\right]_{v=0}^{v=8}du$$
$$=\frac{1}{4}\int_{-4}^4(96-40u)\,du=\frac{1}{4}\left[96u-20u^2\right]_{-4}^4=192$$

13. $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}=6$, $x^2=4u^2$ and the planar ellipse $9x^2+4y^2\le 36$ is the image of the disk $u^2+v^2\le 1$. Thus

$$\iint_R x^2\,dA = \iint_{u^2+v^2\le 1} (4u^2)(6)\,du\,dv = \int_0^{2\pi}\int_0^1 (24r^2\cos^2\theta)\,r\,dr\,d\theta$$

$$= 24\int_0^{2\pi} \cos^2\theta\,d\theta \int_0^1 r^3\,dr = 24\left[\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{2\pi}\left[\frac{1}{4}r^4\right]_0^1$$

$$= 24(\pi)\left(\frac{1}{4}\right) = 6\pi$$

14. $\dfrac{\partial(\mathrm{x,y})}{\partial(\mathrm{u,v})} = \begin{vmatrix} \sqrt{2} & -\sqrt{2/3} \\ \sqrt{2} & \sqrt{2/3} \end{vmatrix} = \dfrac{4}{\sqrt{3}}$, $x^2 - xy + y^2 = 2u^2 + 2v^2$ and the planar ellipse $x^2 - xy + y^2 \le 2$ is the image of the disk $u^2 + v^2 \le 1$. Thus

$$\iint_R (x^2 - xy + y^2)\,dA = \iint_{u^2+v^2\le 1} (2u^2 + 2v^2)\left(\frac{4}{\sqrt{3}}\,du\,dv\right) = \int_0^{2\pi}\int_0^1 \frac{8}{\sqrt{3}}\,r^3\,dr\,d\theta = \frac{4\pi}{\sqrt{3}}$$

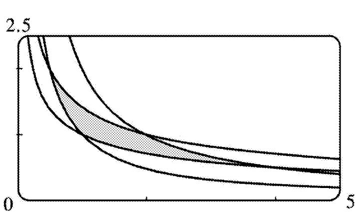
15. $\dfrac{\partial(\mathrm{x,y})}{\partial(\mathrm{u,v})} = \begin{vmatrix} 1/\mathrm{v} & -\mathrm{u/v}^2 \\ 0 & 1 \end{vmatrix} = \dfrac{1}{v}$, $xy = u$, $y = x$ is the image of the parabola $v^2 = u$, $y = 3x$ is the image of the parabola $v^2 = 3u$, and the hyperbolas $xy = 1$, $xy = 3$ are the images of the lines $u = 1$ and $u = 3$ respectively. Thus

$$\iint_R xy\,dA = \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u\left(\frac{1}{v}\right)dv\,du = \int_1^3 u\left(\ln\sqrt{3u} - \ln\sqrt{u}\right)du$$

$$= \int_1^3 u\ln\sqrt{3}\,du = 4\ln\sqrt{3} = 2\ln 3$$

16. Here $y = \dfrac{v}{u}$, $x = \dfrac{u^2}{v}$ so $\dfrac{\partial(\mathrm{x,y})}{\partial(\mathrm{u,v})} = \begin{vmatrix} 2\mathrm{u/v} & -\mathrm{u}^2/\mathrm{v}^2 \\ -\mathrm{v/u}^2 & 1/\mathrm{u} \end{vmatrix} = \dfrac{1}{v}$ and R is the image of the square with vertices $(1,1)$, $(2,1)$, $(2,2)$, and $(1,2)$. So

$$\iint_R y^2\,dA = \int_1^2\int_1^2 \frac{v^2}{u^2}\left(\frac{1}{v}\right)du\,dv = \int_1^2 \frac{v}{2}\,dv = \frac{3}{4}$$



17. **(a)** $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$ and since $u=\dfrac{x}{a}$, $v=\dfrac{y}{b}$, $w=\dfrac{z}{c}$ the solid enclosed by the ellipsoid is the image of the ball $u^2+v^2+w^2 \le 1$. So

$$\iiint_E dV = \iiint_{u^2+v^2+w^2 \le 1} abc\,du\,dv\,dw = (abc)(\text{ volume of the ball }) = \frac{4}{3}\pi abc$$

(b) If we approximate the surface of Earth by the ellipsoid $\dfrac{x^2}{6378^2}+\dfrac{y^2}{6378^2}+\dfrac{z^2}{6356^2}=1$, then we can estimate the volume of Earth by finding the volume of the solid E enclosed by the ellipsoid. From part (a), this is $\iiint_E dV=\frac{4}{3}\pi(6378)(6378)(6356)\approx 1.083\times 10^{12}\text{ km}^3$.

18. $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$ and the solid enclosed by the ellipsoid is the image of the ball $u^2+v^2+w^2 \le 1$. Now $x^2y=(a^2u^2)(bv)$, so

$$\begin{aligned}
\iiint_E x^2y\,dV &= \iiint_{u^2+v^2+w^2 \le 1} (a^2bu^2v)(abc)\,du\,dv\,dw \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^{1}(a^3b^2c)(\rho^2\sin^2\phi\cos^2\theta)(\rho\sin\phi\sin\theta)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= a^3b^2c\int_0^{2\pi}\int_0^{\pi}\int_0^{1}(\rho^5\sin^4\phi\cos^2\theta\sin\theta)\,d\rho\,d\phi\,d\theta \\
&= a^3b^2c\int_0^{2\pi}\cos^2\theta\sin\theta\,d\theta\int_0^{\pi}\sin^4\phi\,d\phi\int_0^{1}\rho^5\,d\rho \\
&= 0 \text{ since } \int_0^{2\pi}\cos^2\theta\sin\theta\,d\theta=0
\end{aligned}$$

19. Letting $u=x-2y$ and $v=3x-y$, we have $x=\frac{1}{5}(2v-u)$ and $y=\frac{1}{5}(v-3u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/5 & 2/5 \\ -3/5 & 1/5 \end{vmatrix} = \dfrac{1}{5}$ and R is the image of the rectangle enclosed by the lines $u=0$, $u=4$, $v=1$, and $v=8$. Thus

$$\iint_R \frac{x-2y}{3x-y}\,dA=\int_0^4\int_1^8 \frac{u}{v}\left|\frac{1}{5}\right|dv\,du=\frac{1}{5}\int_0^4 u\,du\int_1^8 \frac{1}{v}\,dv=\frac{1}{5}\left[\frac{1}{2}u^2\right]_0^4[\ln|v|]_1^8=\frac{8}{5}\ln 8 .$$

20. Letting $u=x+y$ and $v=x-y$, we have $x=\frac{1}{2}(u+v)$ and $y=\frac{1}{2}(u-v)$. Then $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix}=-\frac{1}{2}$ and R is the image of the rectangle enclosed by the lines $u=0$, $u=3$, $v=0$, and $v=2$. Thus

$$\iint_R (x+y)e^{x^2-y^2}\,dA=\int_0^3\int_0^2 ue^{uv}\left|-\frac{1}{2}\right|dv\,du=\frac{1}{2}\int_0^3\left[e^{uv}\right]_{v=0}^{v=2}du=\frac{1}{2}\int_0^3(e^{2u}-1)\,du$$

$$=\frac{1}{2}\left[\frac{1}{2}e^{2u}-u\right]_0^3=\frac{1}{2}\left(\frac{1}{2}e^6-3-\frac{1}{2}\right)=\frac{1}{4}(e^6-7)$$

21. Letting $u=y-x$, $v=y+x$, we have $y=\frac{1}{2}(u+v)$, $x=\frac{1}{2}(v-u)$. Then $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix}=-\frac{1}{2}$ and R is the image of the trapezoidal region with vertices $(-1,1)$, $(-2,2)$, $(2,2)$, and $(1,1)$. Thus

$$\iint_R \cos\frac{y-x}{y+x}\,dA=\int_1^2\int_{-v}^{v}\cos\frac{u}{v}\left|-\frac{1}{2}\right|du\,dv=\frac{1}{2}\int_1^2\left[v\sin\frac{u}{v}\right]_{u=-v}^{u=v}dv$$

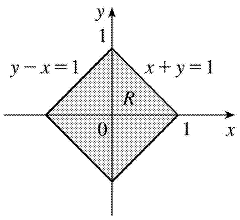
$$=\frac{1}{2}\int_1^2 2v\sin(1)\,dv=\frac{3}{2}\sin 1$$

22. Letting $u=3x$, $v=2y$, we have $9x^2+4y^2=u^2+v^2$, $x=\frac{1}{3}u$, and $y=\frac{1}{2}v$. Then $\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{6}$ and R is the image of the quarter–disk D given by $u^2+v^2\le 1$, $u\ge 0$, $v\ge 0$. Thus

$$\iint_R \sin(9x^2+4y^2)\,dA=\iint_D \frac{1}{6}\sin(u^2+v^2)\,du\,dv=\int_0^{\pi/2}\int_0^1\frac{1}{6}\sin(r^2)\,r\,dr\,d\theta$$

$$=\frac{\pi}{12}\left[-\frac{1}{2}\cos r^2\right]_0^1=\frac{\pi}{24}(1-\cos 1)$$

23. Let $u=x+y$ and $v=-x+y$. Then $u+v=2y\Rightarrow y=\frac{1}{2}(u+v)$ and $u-v=2x\Rightarrow x=\frac{1}{2}(u-v)$.
$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix}=\frac{1}{2}$. Now $|u|=|x+y|\le|x|+|y|\le 1\Rightarrow -1\le u\le 1$, and $|v|=|-x+y|\le|x|+|y|\le 1\Rightarrow -1\le v\le 1$. R is the image of

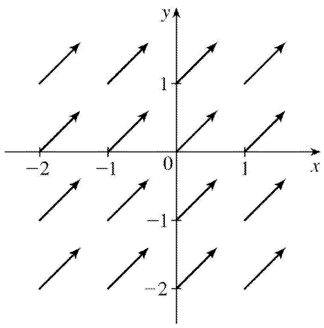


the square region with vertices $(1,1)$, $(1,-1)$, $(-1,-1)$, and $(-1,1)$. So
$\iint_R e^{x+y}\,dA = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} e^u\,du\,dv = \frac{1}{2}\left[e^u\right]_{-1}^{1}\left[v\right]_{-1}^{1} = e - e^{-1}$.

24. Let $u=x+y$ and $v=y$, then $x=u-v$, $y=v$, $\frac{\partial(x,y)}{\partial(u,v)}=1$ and R is the image under T of the triangular region with vertices $(0,0)$, $(1,0)$ and $(1,1)$. Thus
$\iint_R f(x+y)\,dA = \int_0^1\int_0^u (1)\,f(u)\,dv\,du = \int_0^1 f(u)\left[v\right]_{v=0}^{v=u}du = \int_0^1 uf(u)\,du$ as desired.

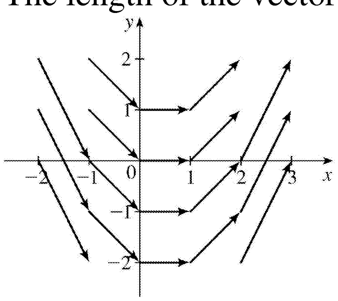
1. $\boldsymbol{F}(x,y)=\frac{1}{2}(\boldsymbol{i}+\boldsymbol{j})$

All vectors in this field are identical, with length $\frac{1}{\sqrt{2}}$ and direction parallel to the line $y=x$.



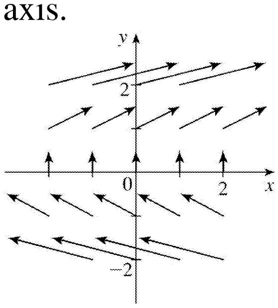
2. $\boldsymbol{F}(x, y)=\boldsymbol{i}+x\,\boldsymbol{j}$

The length of the vector $\boldsymbol{i}+x\,\boldsymbol{j}$ is $\sqrt{1+x^2}$. Vectors are tangent to parabolas opening about the y –axis.



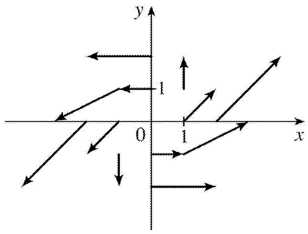
3. $\boldsymbol{F}(x, y)=y\boldsymbol{i}+\frac{1}{2}\,\boldsymbol{j}$.

The length of the vector $y\boldsymbol{i}+\frac{1}{2}\,\boldsymbol{j}$ is $\sqrt{y^2+\frac{1}{4}}$. Vectors are tangent to parabolas opening about the x –axis.



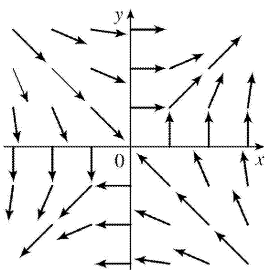
4. $\boldsymbol{F}(x, y)=(x-y)\boldsymbol{i}+x\,\boldsymbol{j}$

The length of the vector $(x-y)\boldsymbol{i}+x\,\boldsymbol{j}$ is $\sqrt{(x-y)^2+x^2}$. Vectors along the line $y=x$ are vertical.



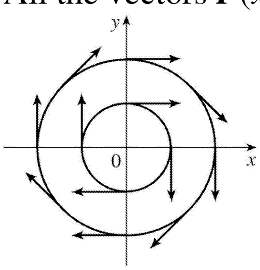
5. $\boldsymbol{F}(x, y) = \dfrac{y\boldsymbol{i} + x\boldsymbol{j}}{\sqrt{x^2 + y^2}}$

The length of the vector $\dfrac{y\boldsymbol{i} + x\boldsymbol{j}}{\sqrt{x^2 + y^2}}$ is 1.



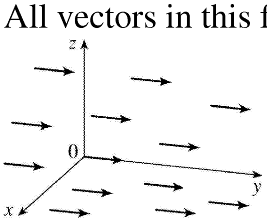
6. $\boldsymbol{F}(x, y) = \dfrac{y\boldsymbol{i} - x\boldsymbol{j}}{\sqrt{x^2 + y^2}}$

All the vectors $\boldsymbol{F}(x, y)$ are unit vectors tangent to circles centered at the origin with radius $\sqrt{x^2 + y^2}$.



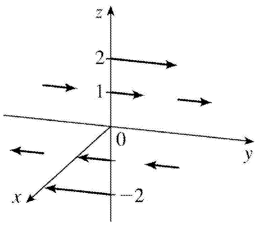
7. $\boldsymbol{F}(x, y, z) = \boldsymbol{j}$

All vectors in this field are parallel to the y –axis and have length 1 .

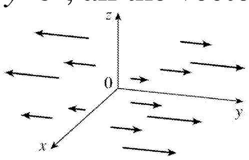


8. $\boldsymbol{F}(x, y, z) = z\boldsymbol{j}$

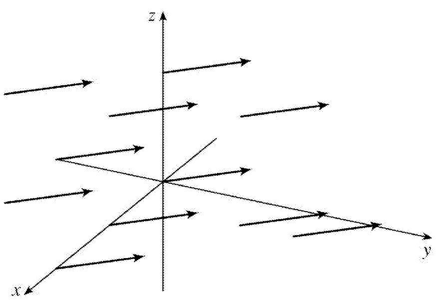
At each point (x, y, z) , $\boldsymbol{F}(x, y, z)$ is a vector of length $|z|$. For $z > 0$, all point in the direction of the positive y –axis while for $z < 0$, all are in the direction of the negative y –axis.



9. $\boldsymbol{F}(x, y, z)=y\boldsymbol{j}$
The length of $\boldsymbol{F}(x, y, z)$ is $|y|$. No vectors emanate from the xz –plane since $y=0$ there. In each plane $y=b$, all the vectors are identical.



10. $\boldsymbol{F}(x, y, z)=\boldsymbol{j}-\boldsymbol{i}$
All vectors in this field have length $\sqrt{2}$ and point in the same direction, parallel to the xy –plane.



11. $\boldsymbol{F}(x, y)=\langle y, x\rangle$ corresponds to graph II. In the first quadrant all the vectors have positive x – and y –components, in the second quadrant all vectors have positive x –components and negative y –components, in the third quadrant all vectors have negative x – and y –components, and in the fourth quadrant all vectors have negative x –components and positive y –components. In addition, the vectors get shorter as we approach the origin.

12. $\boldsymbol{F}(x, y)=\langle 1, \sin y\rangle$ corresponds to graph IV since the x –component of each vector is constant, the vectors are independent of x (vectors along horizontal lines are identical), and the vector field appears to repeat the same pattern vertically.

13. $\boldsymbol{F}(x, y)=\langle x-2, x+1\rangle$ corresponds to graph I since the vectors are independent of y (vectors along vertical lines are identical) and, as we move to the right, both the x – and the y –components get larger.

14. $\boldsymbol{F}(x, y)=\langle y, 1/x\rangle$ corresponds to graph III. As in Exercise 11, all the vectors in the first quadrant have positive x – and y –components, in the second quadrant all vectors have positive x –components and negative y –components, in the third quadrant all vectors have negative x – and y –components,

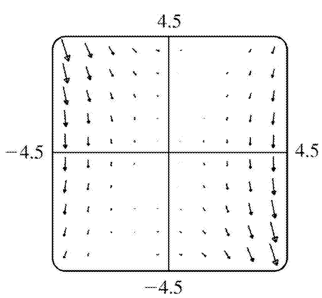
and in the fourth quadrant all vectors have negative x –components and positive y –components. Also, the vectors become longer as we approach the y –axis.

15. $\boldsymbol{F}(x, y, z)=\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}$ corresponds to graph IV, since all vectors have identical length and direction.

16. $\boldsymbol{F}(x, y, z)=\boldsymbol{i}+2\boldsymbol{j}+z\boldsymbol{k}$ corresponds to graph I, since the horizontal vector components remain constant, but the vectors above the xy –plane point generally upward while the vectors below the xy –plane point generally downward.

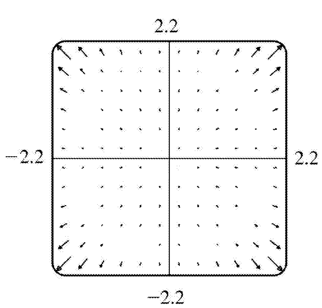
17. $\boldsymbol{F}(x, y, z)=x\boldsymbol{i}+y\boldsymbol{j}+3\boldsymbol{k}$ corresponds to graph III; the projection of each vector onto the xy –plane is $x\boldsymbol{i}+y\boldsymbol{j}$, which points away from the origin, and the vectors point generally upward because their z –components are all 3 .

18. $\boldsymbol{F}(x, y, z)=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ corresponds to graph II; each vector $\boldsymbol{F}(x, y, z)$ has the same length and direction as the position vector of the point (x, y, z) , and therefore the vectors all point directly away from the origin.



19.

The vector field seems to have very short vectors near the line $y=2x$. For $\boldsymbol{F}(x, y)=\langle 0, 0\rangle$ we must have $y^2-2xy=0$ and $3xy-6x^2=0$. The first equation holds if $y=0$ or $y=2x$, and the second holds if $x=0$ or $y=2x$. So both equations hold [and thus $\boldsymbol{F}(x, y)=\mathbf{0}$] along the line $y=2x$.



20.

From the graph, it appears that all of the vectors in the field lie on lines through the origin, and that the vectors have very small magnitudes near the circle $|\boldsymbol{x}|=2$ and near the origin. Note that $\boldsymbol{F}(\boldsymbol{x})=\mathbf{0}\Leftrightarrow r(r-2)=0\Leftrightarrow r=0$ or 2 , so as we suspected, $\boldsymbol{F}(\boldsymbol{x})=\mathbf{0}$ for $|x|=2$ and for $|x|=0$. Note that where $r^2-r<0$, the vectors point towards the origin, and where $r^2-r>0$, they point away from the origin.

21. $\nabla f(x, y)=f_x(x, y)\boldsymbol{i}+f_y(x, y)\boldsymbol{j}=\frac{1}{x+2y}\ \boldsymbol{i}+\frac{2}{x+2y}\ \boldsymbol{j}$

22. $\nabla f(x, y)=f_x(x, y)\boldsymbol{i}+f_y(x, y)\boldsymbol{j}=\left[x^{\alpha}(-\beta e^{-\beta x})+\alpha x^{\alpha-1}e^{-\beta x}\right]\boldsymbol{i}+0\,\boldsymbol{j}=(\alpha-\beta x)x^{\alpha-1}e^{-\beta x}\boldsymbol{i}$

23.

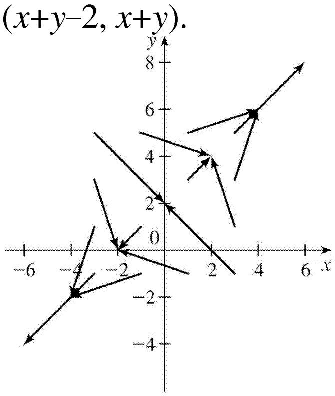
$$\begin{aligned}\nabla f(x, y, z) &= f_x(x, y, z)\boldsymbol{i}+f_y(x, y, z)\boldsymbol{j}+f_z(x, y, z)\boldsymbol{k}\\ &= \frac{x}{\sqrt{x^2+y^2+z^2}}\ \boldsymbol{i}+\frac{y}{\sqrt{x^2+y^2+z^2}}\ \boldsymbol{j}+\frac{z}{\sqrt{x^2+y^2+z^2}}\ \boldsymbol{k}\end{aligned}$$

24.

$$\begin{aligned}\nabla f(x, y, z) &= f_x(x, y, z)\boldsymbol{i}+f_y(x, y, z)\boldsymbol{j}+f_z(x, y, z)\boldsymbol{k}\\ &= \left(\cos\frac{y}{z}\right)\boldsymbol{i}-x\left(\sin\frac{y}{z}\right)\left(\frac{1}{z}\right)\boldsymbol{j}-x\left(\sin\frac{y}{z}\right)\left(-\frac{y}{z^2}\right)\boldsymbol{k}\\ &= \left(\cos\frac{y}{z}\right)\boldsymbol{i}-\frac{x}{z}\left(\sin\frac{y}{z}\right)\boldsymbol{j}+\frac{xy}{z^2}\left(\sin\frac{y}{z}\right)\boldsymbol{k}\end{aligned}$$

25. $f(x, y)=xy-2x \Rightarrow \nabla f(x, y)=(y-2)\boldsymbol{i}+x\boldsymbol{j}$.

The length of $\nabla f(x, y)$ is $\sqrt{(y-2)^2+x^2}$ and $\nabla f(x, y)$ terminates on the line $y=x+2$ at the point $(x+y-2, x+y)$.

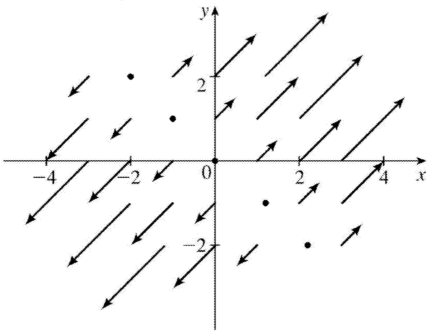


26. $f(x, y)=\frac{1}{4}(x+y)^2 \Rightarrow$

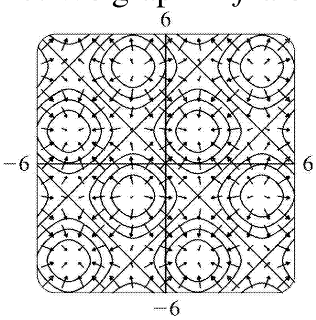
$\nabla f(x, y)=\frac{1}{2}(x+y)\boldsymbol{i}+\frac{1}{2}(x+y)\boldsymbol{j}$.

The length of $\nabla f(x, y)$ is

$\sqrt{\frac{1}{2}(x+y)^2} = \frac{1}{\sqrt{2}}|x+y|$. The vectors are perpendicular to the line $y=-x$ and point away from the line, with length that increases as the distance from the line $y=-x$ increases.

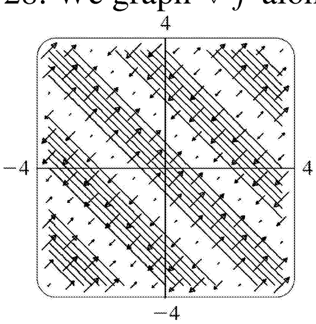


27. We graph ∇f along with a contour map of f .



The graph shows that the gradient vectors are perpendicular to the level curves. Also, the gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.

28. We graph ∇f along with a contour map of f .



The graph shows that the gradient vectors are perpendicular to the level curves. Also, the gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.

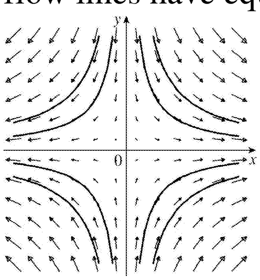
29. $f(x, y)=xy \Rightarrow \nabla f(x, y)=y\mathbf{i}+x\mathbf{j}$. In the first quadrant, both components of each vector are positive, while in the third quadrant both components are negative. However, in the second quadrant each vector's x –component is positive while its y –component is negative (and vice versa in the fourth quadrant). Thus, ∇f is graph IV.

30. $f(x, y)=x^2-y^2 \Rightarrow \nabla f(x, y)=2x\boldsymbol{i}-2y\boldsymbol{j}$. In the first quadrant, the x –component of each vector is positive while the y –component is negative. The other three quadrants are similar, where the x –component of each vector has the same sign as the x –value of its initial point, and the y –component has sign opposite that of the y –value of the initial point. Thus, ∇f is graph III.

31. $f(x, y)=x^2+y^2 \Rightarrow \nabla f(x, y)=2x\boldsymbol{i}+2y\boldsymbol{j}$. Thus, each vector $\nabla f(x, y)$ has the same direction and twice the length of the position vector of the point (x, y) , so the vectors all point directly away from the origin and their lengths increase as we move away from the origin. Hence, ∇f is graph II.

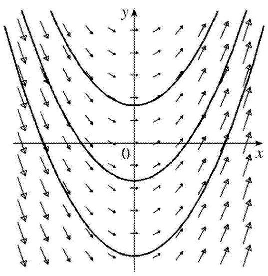
32. $f(x, y)=\sqrt{x^2+y^2} \Rightarrow \nabla f(x, y)=\dfrac{x}{\sqrt{x^2+y^2}}\boldsymbol{i}+\dfrac{y}{\sqrt{x^2+y^2}}\boldsymbol{j}$. Then $|\nabla f(x, y)|=\dfrac{1}{\sqrt{x^2+y^2}}\sqrt{x^2+y^2}=1$, so all vectors are unit vectors. In addition, each vector $\nabla f(x, y)$ has the same direction as the position vector of the point (x,y) , so the vectors all point directly away from the origin. Hence, ∇f is graph I.

33. **(a)** We sketch the vector field $\boldsymbol{F}(x, y)=x\boldsymbol{i}-y\boldsymbol{j}$ along with several approximate flow lines.The flow lines appear to be hyperbolas with shape similar to the graph of $y=\pm 1/x$, so we might guess that the flow lines have equations $y=C/x$.



(b) If $x=x(t)$ and $y=y(t)$ are parametric equations of a flow line, then the velocity vector of the flow line at the point (x, y) is $x'(t)\boldsymbol{i}+y'(t)\boldsymbol{j}$. Since the velocity vectors coincide with the vectors in the vector field, we have $x'(t)\boldsymbol{i}+y'(t)\boldsymbol{j}=x\boldsymbol{i}-y\boldsymbol{j} \Rightarrow dx/dt=x$, $dy/dt=-y$. To solve these differential equations, we know $dx/dt=x \Rightarrow dx/x=dt \Rightarrow \ln|x|=t+C \Rightarrow x=\pm e^{t+C}=Ae^t$ for some constant A , and $dy/dt=-y \Rightarrow dy/y=-dt \Rightarrow \ln|y|=-t+K \Rightarrow y=\pm e^{-t+K}=Be^{-t}$ for some constant B . Therefore $xy=Ae^tBe^{-t}=AB=$ constant. If the flow line passes through (1, 1) then (1)(1)= constant $=1 \Rightarrow xy=1 \Rightarrow y=1/x$, $x>0$.

34. **(a)** We sketch the vector field $\boldsymbol{F}(x, y)=\boldsymbol{i}+x\boldsymbol{j}$ along with several approximate flow lines.The flow lines appear to be parabolas.



(b) If $x=x(t)$ and $y=y(t)$ are parametric equations of a flow line, then the velocity vector of the flow line at the point (x, y) is $x'(t)\mathbf{i}+y'(t)\mathbf{j}$. Since the velocity vectors coincide with the vectors in the vector field, we have $x'(t)\mathbf{i}+y'(t)\mathbf{j}=\mathbf{i}+x\mathbf{j}\Rightarrow \frac{dx}{dt}=1$, $\frac{dy}{dt}=x$. Thus $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{x}{1}=x$.

(c) From part (b), $dy/dx=x$. Integrating, we have $y=\frac{1}{2}x^2+c$. Since the particle starts at the origin, we know (0, 0) is on the curve, so $0=0+c\Rightarrow c=0$ and the path the particle follows is $y=\frac{1}{2}x^2$.

1. $x=t^2$ and $y=t$, $0\le t\le 2$, so by Formula 3

$$\int_C y\,ds = \int_0^2 t\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_0^2 t\sqrt{(2t)^2+(1)^2}\,dt$$

$$= \int_0^2 t\sqrt{4t^2+1}\,dt= \frac{1}{12}(4t^2+1)^{3/2}\Big]_0^2= \frac{1}{12}(17\sqrt{17}-1)$$

2.

$$\int_C \frac{y}{x}\,ds =\int_{1/2}^1 \frac{t^3}{t^4}\sqrt{(4t^3)^2+(3t^2)^2}\,dt=\int_{1/2}^1 \frac{1}{t}\sqrt{16t^6+9t^4}\,dt=\int_{1/2}^1 t\sqrt{16t^2+9}\,dt$$

$$= \frac{1}{48}(16t^2+9)^{3/2}\Big]_{1/2}^1= \frac{1}{48}(25^{3/2}-13^{3/2})= \frac{1}{48}(125-13\sqrt{13})$$

3. Parametric equations for C are $x=4\cos t$, $y=4\sin t$, $-\frac{\pi}{2}\le t\le \frac{\pi}{2}$. Then

$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2}(4\cos t)(4\sin t)^4\sqrt{(-4\sin t)^2+(4\cos t)^2}\,dt$$

$$= \int_{-\pi/2}^{\pi/2}4^5\cos t\sin^4 t\sqrt{16(\sin^2 t+\cos^2 t)}\,dt$$

$$= 4^5\int_{-\pi/2}^{\pi/2}(\sin^4 t\cos t)(4)\,dt=(4)^6\left[\frac{1}{5}\sin^5 t\right]_{-\pi/2}^{\pi/2}= \frac{2\cdot 4^6}{5}=1638.4$$

4. Parametric equations for C are $x=1+3t$, $y=2+5t$, $0\le t\le 1$. Then

$$\int_C ye^x\,ds=\int_0^1(2+5t)e^{1+3t}\sqrt{3^2+5^2}\,dt=\sqrt{34}\int_0^1(2+5t)e^{1+3t}\,dt$$

Integrating by parts with $u=2+5t \Rightarrow du=5\,dt$, $dv=e^{1+3t} \Rightarrow v=\frac{1}{3}e^{1+3t}\,dt$ gives

$$\int_C ye^x\,ds = \sqrt{34}\left[\frac{1}{3}(2+5t)e^{1+3t}-\frac{5}{9}e^{1+3t}\right]_0^1$$

$$= \sqrt{34}\left[\left(\frac{7}{3}-\frac{5}{9}\right)e^4-\left(\frac{2}{3}-\frac{5}{9}\right)e\right]= \frac{\sqrt{34}}{9}(16e^4-e)$$

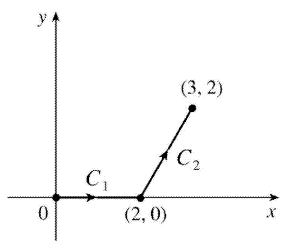
5. If we choose x as the parameter, parametric equations for C are $x=x$, $y=x^2$ for $1\le x\le 3$ and

$$\int_C (xy+\ln x)\,dy = \int_1^3 (x\cdot x^2+\ln x)2x\,dx = \int_1^3 2(x^4+x\ln x)\,dx$$

$$= 2\left[\frac{1}{5}x^5+\frac{1}{2}x^2\ln x-\frac{1}{4}x^2\right]_1^3 \text{ (by integrating by parts in the second term)}$$

$$= 2\left(\frac{243}{5}+\frac{9}{2}\ln 3-\frac{9}{4}-\frac{1}{5}+\frac{1}{4}\right)=\frac{464}{5}+9\ln 3$$

6. Choosing y as the parameter, we have $x=e^y$, $y=y$, $0\le y\le 1$. Then
$\int_c xe^y\,dx=\int_0^1 e^y(e^y)e^y\,dy=\int_0^1 e^{3y}\,dy= \left.\frac{1}{3}e^{3y}\right]_0^1=\frac{1}{3}(e^3-1)$.

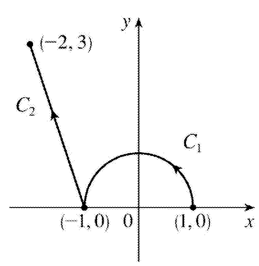


7.
$C=C_1+C_2$
On C_1 : $x=x$, $y=0\Rightarrow dy=0\,dx$, $0\le x\le 2$.
On C_2 : $x=x$, $y=2x-4\Rightarrow dy=2\,dx$, $2\le x\le 3$.
Then

$$\int_C xy\,dx+(x-y)\,dy = \int_{C_1} xy\,dx+(x-y)\,dy+\int_{C_2} xy\,dx+(x-y)\,dy$$

$$=\int_0^2 (0+0)\,dx+\int_2^3 dx$$

$$=\int_2^3 (2x^2-6x+8)\,dx=\frac{17}{3}$$



8.
$C=C_1+C_2$
On C_1 : $x=\cos t\Rightarrow dx=-\sin t\,dt$, $y=\sin t\Rightarrow$

$dy = \cos t\, dt$, $0 \le t \le \pi$.
On C_2 : $x = -1 - t \Rightarrow dx = -dt$, $y = 3t \Rightarrow$
$dy = 3\,dt$, $0 \le t \le 1$.

Then

$$\int_C \sin x\,dx + \cos y\,dy = \int_{C_1} \sin x\,dx + \cos y\,dy + \int_{C_2} \sin x\,dx + \cos y\,dy$$
$$= \int_0^\pi \sin(\cos t)(-\sin t\,dt) + \cos(\sin t)\cos t\,dt$$
$$+ \int_0^1 \sin(-1-t)(-dt) + \cos(3t)(3\,dt)$$
$$= \left[-\cos(\cos t) + \sin(\sin t)\right]_0^\pi + \left[-\cos(-1-t) + \sin(3t)\right]_0^1$$
$$= -\cos(\cos\pi) + \sin(\sin\pi) + \cos(\cos 0) - \sin(\sin 0)$$
$$-\cos(-2) + \sin(3) + \cos(-1) - \sin(0)$$
$$= -\cos(-1) + \sin 0 + \cos(1) - \sin 0 - \cos(-2) + \sin 3 + \cos(-1)$$
$$= -\cos 1 + \cos 1 - \cos 2 + \sin 3 + \cos 1 = \cos 1 - \cos 2 + \sin 3$$

where we have used the identity $\cos(-\theta) = \cos\theta$.

9. $x = 4\sin t$, $y = 4\cos t$, $z = 3t$, $0 \le t \le \frac{\pi}{2}$. Then by Formula 9,

$$\int_C xy^3\,ds = \int_0^{\pi/2} (4\sin t)(4\cos t)^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$
$$= \int_0^{\pi/2} 4^4 \cos^3 t \sin t \sqrt{(4\cos t)^2 + (-4\sin t)^2 + (3)^2}\,dt$$
$$= \int_0^{\pi/2} 256\cos^3 t \sin t \sqrt{16(\cos^2 t + \sin^2 t + 9)}\,dt$$
$$= 1280\int_0^{\pi/2} \cos^3 t \sin t\,dt = -320\cos^4 t\Big]_0^{\pi/2} = 320$$

10. Parametric equations for C are $x = 4t$, $y = 6 - 5t$, $z = -1 + 6t$, $0 \le t \le 1$. Then

$$\int_C x^2 z\,ds = \int_0^1 (4t)^2 (6t-1)\sqrt{4^2 + (-5)^2 + 6^2}\,dt = \sqrt{77}\int_0^1 (96t^3 - 16t^2)\,dt$$
$$= \sqrt{77}\left[96\cdot\frac{t^4}{4} - 16\cdot\frac{t^3}{3}\right]_0^1 = \frac{56}{3}\sqrt{77}$$

11. Parametric equations for C are $x=t$, $y=2t$, $z=3t$, $0\leq t\leq 1$. Then

$$\int_C xe^{yz}\,ds = \int_0^1 te^{(2t)(3t)}\sqrt{1^2+2^2+3^2}\,dt=\sqrt{14}\int_0^1 te^{6t^2}\,dt$$

$$= \sqrt{14}\left[\frac{1}{12}e^{6t^2}\right]_0^1=\frac{\sqrt{14}}{12}(e^6-1)$$

12. $\sqrt{(dx/dt)^2+(dy/dt)^2+(dz/dt)^2}=\sqrt{1^2+(2t)^2+(3t^2)^2}=\sqrt{1+4t^2+9t^4}$. Then

$$\int_C(2x+9z)\,ds=\int_0^1(2t+9t^3)\sqrt{1+4t^2+9t^4}\,dt\ \left[\text{ let } u=1+4t^2+9t^4\Rightarrow \frac{1}{4}\,du=(2t+9t^3)\,dt\ \right]$$

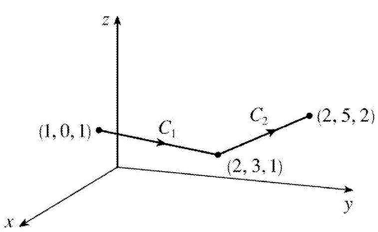
$$=\int_1^{14}\frac{1}{4}\sqrt{u}\,du=\frac{1}{6}u^{3/2}\Big]_1^{14}=\frac{1}{6}(14^{3/2}-1)$$

13. $\int_C x^2y\sqrt{z}\,dz=\int_0^1(t^3)^2(t)\sqrt{t^2}\cdot 2t\,dt=\int_0^1 2t^9\,dt=\frac{1}{5}t^{10}\Big]_0^1=\frac{1}{5}$

14.

$$\int_C z\,dx+x\,dy+y\,dz=\int_0^1 t^2\cdot 2t\,dt+t^2\cdot 3t^2\,dt+t^3\cdot 2t\,dt=\int_0^1(2t^3+5t^4)\,dt$$

$$=\left[\frac{1}{2}t^4+t^5\right]_0^1=\frac{1}{2}+1=\frac{3}{2}$$



15.

On C_1 : $x=1+t\Rightarrow dx=dt$, $y=3t\Rightarrow dy=3\,dt$, $z=1$
$\Rightarrow dz=0\,dt$, $0\leq t\leq 1$.

On C_2 : $x=2\Rightarrow dx=0\,dt$, $y=3+2t\Rightarrow$
$dy=2\,dt$, $z=1+t\Rightarrow dz=dt$, $0\leq t\leq 1$.

Then $\int_C(x+yz)\,dx+2x\,dy+xyz\,dz$

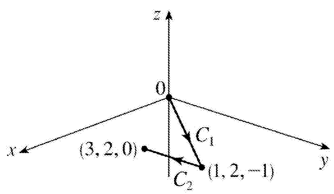
$$=\int_{C_1}(x+yz)\,dx+2x\,dy+xyz\,dz+\int_{C_2}(x+yz)\,dx+2x\,dy+xyz\,dz$$

$=\int_0^1 (1+t+(3t)(1))\,dt+2(1+t)\cdot 3\,dt+(1+t)(3t)(1)\cdot 0\,dt$

$+\int_0^1 (2+(3+2t)(1+t))\cdot 0\,dt+2(2)\cdot 2\,dt+(2)(3+2t)(1+t)\,dt$

$=\int_0^1 (10t+7)\,dt+\int_0^1 (4t^2+10t+14)\,dt$

$=\left[5t^2+7t\right]_0^1+\left[\frac{4}{3}t^3+5t^2+14t\right]_0^1=12+\frac{61}{3}=\frac{97}{3}$

16. 

On C_1 : $x=t \Rightarrow dx=dt$, $y=2t \Rightarrow dy=2\,dt$, $z=-t$
$\Rightarrow dz=-dt$, $0\le t\le 1$.
On C_2 : $x=1+2t \Rightarrow dx=2\,dt$, $y=2 \Rightarrow$
$dy=0\,dt$, $z=-1+t \Rightarrow dz=dt$, $0\le t\le 1$.

Then $\int_C x^2\,dx+y^2\,dy+z^2\,dz$
$=\int_{C_1} x^2\,dx+y^2\,dy+z^2\,dz+\int_{C_2} x^2\,dx+y^2\,dy+z^2\,dz$

$=\int_0^1 t^2\,dt+(2t)^2\cdot 2\,dt+(-t)^2(-dt)+\int_0^1 (1+2t)^2\cdot 2\,dt+2^2\cdot 0\,dt+(-1+t)^2\,dt$

$=\int_0^1 8t^2\,dt+\int_0^1 (9t^2+6t+3)\,dt=\left[\frac{8}{3}t^3\right]_0^1+\left[3t^3+3t^2+3t\right]_0^1=\frac{35}{3}$

17. **(a)** Along the line $x=-3$, the vectors of $\boldsymbol{F}$ have positive y –components, so since the path goes upward, the integrand $\boldsymbol{F}\cdot\boldsymbol{T}$ is always positive. Therefore $\int_{C_1} \boldsymbol{F}\cdot d\boldsymbol{r}=\int_{C_1} \boldsymbol{F}\cdot\boldsymbol{T}\,ds$ is positive.

(b) All of the (nonzero) field vectors along the circle with radius 3 are pointed in the clockwise direction, that is, opposite the direction to the path. So $\boldsymbol{F}\cdot\boldsymbol{T}$ is negative, and therefore $\int_{C_2} \boldsymbol{F}\cdot d\boldsymbol{r}=\int_{C_2} \boldsymbol{F}\cdot\boldsymbol{T}\,ds$ is negative.

18. Vectors starting on

C_1 point in roughly the same direction as C_1, so the tangential component $\boldsymbol{F}\cdot\boldsymbol{T}$ is positive. Then $\int_{C_1}\boldsymbol{F}\cdot d\boldsymbol{r}=\int_{C_1}\boldsymbol{F}\cdot\boldsymbol{T}\,ds$ is positive. On the other hand, no vectors starting on C_2 point in the same direction as C_2, while some vectors point in roughly the opposite direction, so we would expect $\int_{C_2}\boldsymbol{F}\cdot d\boldsymbol{r}=\int_{C_2}\boldsymbol{F}\cdot\boldsymbol{T}\,ds$ to be negative.

19. $\boldsymbol{r}(t)=t^2\boldsymbol{i}-t^3\boldsymbol{j}$, so $\boldsymbol{F}(\boldsymbol{r}(t))=(t^2)^2(-t^3)^3\boldsymbol{i}-(-t^3)\sqrt{t^2}\,\boldsymbol{j}=-t^{13}\boldsymbol{i}+t^4\boldsymbol{j}$ and $\boldsymbol{r}'(t)=2t\,\boldsymbol{i}-3t^2\boldsymbol{j}$.
Thus $\int_C\boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^1\boldsymbol{F}(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)dt=\int_0^1(-2t^{14}-3t^6)dt=\left[-\frac{2}{15}t^{15}-\frac{3}{7}t^7\right]_0^1=-\frac{59}{105}$.

20. $\boldsymbol{F}(\boldsymbol{r}(t))=(t^2)(t^3)\boldsymbol{i}+(t)(t^3)\boldsymbol{j}+(t)(t^2)\boldsymbol{k}=t^5\boldsymbol{i}+t^4\boldsymbol{j}+t^3\boldsymbol{k}$, $\boldsymbol{r}'(t)=\boldsymbol{i}+2t\,\boldsymbol{j}+3t^2\boldsymbol{k}$.
Thus $\int_C\boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^2\boldsymbol{F}(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)dt=\int_0^2(t^5+2t^5+3t^5)dt=\left.t^6\right]_0^2=64$.

21.

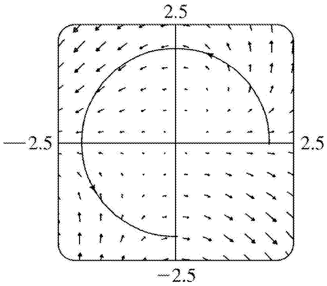
$$\int_C\boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^1\left\langle\sin t^3,\cos(-t^2),\mathrm{t}^4\right\rangle\cdot\left\langle 3t^2,-2\mathrm{t},1\right\rangle dt$$
$$=\int_0^1(3t^2\sin t^3-2t\cos t^2+t^4)dt=\left[-\cos t^3-\sin t^2+\frac{1}{5}t^5\right]_0^1=\frac{6}{5}-\cos 1-\sin 1$$

22.

$$\int_C\boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^\pi\langle\cos \mathrm{t},\sin \mathrm{t},-t\rangle\cdot\langle 1,\cos \mathrm{t},-\sin t\rangle\,dt=\int_0^\pi(\cos t+\sin t\cos t+t\sin t)dt$$
$$=\left[\sin t+\frac{1}{2}\sin^2 t+(\sin t-t\cos t)\right]_0^\pi=\pi$$

23. We graph $\boldsymbol{F}(x,y)=(x-y)\boldsymbol{i}+xy\,\boldsymbol{j}$ and the curve C. We see that most of the vectors starting on C point in roughly the same direction as C, so for these portions of C the tangential component $\boldsymbol{F}\cdot\boldsymbol{T}$ is positive. Although some vectors in the third quadrant which start on C point in roughly the opposite direction, and hence give negative tangential components, it seems reasonable that the effect of these portions of C is outweighed by the positive tangential components. Thus, we would expect $\int_C\boldsymbol{F}\cdot d\boldsymbol{r}=\int_C\boldsymbol{F}\cdot\boldsymbol{T}\,ds$ to be positive.

To verify, we evaluate $\int_C\boldsymbol{F}\cdot d\boldsymbol{r}$. The curve C can be represented by $\boldsymbol{r}(t)=2\cos t\,\boldsymbol{i}+2\sin t\,\boldsymbol{j}$, $0\le t\le\frac{3\pi}{2}$, so $\boldsymbol{F}(\boldsymbol{r}(t))=(2\cos t-2\sin t)\boldsymbol{i}+4\cos t\sin t\,\boldsymbol{j}$ and $\boldsymbol{r}'(t)=-2\sin t\,\boldsymbol{i}+2\cos t\,\boldsymbol{j}$. Then



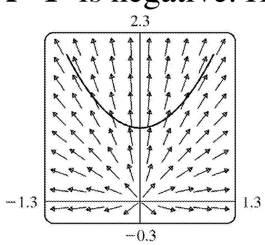
$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^{3\pi/2} \boldsymbol{F}(\boldsymbol{r}(t))\cdot \boldsymbol{r}'(t)\,dt$$

$$= \int_0^{3\pi/2} [-2\sin t(2\cos t-2\sin t)+2\cos t(4\cos t\sin t)]\,dt$$

$$= 4\int_0^{3\pi/2} (\sin^2 t-\sin t\cos t+2\sin t\cos^2 t)\,dt$$

$$= 3\pi+\frac{2}{3} \quad \text{[using a CAS]}$$

24. We graph $\boldsymbol{F}(x, y)=\dfrac{x}{\sqrt{x^2+y^2}}\,\boldsymbol{i}+\dfrac{y}{\sqrt{x^2+y^2}}\,\boldsymbol{j}$ and the curve C. In the first quadrant, each vector starting on C points in roughly the same direction as C, so the tangential component $\boldsymbol{F}\cdot\boldsymbol{T}$ is positive. In the second quadrant, each vector starting on C points in roughly the direction opposite to C, so $\boldsymbol{F}\cdot\boldsymbol{T}$ is negative. Here, it appears that the tangential components in the first and second quadrants



counteract each other, so it seems reasonable to guess that $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds$ is zero. To verify, we evaluate $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$. The curve C can be represented by $\boldsymbol{r}(t)=t\,\boldsymbol{i}+(1+t^2)\boldsymbol{j}$, $-1\le t\le 1$, so

$\boldsymbol{F}(\boldsymbol{r}(t))=\dfrac{t}{\sqrt{t^2+(1+t^2)^2}}\,\boldsymbol{i}+\dfrac{1+t^2}{\sqrt{t^2+(1+t^2)^2}}\,\boldsymbol{j}$ and $\boldsymbol{r}'(t)=\boldsymbol{i}+2t\,\boldsymbol{j}$. Then

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_{-1}^{1} \boldsymbol{F}(\boldsymbol{r}(t))\cdot \boldsymbol{r}'(t)\,dt$$

$$= \int_{-1}^{1} \left(\frac{t}{\sqrt{t^2+(1+t^2)^2}} + \frac{2t(1+t^2)}{\sqrt{t^2+(1+t^2)^2}} \right) dt$$

$$= \int_{-1}^{1} \frac{t\,(3+2t^2)}{\sqrt{t^4+3t^2+1}}\, dt=0 \quad \text{[since the integrand is an odd function]}$$

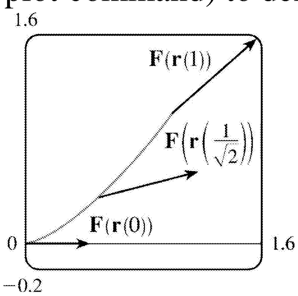
25. **(a)** $\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^1 \left\langle e^{t^2-1}, t^5 \right\rangle \cdot \left\langle 2t, 3t^2 \right\rangle dt = \int_0^1 \left(2te^{t^2-1} + 3t^7 \right) dt = \left[e^{t^2-1} + \frac{3}{8}t^8 \right]_0^1 = \frac{11}{8} - 1/e$

(b) $\boldsymbol{r}(0)=\boldsymbol{0}$, $\boldsymbol{F}(\boldsymbol{r}(0))=\left\langle e^{-1}, 0 \right\rangle$;

$\boldsymbol{r}\left(\frac{1}{\sqrt{2}} \right)=\left\langle \frac{1}{2}, \frac{1}{2\sqrt{2}} \right\rangle$, $\boldsymbol{F}\left(\boldsymbol{r}\left(\frac{1}{\sqrt{2}} \right)\right)=\left\langle e^{-1/2}, \frac{1}{4\sqrt{2}} \right\rangle$;

$\boldsymbol{r}(1)=\langle 1, 1\rangle$, $\boldsymbol{F}(\boldsymbol{r}(1))=\langle 1, 1\rangle$.

In order to generate the graph with Maple, we use the PLOT command (not to be confused with the plot command) to define each of the vectors. For example,

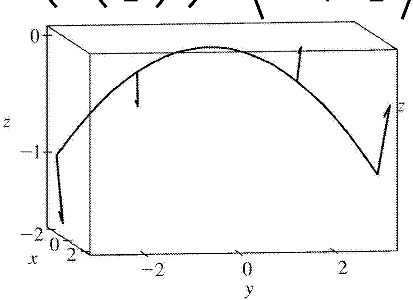


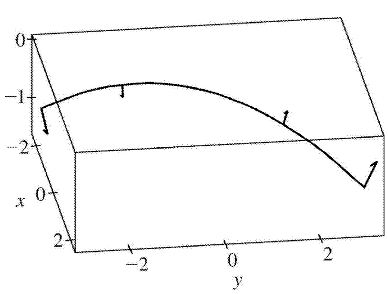
v1:=PLOT(CURVES([[0, 0], [evalf(1/exp(1)), 0]])); generates the vector from the vector field at the point (0, 0) (but without an arrowhead) and gives it the name v1. To show everything on the same screen, we use the display command. In Mathematica, we use ListPlot (with the PlotJoined – > True option) to generate the vectors, and then Show to show everything on the same screen.

26. **(a)** $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_{-1}^{1}\left\langle 2t, t^2, 3t\right\rangle\cdot\langle 2, 3, -2t\rangle dt=\int_{-1}^{1}(4t+3t^2-6t^2)dt=\left[2t^2-t^3\right]_{-1}^{1}=-2$

(b) Now $\boldsymbol{F}(\boldsymbol{r}(t))=\left\langle 2t, t^2, 3t\right\rangle$, so $\boldsymbol{F}(\boldsymbol{r}(-1))=\langle -2, 1, -3\rangle$, $\boldsymbol{F}\left(\boldsymbol{r}\left(-\frac{1}{2}\right)\right)=\left\langle -1, \frac{1}{4}, -\frac{3}{2}\right\rangle$,

$\boldsymbol{F}\left(\boldsymbol{r}\left(\frac{1}{2}\right)\right)=\left\langle 1, \frac{1}{4}, \frac{3}{2}\right\rangle$, and $\boldsymbol{F}(\boldsymbol{r}(1))=\langle 2, 1, 3\rangle$.





27. The part of the astroid that lies in the quadrant is parametrized by $x=\cos^3 t$, $y=\sin^3 t$, $0\le t\le \frac{\pi}{2}$.
Now $\frac{dx}{dt}=3\cos^2 t(-\sin t)$ and $\frac{dy}{dt}=3\sin^2 t\cos t$, so
$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}=3\cos t\sin t\sqrt{\cos^2 t+\sin^2 t}=3\cos t\sin t$.
Therefore $\int_C x^3y^5\,ds=\int_0^{\pi/2}\cos^9 t\sin^{15} t(3\cos t\sin t)\,dt=\frac{945}{16,777,216}\pi$.

28. We parametrize the line as $\mathrm{r}(t)=\langle 1,2,1\rangle+t[\langle 6,4,5\rangle-\langle 1,2,1\rangle]=(1+5t)\boldsymbol{i}+(2+2t)\boldsymbol{j}+(1+4t)\boldsymbol{k}$,
$0\le t\le 1$. Using a CAS, we calculate

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^1\left\langle (1+5t)^4e^{2+2t},\ \ln(1+4t),\sqrt{(2+2t)^2+(1+4t)^2}\right\rangle\cdot\langle 5,2,4\rangle\,dt$$

$$=\frac{5235e^4}{4}-\frac{6285e^2}{4}+\frac{9\sqrt{5}\sinh^{-1}\left(\frac{14}{3}\right)}{25}-\frac{9\sqrt{5}\sinh^{-1}\left(\frac{4}{3}\right)}{25}+\frac{5\ln 5}{2}+\frac{14\sqrt{41}}{5}-\frac{4\sqrt{5}}{5}-2$$

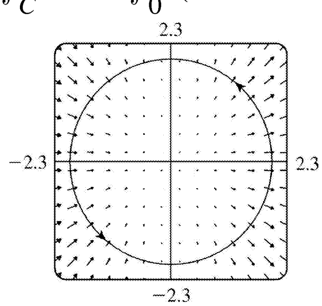
$$=\frac{5235e^4}{4}-\frac{6285e^2}{4}-\frac{18\sqrt{5}\ln 3}{25}+\frac{9\sqrt{5}\ln(14+\sqrt{205})}{25}+\frac{5\ln 5}{2}+\frac{14\sqrt{41}-4\sqrt{5}}{5}-2$$

The first answer is the one given by Maple. The two answers are equivalent by Equation 7.6.3 .

29. A calculator or CAS gives $\int_C x\sin y\,ds=\int_1^2\ln t\sin(e^{-t})\sqrt{(1/t)^2+(-e^{-t})^2}\,dt\approx 0.052$.

30. **(a)** We parametrize the circle C as $\boldsymbol{r}(t)=2\cos t\,\boldsymbol{i}+2\sin t\,\boldsymbol{j}$, $0\le t\le 2\pi$. So
$\boldsymbol{F}(\boldsymbol{r}(t))=\left\langle 4\cos^2 \mathrm{t},\ 4\cos t\sin t\right\rangle$, $\boldsymbol{r}'(t)=\langle -2\sin \mathrm{t},\ 2\cos t\rangle$, and
$W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^{2\pi}(-8\cos^2 t\sin t+8\cos^2 t\sin t)\,dt=0$.

(b)



From the graph, we see that all of the vectors in the field are perpendicular to the path. This indicates that the field does no work on the particle, since the field never pulls the particle in the direction in which it is going. In other words, at any point along C , $\boldsymbol{F}\cdot\boldsymbol{T}=0$, and so certainly $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=0$.

31. We use the parametrization $x=2\cos t$, $y=2\sin t$, $-\frac{\pi}{2}\leq t\leq\frac{\pi}{2}$. Then

$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\sqrt{(-2\sin t)^2+(2\cos t)^2}\,dt=2\,dt$, so $m=\int_C k\,ds=2k\int_{-\pi/2}^{\pi/2}dt=2k(\pi)$,

$\bar{x}=\frac{1}{2\pi k}\int_C xk\,ds=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(2\cos t)2\,dt=\frac{1}{2\pi}\left[4\sin t\right]_{-\pi/2}^{\pi/2}=\frac{4}{\pi}$, $\bar{y}=\frac{1}{2\pi k}\int_C yk\,ds=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(2\sin t)2\,dt=0$.

Hence $(\bar{x},\bar{y})=\left(\frac{4}{\pi},0\right)$.

32. We use the parametrization $x=r\cos t$, $y=r\sin t$, $0\leq t\leq\frac{\pi}{2}$. Then

$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\sqrt{(-r\sin t)^2+(r\cos t)^2}\,dt=r\,dt$, so

$m=\int_C(x+y)ds=\int_0^{\pi/2}(r\cos t+r\sin t)r\,dt=r^2\left[\sin t-\cos t\right]_0^{\pi/2}=2r^2$,

$$\bar{x}=\frac{1}{2r^2}\int_C x(x+y)\,ds=\frac{1}{2r^2}\int_0^{\pi/2}(r^2\cos^2 t+r^2\cos t\sin t)r\,dt=\frac{r}{2}\left[\frac{t}{2}+\frac{\sin 2t}{4}-\frac{\cos 2t}{4}\right]_0^{\pi/2}$$

$=\frac{r(\pi+2)}{8}$, and

$$\bar{y}=\frac{1}{2r^2}\int_C y(x+y)\,ds=\frac{1}{2r^2}\int_0^{\pi/2}(r^2\sin t\cos t+r^2\sin^2 t)r\,dt$$

$$=\frac{r}{2}\left[-\frac{\cos 2t}{4}+\frac{t}{2}-\frac{\sin 2t}{4}\right]_0^{\pi/2}=\frac{r(\pi+2)}{8}\ .$$

Therefore $(\bar{x},\bar{y})=\left(\frac{r(\pi+2)}{8},\frac{r(\pi+2)}{8}\right)$.

33. **(a)** $\bar{x}=\frac{1}{m}\int_C x\rho(x,y,z)\,ds$, $\bar{y}=\frac{1}{m}\int_C y\rho(x,y,z)\,ds$, $\bar{z}=\frac{1}{m}\int_C z\rho(x,y,z)\,ds$ where $m=\int_C\rho(x,y,z)\,ds$.

(b) $m=\int_C k\,ds=k\int_0^{2\pi}\sqrt{4\sin^2 t+4\cos^2 t+9}\,dt=k\sqrt{13}\int_0^{2\pi}dt=2\pi k\sqrt{13}$, $\bar{x}=\frac{1}{2\pi k\sqrt{13}}\int_0^{2\pi}k2\sqrt{13}\sin t\,dt=0$,

$\bar{y}=\frac{1}{2\pi k\sqrt{13}}\int_0^{2\pi}k2\sqrt{13}\cos t\,dt=0$, $\bar{z}=\frac{1}{2\pi k\sqrt{13}}\int_0^{2\pi}(k\sqrt{13})(3t)\,dt=\frac{3}{2\pi}(2\pi^2)=3\pi$. Hence

$(\overline{x}, \overline{y}, \overline{z})=(0, 0, 3\pi)$.

34.

$$m = \int_C (x^2+y^2+z^2)ds = \int_0^{2\pi} (t^2+1)\sqrt{(1)^2+(-\sin t)^2+(\cos t)^2}\, dt = \int_0^{2\pi} (t^2+1)\sqrt{2}\, dt$$

$$= \sqrt{2}\left(\frac{8}{3}\pi^3+2\pi\right),$$

$$\overline{x} = \frac{1}{\sqrt{2}\left(\frac{8}{3}\pi^3+2\pi\right)}\int_0^{2\pi}\sqrt{2}\,(t^3+t)\,dt = \frac{4\pi^4+2\pi^2}{\frac{8}{3}\pi^3+2\pi} = \frac{3\pi(2\pi^2+1)}{4\pi^2+3},$$

$$\overline{y} = \frac{3}{2\sqrt{2}\pi(4\pi^2+3)}\int_0^{2\pi}(\sqrt{2}\cos t)(t^2+1)\,dt = 0 \text{ , and}$$

$$\overline{z} = \frac{3}{2\sqrt{2}\pi(4\pi^2+3)}\int_0^{2\pi}(\sqrt{2}\sin t)(t^2+1)\,dt = 0 \text{ . Hence } (\overline{x}, \overline{y}, \overline{z}) = \left(\frac{3\pi(2\pi^2+1)}{4\pi^2+3}, 0, 0\right).$$

35.

From Example 3, $\rho(x, y)=k(1-y)$, $x=\cos t$, $y=\sin t$, and $ds=dt$, $0\le t\le \pi \Rightarrow$

$$I_x = \int_C y^2 \rho(x, y)ds = \int_0^{\pi}\sin^2 t\, dt = k\int_0^{\pi}(\sin^2 t - \sin^3 t)\,dt$$

$$= \frac{1}{2}k\int_0^{\pi}(1-\cos 2t)\,dt - k\int_0^{\pi}(1-\cos^2 t)\sin t\, dt \quad \text{[Let } u=t, du=-tdt-3pt \text{ in the second integral]}$$

$$= k\left[\frac{\pi}{2}+\int_1^{-1}(1-u^2)\,du\right] = k\left(\frac{\pi}{2}-\frac{4}{3}\right)$$

$$I_y = \int_C x^2 \rho(x, y)ds = k\int_0^{\pi}\cos^2 t(1-\sin t)\,dt = \frac{k}{2}\int_0^{\pi}(1+\cos 2t)\,dt - k\int_0^{\pi}\cos^2 t\sin t\, dt$$

$$= k\left(\frac{\pi}{2}-\frac{2}{3}\right) \text{ , using the same substitution as above.}$$

36.

The wire is given as $x=2\sin t$, $y=2\cos t$, $z=3t$, $0\le t\le 2\pi$ with $\rho(x, y, z)=k$. Then

$$ds=\sqrt{(2\cos t)^2+(-2\sin t)^2+3^2}=\sqrt{4(\cos^2 t+\sin^2 t)+9}=\sqrt{13} \text{ and}$$

$$I_x = \int_C (y^2+z^2)\rho(x, y, z)ds = \int_0^{2\pi}(4\cos^2 t+9t^2)(k)\sqrt{13}\, dt = \sqrt{13}\,k\left[4\left(\frac{1}{2}t+\frac{1}{4}\sin 2t\right)+3t^3\right]_0^{2\pi}$$

$=\sqrt{13}\,k(4\pi+24\pi^3)=4\sqrt{13}\,\pi k(1+6\pi^2)$

$I_y=\int_C (x^2+z^2)\rho(x,y,z)\,ds=\int_0^{2\pi}\left(4\sin^2 t+9t^2\right)(k)\sqrt{13}\,dt=\sqrt{13}\,k\left[4\left(\frac{1}{2}t-\frac{1}{4}\sin 2t\right)+3t^3\right]_0^{2\pi}$

$=\sqrt{13}\,k(4\pi+24\pi^3)=4\sqrt{13}\,\pi k(1+6\pi^2)$

$I_z=\int_C (x^2+y^2)\rho(x,y,z)\,ds=\int_0^{2\pi}(4\sin^2 t+4\cos^2 t)(k)\sqrt{13}\,dt=4\sqrt{13}\,k\int_0^{2\pi}dt=8\pi\sqrt{13}\,k$

37.

$W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^{2\pi}\langle t-\sin t,\,3-\cos t\rangle\cdot\langle 1-\cos t,\,\sin t\rangle\,dt$

$=\int_0^{2\pi}(t-t\cos t-\sin t+\sin t\cos t+3\sin t-\sin t\cos t)\,dt$

$=\int_0^{2\pi}(t-t\cos t+2\sin t)\,dt=\left[\frac{1}{2}t^2-(t\sin t+\cos t)-2\cos t\right]_0^{2\pi}$ [by integrating by parts in the second term]

$=2\pi^2$

38.

$x=x$, $y=x^2$, $-1\le x\le 2$,

$W=\int_{-1}^{2}\left\langle x\sin x^2,\,x^2\right\rangle\cdot\langle 1,\,2x\rangle\,dx=\int_{-1}^{2}(x\sin x^2+2x^3)\,dx=\left[-\frac{1}{2}\cos x^2+\frac{1}{2}x^4\right]_{-1}^{2}$

$=\frac{1}{2}(15+\cos 1-\cos 4)$

39.

$\boldsymbol{r}(t)=\langle 1+2t,\,4t,\,2t\rangle$, $0\le t\le 1$,

$W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^1\langle 6t,\,1+4t,\,1+6t\rangle\cdot\langle 2,4,2\rangle\,dt=\int_0^1(12t+4(1+4t)+2(1+6t))\,dt$

$=\int_0^1(40t+6)\,dt=\left[20t^2+6t\right]_0^1=26$

40.

$\boldsymbol{r}(t)=2\boldsymbol{i}+t\,\boldsymbol{j}+5t\,\boldsymbol{k}$, $0\le t\le 1$. Therefore

$W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^1\frac{K\langle 2,t,5t\rangle}{(4+26t^2)^{3/2}}\cdot\langle 0,1,5\rangle\,dt=K\int_0^1\frac{26t}{(4+26t^2)^{3/2}}\,dt$

$$= K\left[-(4+26t^2)^{-1/2}\right]_0^1 = K\left(\frac{1}{2} - \frac{1}{\sqrt{30}}\right)$$

41.

Let $\boldsymbol{F}=185\boldsymbol{k}$. To parametrize the staircase, let

$x=20\cos t$, $y=20\sin t$, $z=\frac{90}{6\pi}t=\frac{15}{\pi}t$, $0\le t\le 6\pi \Rightarrow$

$$W = \int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^{6\pi}\langle 0, 0, 185\rangle\cdot\left\langle -20\sin t, 20\cos t, \frac{15}{\pi}\right\rangle dt = (185)\frac{15}{\pi}\int_0^{6\pi} dt = (185)(90)$$

$$\approx 1.67\times 10^4 \text{ ft–lb}$$

42.

This time m is a function of t : $m=185-\frac{9}{6\pi}t=185-\frac{3}{2\pi}t$. So let $\boldsymbol{F}=\left(185-\frac{3}{2\pi}t\right)\boldsymbol{k}$. To parametrize the staircase, let $x=20\cos t$, $y=20\sin t$, $z=\frac{90}{6\pi}t=\frac{15}{\pi}t$, $0\le t\le 6\pi$. Therefore

$$W = \int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^{6\pi}\left\langle 0, 0, 185-\frac{3}{2\pi}t\right\rangle\cdot\left\langle -20\sin t, 20\cos t, \frac{15}{\pi}\right\rangle dt = \frac{15}{\pi}\int_0^{6\pi}\left(185-\frac{3}{2\pi}t\right)dt$$

$$= \frac{15}{\pi}\left[185t-\frac{3}{4\pi}t^2\right]_0^{6\pi} = 90\left(185-\frac{9}{2}\right)\approx 1.62\times 10^4 \text{ ft–lb}$$

43. **(a)** $\boldsymbol{r}(t)=\langle\cos t, \sin t\rangle$, $0\le t\le 2\pi$, and let $\boldsymbol{F}=\langle a, b\rangle$. Then

$$W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^{2\pi}\langle a, b\rangle\cdot\langle -\sin t, \cos t\rangle\, dt$$

$$=\int_0^{2\pi}(-a\sin t+b\cos t)dt=\left[a\cos t+b\sin t\right]_0^{2\pi}$$

$$=a+0-a+0=0$$

(b) Yes. $\boldsymbol{F}(x, y)=k\,\boldsymbol{x}=\langle kx, ky\rangle$ and

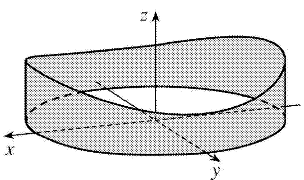
$$W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^{2\pi}\langle k\cos t, k\sin t\rangle\cdot\langle -\sin t, \cos t\rangle\, dt$$

$$=\int_0^{2\pi}(-k\sin t\cos t+k\sin t\cos t)dt=\int_0^{2\pi}0\,dt=0$$

44. Consider the base of the fence in the xy –plane, centered at the origin, with the height given by $z=h(x, y)$. The fence can be graphed using the parametric equations

$x=10\cos u$, $y=10\sin u$,

$$\begin{aligned} z &= v\left[4+0.01((10\cos u)^2-(10\sin u)^2)\right] \\ &= v(4+\cos^2 u-\sin^2 u) \\ &= v(4+\cos 2u),\ 0\le u\le 2\pi,\ 0\le v\le 1 . \end{aligned}$$



The area of the fence is $\int_C h(x, y)\,ds$ where C , the base of the fence, is given by $x=10\cos t$, $y=10\sin t$, $0\le t\le 2\pi$. Then

$$\begin{aligned} \int_C h(x, y)\,ds &= \int_0^{2\pi}\left[4+0.01((10\cos t)^2-(10\sin t)^2)\right]\sqrt{(-10\sin t)^2+(10\cos t)^2}\,dt \\ &= \int_0^{2\pi}(4+\cos 2t)\sqrt{100}\,dt=10\left[4t+\frac{1}{2}\sin 2t\right]_0^{2\pi} \\ &= 10(8\pi)=80\pi\,\mathrm{m}^2 \end{aligned}$$

If we paint both sides of the fence, the total surface area to cover is $160\pi\ \mathrm{m}^2$, and since 1 L of paint covers $100\ \mathrm{m}^2$, we require $\dfrac{160\pi}{100}=1.6\pi\approx 5.03$ L of paint.

45. The work done in moving the object is $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds$. We can approximate this integral by dividing C into 7 segments of equal length $\Delta s=2$ and approximating $\boldsymbol{F}\cdot\boldsymbol{T}$, that is, the tangential component of force, at a point (x_i^*, y_i^*) on each segment. Since C is composed of straight line segments, $\boldsymbol{F}\cdot\boldsymbol{T}$ is the scalar projection of each force vector onto C . If we choose (x_i^*, y_i^*) to be the point on the segment closest to the origin, then the work done is $\int_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds\approx\sum_{i=1}^{7}\left[\boldsymbol{F}(x_i^*, y_i^*)\cdot\boldsymbol{T}(x_i^*, y_i^*)\right]\Delta s=[2+2+2+2+1+1+1](2)=22$. Thus, we estimate the work done to be approximately 22 J.

46. Use the orientation pictured in the figure. Then since $\boldsymbol{B}$ is tangent to any circle that lies in the plane perpendicular to the wire, $\boldsymbol{B}=|\boldsymbol{B}|\boldsymbol{T}$ where $\boldsymbol{T}$ is the unit tangent to the circle C : $x=r\cos\theta$, $y=r\sin\theta$. Thus

$\boldsymbol{B}=|\boldsymbol{B}|\left\langle -\sin\theta , \cos\theta \right\rangle$. Then
$\int_C \boldsymbol{B}\cdot d\boldsymbol{r}=\int_0^{2\pi}|\boldsymbol{B}|\left\langle -\sin\theta , \cos\theta \right\rangle\cdot\left\langle -r\sin\theta , r\cos\theta \right\rangle d\theta=\int_0^{2\pi}|\boldsymbol{B}|r\, d\theta=2\pi r|\boldsymbol{B}|$. (Note that $|\boldsymbol{B}|$ here is the magnitude of the field at a distance r from the wire's center). But by Ampere's Law $\int_C \boldsymbol{B}\cdot d\boldsymbol{r}=\mu_0 I$.

Hence $|\boldsymbol{B}|=\dfrac{\mu_0 I}{2\pi r}$.

1. C appears to be a smooth curve, and since ∇f is continuous, we know f is differentiable. Then Theorem 2 says that the value of $\int_C \nabla f \cdot d\mathbf{r}$ is simply the difference of the values of f at the terminal and initial points of C . From the graph, this is $50-10=40$.

2. C is represented by the vector function $\mathbf{r}(t)=(t^2+1)\mathbf{i}+(t^3+t)\mathbf{j}$, $0\le t\le 1$, so $\mathbf{r}'(t)=2t\,\mathbf{i}+(3t^2+1)\mathbf{j}$. Since $3t^2+1\neq 0$, we have $\mathbf{r}'(t)\neq\mathbf{0}$, thus C is a smooth curve. ∇f is continuous, and hence f is differentiable, so by Theorem 2 we have $\int_C \nabla f\cdot d\mathbf{r}=f(\mathbf{r}(1))-f(\mathbf{r}(0))=f(2, 2)-f(1, 0)=9-3=6$.

3. $\partial(6x+5y)/\partial y=5=\partial(5x+4y)/\partial x$ and the domain of $\mathbf{F}$ is R^2 which is open and simply-connected, so by Theorem 6 $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\nabla f=\mathbf{F}$, that is, $f_x(x, y)=6x+5y$ and $f_y(x, y)=5x+4y$. But $f_x(x, y)=6x+5y$ implies $f(x,y)=3x^2+5xy+g(y)$ and differentiating both sides of this equation with respect to y gives $f_y(x, y)=5x+g'(y)$. Thus $5x+4y=5x+g'(y)$ so $g'(y)=4y$ and $g(y)=2y^2+K$ where K is a constant. Hence $f(x, y)=3x^2+5xy+2y^2+K$ is a potential function for $\mathbf{F}$.

4. $\partial(x^3+4xy)/\partial y=4x$, $\partial(4xy-y^3)/\partial x=4y$. Since these are not equal, $\mathbf{F}$ is not conservative.

5. $\partial(xe^y)/\partial y=xe^y$,$\partial(ye^x)/\partial x=ye^x$. Since these are not equal, $\mathbf{F}$ is not conservative.

6. $\partial(e^y)/\partial y=e^y=\partial(xe^y)/\partial x$ and the domain of $\mathbf{F}$ is R^2 . Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f=\mathbf{F}$. Then $f_x(x, y)=e^y$ implies $f(x, y)=xe^y+g(y)$ and $f_y(x, y)=xe^y+g'(y)$. But $f_y(x, y)=xe^y$ so $g'(y)=0\Rightarrow g(y)=K$. Then $f(x, y)=xe^y+K$ is a potential function for $\mathbf{F}$.

7. $\partial(2x\cos y-y\cos x)/\partial y=-2x\sin y-\cos x=\partial(-x^2\sin y-\sin x)/\partial x$ and the domain of $\mathbf{F}$ is R^2 . Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f=\mathbf{F}$. Then $f_x(x, y)=2x\cos y-y\cos x$ implies $f(x, y)=x^2\cos y-y\sin x+g(y)$ and $f_y(x, y)=-x^2\sin y-\sin x+g'(y)$. But $f_y(x, y)=-x^2\sin y-\sin x$ so $g'(y)=0\Rightarrow g(y)=K$. Then $f(x, y)=x^2\cos y-y\sin x+K$ is a potential function for $\mathbf{F}$.

8. $\partial(1+2xy+\ln x)/\partial y=2x=\partial(x^2)/\partial x$ and the domain of $\mathbf{F}$ is $\{ (x, y) \mid x>0 \}$ which is open and simply-connected. Hence $\mathbf{F}$ is conservative, so there exists a function f such that $\nabla f=\mathbf{F}$. Then $f_x(x, y)=1+2xy+\ln x$ implies $f(x, y)=x+x^2y+x\ln x-x+g(y)$ and $f_y(x, y)=x^2+g'(y)$. But $f_y(x, y)=x^2$ so $g'(y)=0\Rightarrow g(y)=K$. Then $f(x,y)=x^2y+x\ln x+K$ is a potential function for $\mathbf{F}$.

9. $\partial(ye^{x}+\sin y)/\partial y=e^{x}+\cos y=\partial(e^{x}+x\cos y)/\partial x$ and the domain of $\boldsymbol{F}$ is R^{2} . Hence $\boldsymbol{F}$ is conservative so there exists a function f such that $\nabla f=\boldsymbol{F}$. Then $f_{x}(x, y)=ye^{x}+\sin y$ implies $f(x, y)=ye^{x}+x\sin y+g(y)$ and $f_{y}(x, y)=e^{x}+x\cos y+g'(y)$. But $f_{y}(x, y)=e^{x}+x\cos y$ so $g(y)=K$ and $f(x, y)=ye^{x}+x\sin y+K$ is a potential function for $\boldsymbol{F}$.

10. $\dfrac{\partial(xy\cosh xy+\sinh xy)}{\partial y}=x^{2}y\sinh xy+x\cosh xy+x\cosh xy=x^{2}y\sinh xy+2x\cosh xy=\dfrac{\partial(x^{2}\cosh xy)}{\partial x}$ and the domain of $\boldsymbol{F}$ is R^{2} . Thus $\boldsymbol{F}$ is conservative, so there exists a function f such that $\nabla f=\boldsymbol{F}$. Then $f_{x}(x, y)=xy\cosh xy+\sinh xy$ implies $f(x, y)=x\sinh xy+g(y)\Rightarrow f_{y}(x, y)=x^{2}\cosh xy+g'(y)$. But $f_{y}(x, y)=x^{2}\cosh xy$ so $g(y)=K$ and $f(x, y)=x\sinh xy+K$ is a potential function for $\boldsymbol{F}$.

11. **(a)** $\boldsymbol{F}$ has continuous first–order partial derivatives and $\dfrac{\partial}{\partial y}2xy=2x=\dfrac{\partial}{\partial x}(x^{2})$ on R^{2} , which is open and simply–connected. Thus, $\boldsymbol{F}$ is conservative by Theorem 6. Then we know that the line integral of $\boldsymbol{F}$ is independent of path; in particular, the value of $\int_{C}\boldsymbol{F}\cdot d\boldsymbol{r}$ depends only on the endpoints of C . Since all three curves have the same initial and terminal points, $\int_{C}\boldsymbol{F}\cdot d\boldsymbol{r}$ will have the same value for each curve.

(b) We first find a potential function f , so that $\nabla f=\boldsymbol{F}$. We know $f_{x}(x, y)=2xy$ and $f_{y}(x, y)=x^{2}$. Integrating $f_{x}(x, y)$ with respect to x , we have $f(x, y)=x^{2}y+g(y)$. Differentiating both sides with respect to y gives $f_{y}(x, y)=x^{2}+g'(y)$, so we must have $x^{2}+g'(y)=x^{2}\Rightarrow g'(y)=0\Rightarrow g(y)=K$, a constant. Thus $f(x, y)=x^{2}y+K$. All three curves start at $(1, 2)$ and end at $(3, 2)$, so by Theorem 2, $\int_{C}\boldsymbol{F}\cdot d\boldsymbol{r}=f(3, 2)-f(1, 2)=18-2=16$ for each curve.

12. **(a)** $f_{x}(x, y)=y$ implies $f(x, y)=xy+g(y)$ and $f_{y}(x, y)=x+g'(y)$. But $f_{y}(x, y)=x+2y$ so $g'(y)=2y\Rightarrow g(y)=y^{2}+K$. We can take $K=0$, so $f(x, y)=xy+y^{2}$.

(b) $\int_{C}\boldsymbol{F}\cdot d\boldsymbol{r}=f(2, 1)-f(0, 1)=3-1=2$.

13. **(a)** $f_{x}(x, y)=x^{3}y^{4}$ implies $f(x, y)=\frac{1}{4}x^{4}y^{4}+g(y)$ and $f_{y}(x, y)=x^{4}y^{3}+g'(y)$. But $f_{y}(x, y)=x^{4}y^{3}$ so $g'(y)=0\Rightarrow g(y)=K$, a constant. We can take $K=0$, so $f(x, y)=\frac{1}{4}x^{4}y^{4}$.

(b) The initial point of C is $\boldsymbol{r}(0)=(0, 1)$ and the terminal point is $\boldsymbol{r}(1)=(1, 2)$, so
$\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(1, 2)-f(0, 1)=4-0=4$.

14. **(a)** $f_x(x, y)=y^2/(1+x^2)$ implies $f(x, y)=y^2\arctan x+g(y)\Rightarrow f_y(x, y)=2y\arctan x+g'(y)$. But $f_y(x, y)=2y\arctan x$ so $g'(y)=0\Rightarrow g(y)=K$. We can take $K=0$, so $f(x, y)=y^2\arctan x$.
(b) The initial point of C is $\boldsymbol{r}(0)=(0, 0)$ and the terminal point is $\boldsymbol{r}(1)=(1, 2)$, so
$\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(1, 2)-f(0, 0)=4\arctan 1-0=4\cdot \frac{\pi}{4}=\pi$.

15. **(a)** $f_x(x, y, z)=yz$ implies $f(x, y, z)=xyz+g(y, z)$ and so $f_y(x, y, z)=xz+g_y(y, z)$. But $f_y(x, y, z)=xz$ so $g_y(y, z)=0\Rightarrow g(y, z)=h(z)$. Thus $f(x, y, z)=xyz+h(z)$ and $f_z(x, y, z)=xy+h'(z)$. But $f_z(x, y, z)=xy+2z$, so $h'(z)=2z\Rightarrow h(z)=z^2+K$. Hence $f(x, y, z)=xyz+z^2$ (taking $K=0$).
(b) $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(4, 6, 3)-f(1, 0, -2)=81-4=77$.

16. **(a)** $f_x(x, y, z)=2xz+y^2$ implies $f(x, y, z)=x^2z+xy^2+g(y, z)$ and so $f_y(x, y, z)=2xy+g_y(y, z)$. But $f_y(x, y, z)=2xy$ so $g_y(y, z)=0 \Rightarrow g(y, z)=h(z)$. Thus $f(x, y, z)=x^2z+xy^2+h(z)$ and $f_z(x, y, z)=x^2+h'(z)$. But $f_z(x, y, z)=x^2+3z^2$, so $h'(z)=3z^2\Rightarrow h(z)=z^3+K$. Hence $f(x, y, z)=x^2z+xy^2+z^3$ (taking $K=0$).
(b) $t=0$ corresponds to the point $(0, 1, -1)$ and $t=1$ corresponds to $(1, 2, 1)$, so
$\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(1, 2, 1)-f(0, 1, -1)=6-(-1)=7$.

17. **(a)** $f_x(x, y, z)=y^2\cos z$ implies $f(x, y, z)=xy^2\cos z+g(y, z)$ and so $f_y(x, y, z)=2xy\cos z+g_y(y, z)$. But $f_y(x, y, z)=2xy\cos z$ so $g_y(y, z)=0\Rightarrow g(y, z)=h(z)$. Thus $f(x, y, z)=xy^2\cos z+h(z)$ and $f_z(x, y, z)=-xy^2\sin z+h'(z)$. But $f_z(x, y, z)=-xy^2\sin z$, so $h'(z)=0\Rightarrow h(z)=K$. Hence $f(x, y, z)=xy^2\cos z$ (taking $K=0$).
(b) $\boldsymbol{r}(0)=\langle 0, 0, 0\rangle$, $\boldsymbol{r}(\pi)=\langle \pi^2, 0, \pi\rangle$ so $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(\pi^2, 0, \pi)-f(0, 0, 0)=0-0=0$.

18. **(a)** $f_x(x, y, z)=e^y$ implies $f(x, y, z)=xe^y+g(y, z)$ and so $f_y(x, y, z)=xe^y+g_y(y, z)$. But $f_y(x, y, z)=xe^y$ so $g_y(y, z)=0\Rightarrow g(y, z)=h(z)$. Thus $f(x, y, z)=xe^y+h(z)$ and $f_z(x, y, z)=0+h'(z)$. But $f_z(x, y, z)=(z+1)e^z$, so

$h'(z)=(z+1)e^{z} \Rightarrow h(z)=ze^{z}+K$ (using integration by parts). Hence $f(x, y, z)=xe^{y}+ze^{z}$ (taking $K=0$).
(b) $\boldsymbol{r}(0)=\langle 0, 0, 0\rangle$, $\boldsymbol{r}(1)=\langle 1, 1, 1\rangle$ so $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(1, 1, 1)-f(0, 0, 0)=2e-0=2e$.

19. Here $\boldsymbol{F}(x, y)=\tan y\boldsymbol{i}+x\sec^{2}y\boldsymbol{j}$. Then $f(x, y)=x\tan y$ is a potential function for $\boldsymbol{F}$, that is, $\nabla f=\boldsymbol{F}$ so $\boldsymbol{F}$ is conservative and thus its line integral is independent of path. Hence
$\int_C \tan y\,dx+x\sec^{2}y\,dy=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f\left(2, \frac{\pi}{4}\right)-f(1, 0)=2\tan\frac{\pi}{4}-\tan 0=2$.

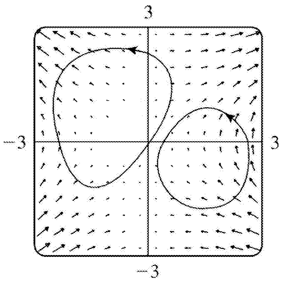
20. Here $\boldsymbol{F}(x, y)=(1-ye^{-x})\boldsymbol{i}+e^{-x}\boldsymbol{j}$. Then $f(x, y)=x+ye^{-x}$ is a potential function for $\boldsymbol{F}$, that is, $\nabla f=\boldsymbol{F}$ so $\boldsymbol{F}$ is conservative and thus its line integral is independent of path. Hence
$\int_C (1-ye^{-x})dx+e^{-x}dy=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(1, 2)-f(0, 1)=(1+2e^{-1})-1=2/e$.

21. $\boldsymbol{F}(x, y)=2y^{3/2}\boldsymbol{i}+3x\sqrt{y}\,\boldsymbol{j}$, $W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$. Since $\partial(2y^{3/2})/\partial y=3\sqrt{y}=\partial(3x\sqrt{y})/\partial x$, there exists a function f such that $\nabla f=\boldsymbol{F}$. In fact, $f_x(x, y)=2y^{3/2}\Rightarrow f(x, y)=2xy^{3/2}+g(y)\Rightarrow f_y(x, y)=3xy^{1/2}+g'(y)$. But $f_y(x, y)=3x\sqrt{y}$ so $g'(y)=0$ or $g(y)=K$. We can take $K=0\Rightarrow f(x, y)=2xy^{3/2}$. Thus
$W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(2, 4)-f(1, 1)=2(2)(8)-2(1)=30$.

22. $\boldsymbol{F}(x, y)=\frac{y^2}{x^2}\boldsymbol{i}-\frac{2y}{x}\boldsymbol{j}$, $W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$. Since $\frac{\partial}{\partial y}\left(\frac{y^2}{x^2}\right)=\frac{2y}{x^2}=\frac{\partial}{\partial x}\left(-\frac{2y}{x}\right)$, there exists a function f such that $\nabla f=\boldsymbol{F}$. In fact, $f_x=y^2/x^2\Rightarrow f(x, y)=-y^2/x+g(y)\Rightarrow f_y=-2y/x+g'(y)\Rightarrow g'(y)=0$, so we can take $f(x, y)=-y^2/x$ as a potential function for $\boldsymbol{F}$. Thus
$W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(4, -2)-f(1, 1)=-[(-2)^2/4]+(1/1)=0$.

23. We know that if the vector field (call it $\boldsymbol{F}$) is conservative, then around any closed path C , $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=0$. But take C to be some circle centered at the origin, oriented counterclockwise. All of the field vectors along C oppose motion along C , so the integral around C will be negative. Therefore the field is not conservative.

24.

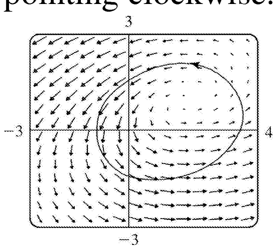


From the graph, it appears that $\boldsymbol{F}$ is conservative, since around all closed paths, the number and size of the field vectors pointing in directions similar to that of the path seem to be roughly the same as the number and size of the vectors pointing in the opposite direction. To check, we calculate

$$\frac{\partial}{\partial y}(2xy+\sin y)=2x+\cos y\ ,\quad \frac{\partial}{\partial x}(x^2+x\cos y)=2x+\cos y$$

Thus $\boldsymbol{F}$ is conservative, by Theorem 6.

25. From the graph, it appears that $\boldsymbol{F}$ is not conservative. For example, any closed curve containing the point (2, 1) seems to have many field vectors pointing counterclockwise along it, and none pointing clockwise. So along this path the integral $\int \boldsymbol{F}\cdot d\boldsymbol{r}\neq 0$. To confirm our guess, we calculate



$$\frac{\partial}{\partial y}\left(\frac{x-2y}{\sqrt{1+x^2+y^2}}\right)=(x-2y)\left[\frac{-y}{(1+x^2+y^2)^{3/2}}\right]-\frac{2}{\sqrt{1+x^2+y^2}}=\frac{-2-2x^2-xy}{(1+x^2+y^2)^{3/2}}\ ,$$

$$\frac{\partial}{\partial x}\left(\frac{x-2}{\sqrt{1+x^2+y^2}}\right)=(x-2)\left[\frac{-x}{(1+x^2+y^2)^{3/2}}\right]+\frac{1}{\sqrt{1+x^2+y^2}}=\frac{1+y^2+2x}{(1+x^2+y^2)^{3/2}}\ .$$

These are not equal, so the field is not conservative, by Theorem 5.

26. $\nabla f(x,y)=\cos(x-2y)\boldsymbol{i}-2\cos(x-2y)\boldsymbol{j}$

(a) We use Theorem 2: $\int_{C_1}\boldsymbol{F}\cdot d\boldsymbol{r}=\int_{C_1}\nabla f\cdot d\boldsymbol{r}=f(\boldsymbol{r}(b))-f(\boldsymbol{r}(a))$ where C_1 starts at $t=a$ and ends at $t=b$.

So because $f(0,0)=\sin 0=0$ and $f(\pi,\pi)=\sin(\pi-2\pi)=0$, one possible curve C_1 is the straight line from (0, 0) to (π,π) ; that is, $\boldsymbol{r}(t)=\pi t\boldsymbol{i}+\pi t\boldsymbol{j}$, $0\leq t\leq 1$.

(b) From (a), $\int_{C_2}\boldsymbol{F}\cdot d\boldsymbol{r}=f(\boldsymbol{r}(b))-f(\boldsymbol{r}(a))$. So because $f(0,0)=\sin 0=0$ and $f\left(\frac{\pi}{2},0\right)=1$, one possible curve C_2 is $\boldsymbol{r}(t)=\frac{\pi}{2}t\boldsymbol{i}$, $0\leq t\leq 1$, the straight line from (0, 0) to $\left(\frac{\pi}{2},0\right)$.

27. Since $\boldsymbol{F}$ is conservative, there exists a function f such that $\boldsymbol{F}=\nabla f$, that is, $P=f_x$, $Q=f_y$, and $R=f_z$. Since P, Q and R have continuous first order partial derivatives, Clairaut's Theorem says that $\partial P/\partial y=f_{xy}=f_{yx}=\partial Q/\partial x$, $\partial P/\partial z=f_{xz}=f_{zx}=\partial R/\partial x$, and $\partial Q/\partial z=f_{yz}=f_{zy}=\partial R/\partial y$.

28. Here $\boldsymbol{F}(x, y, z)=y\boldsymbol{i}+x\boldsymbol{j}+xyz\boldsymbol{k}$. Then using the notation of Exercise 27, $\partial P/\partial z=0$ while $\partial R/\partial x=yz$. Since these aren't equal, $\boldsymbol{F}$ is not conservative. Thus by Theorem 4, the line integral of $\boldsymbol{F}$ is not independent of path.

29. $D=\{ (x, y) \mid x>0,\ y>0 \}$= the first quadrant (excluding the axes).
(a) D is open because around every point in D we can put a disk that lies in D.
(b) D is connected because the straight line segment joining any two points in D lies in D.
(c) D is simply–connected because it's connected and has no holes.

30. $D=\{ (x, y) \mid x\neq 0 \}$ consists of all points in the xy –plane except for those on the y –axis.
(a) D is open.
(b) Points on opposite sides of the y –axis cannot be joined by a path that lies in D, so D is not connected.
(c) D is not simply–connected because it is not connected.

31. $D=\{ (x, y) \mid 1<x^2+y^2<4 \}$= the annular region between the circles with center (0, 0) and radii 1 and 2 .
(a) D is open.
(b) D is connected.
(c) D is not simply–connected. For example, $x^2+y^2=(1.5)^2$ is simple and closed and lies within D but encloses points that are not in D. (Or we can say, D has a hole, so is not simply–connected.)

32. $D=\{ (x, y) \mid x^2+y^2\leq 1 \text{ or } 4\leq x^2+y^2\leq 9 \}$= the points on or inside the circle $x^2+y^2=1$, together with the points on or between the circles $x^2+y^2=4$ and $x^2+y^2=9$.
(a) D is not open because, for instance, no disk with center (0, 2) lies entirely within D.
(b) D is not connected because, for example, (0, 0) and (0, 2.5) lie in D but cannot be joined by a path that lies entirely in D.
(c) D is not simply–connected because, for example, $x^2+y^2=9$ is a simple closed curve in D but encloses points that are not in D.

33. **(a)** $P=-\dfrac{y}{x^2+y^2}$, $\dfrac{\partial P}{\partial y}=\dfrac{y^2-x^2}{(x^2+y^2)^2}$ and $Q=\dfrac{x}{x^2+y^2}$, $\dfrac{\partial Q}{\partial x}=\dfrac{y^2-x^2}{(x^2+y^2)^2}$. Thus $\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}$.
(b) C_1 : $x=\cos t$, $y=\sin t$, $0\leq t\leq \pi$, C_2 : $x=\cos t$, $y=\sin t$, $t=2\pi$ to $t=\pi$. Then

$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^{\pi} \frac{(-\sin t)(-\sin t) + (\cos t)(\cos t)}{\cos^2 t + \sin^2 t}\, dt = \int_0^{\pi} dt = \pi$ and $\int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{2\pi}^{\pi} dt = -\pi$

Since these aren't equal, the line integral of $\boldsymbol{F}$ isn't independent of path. (Or notice that $\int_{C_3} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^{2\pi} dt = 2\pi$ where C_3 is the circle $x^2 + y^2 = 1$, and apply the contrapositive of Theorem 3). This doesn't contradict Theorem 6, since the domain of $\boldsymbol{F}$, which is R^2 except the origin, isn't simply–connected.

34. **(a)** Here $\boldsymbol{F}(\boldsymbol{r}) = c\boldsymbol{r}/|\boldsymbol{r}|^3$ and $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Then $f(\boldsymbol{r}) = -c/|\boldsymbol{r}|$ is a potential function for $\boldsymbol{F}$, that is, $\nabla f = \boldsymbol{F}$. (See the discussion of gradient fields in Section 17.1). Hence $\boldsymbol{F}$ is conservative and its line integral is independent of path. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$.

$$W = \int_C \boldsymbol{F} \cdot d\boldsymbol{r} = f(P_2) - f(P_1) = -\frac{c}{(x_2^2 + y_2^2 + z_2^2)^{1/2}} + \frac{c}{(x_1^2 + y_1^2 + z_1^2)^{1/2}} = c\left(\frac{1}{d_1} - \frac{1}{d_2}\right).$$

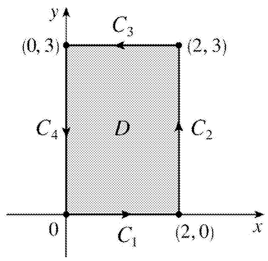
(b) In this case, $c = -(mMG) \Rightarrow$

$$\begin{aligned} W &= -mMG\left(\frac{1}{1.52\times 10^8} - \frac{1}{1.47\times 10^8}\right) \\ &= -(5.97\times 10^{24})(1.99\times 10^{30})(6.67\times 10^{-11})(-2.2377\times 10^{-10}) \approx 1.77\times 10^{35}\,\mathrm{J} \end{aligned}$$

(c) In this case, $c = \varepsilon qQ \Rightarrow$

$$W = \varepsilon qQ\left(\frac{1}{10^{-12}} - \frac{1}{5\times 10^{-13}}\right) = (8.985\times 10^{10})(1)(-1.6\times 10^{-19})(-10^{12}) \approx 1.4\times 10^4\ \mathrm{J}.$$

1. **(a)**



C_1 : $x=t \Rightarrow dx=dt$, $y=0 \Rightarrow dy=0\,dt$, $0\le t\le 2$.

C_2 : $x=2 \Rightarrow dx=0\,dt$, $y=t \Rightarrow dy=dt$, $0\le t\le 3$.

C_3 : $x=2-t \Rightarrow dx=-dt$, $y=3 \Rightarrow dy=0\,dt$, $0\le t\le 2$.

C_4 : $x=0 \Rightarrow dx=0\,dt$, $y=3-t \Rightarrow dy=-dt$, $0\le t\le 3$.

Thus

$$\oint_C xy^2\,dx+x^3\,dy \quad \oint_{C_1+C_2+C_3+C_4} xy^2\,dx+x^3\,dy$$
$$=\int_0^2 0\,dt+\int_0^3 8\,dt+\int_0^2 -9(2-t)\,dt+\int_0^3 0\,dt$$
$$=0+24-18+0=6$$

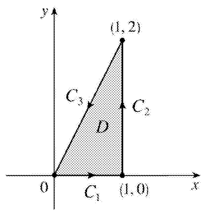
(b)

$$\oint_C xy^2\,dx+x^3\,dy=\iint_D \left[\frac{\partial}{\partial x}(x^3)-\frac{\partial}{\partial y}(xy^2)\right]dA=\int_0^2\int_0^3(3x^2-2xy)\,dy\,dx$$
$$=\int_0^2(9x^2-9x)\,dx=24-18=6$$

2. **(a)** $x=\cos t$, $y=\sin t$, $0\le t\le 2\pi$. Then $\oint_C y\,dx-x\,dy=\int_0^{2\pi}[\sin t(-\sin t)-\cos t(\cos t)]dt=-\int_0^{2\pi} dt=-2\pi$.

(b) $\oint_C y\,dx-x\,dy=\iint_D\left[\frac{\partial}{\partial x}(-x)-\frac{\partial}{\partial y}(y)\right]dA=-2\iint_D dA=-2A(D)=-2\pi(1)^2=-2\pi$

3. **(a)**



C_1 : $x=t \Rightarrow dx=dt$, $y=0 \Rightarrow dy=0\,dt$, $0\le t\le 1$

C_2 : $x=1 \Rightarrow dx=0\,dt$, $y=t \Rightarrow dy=dt$, $0\le t\le 2$.

C_3 : $x=1-t \Rightarrow dx=-dt$, $y=2-2t \Rightarrow dy=-2\,dt$, $0\le t\le 1$.

Thus

$$\oint_C xy\,dx+x^2y^3\,dy = \oint_{C_1+C_2+C_3} xy\,dx+x^2y^3\,dy$$
$$=\int_0^1 0\,dt+\int_0^2 t^3\,dt+\int_0^1 \Big[-(1-t)(2-2t)-2(1-t)^2(2-2t)^3\Big]dt$$
$$=0+\Big[\frac{1}{4}t^4\Big]_0^2+\Big[\frac{2}{3}(1-t)^3+\frac{8}{3}(1-t)^6\Big]_0^1=4-\frac{10}{3}=\frac{2}{3}$$

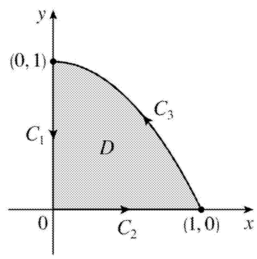
(b)

$$\oint_C xy\,dx+x^2y^3\,dy=\iint_D\Big[\frac{\partial}{\partial x}(x^2y^3)-\frac{\partial}{\partial y}(xy)\Big]dA=\int_0^1\int_0^{2x}(2xy^3-x)\,dy\,dx$$
$$=\int_0^1\Big[\frac{1}{2}xy^4-xy\Big]_{y=0}^{y=2x}dx=\int_0^1(8x^5-2x^2)\,dx=\frac{4}{3}-\frac{2}{3}=\frac{2}{3}$$

4. (a)

C_1:x=0 $\Rightarrow dx=0\,dt$
, $y=1-t \Rightarrow$
$dy=-dt$,
$0\le t\le 1$
C_2:x=$t \Rightarrow dx=dt$
, $y=0\Rightarrow$
$dy=0\,dt$,
$0\le t\le 1$
C_3:x=$1-t \Rightarrow dx=-dt$
,
$y=1-(1-t)^2=2t-t^2$
$\Rightarrow$
$dy=(2-2t)\,dt$
, $0\le t\le 1$

Thus



$$\oint_C x\,dx+y\,dy=\oint_{C_1+C_2+C_3} x\,dx+y\,dy$$

$$=\int_0^1 (0\,dt+(1-t)(-dt))+\int_0^1 (t\,dt+0\,dt)+\int_0^1 ((1-t)(-dt)+(2t-t^2)(2-2t)\,dt)$$

$$=\left[\frac{1}{2}t^2-t\right]_0^1+\left[\frac{1}{2}t^2\right]_0^1+\left[\frac{1}{2}t^4-2t^3+\frac{5}{2}t^2-t\right]_0^1$$

$$=-\frac{1}{2}+\frac{1}{2}+\left(\frac{1}{2}-2+\frac{5}{2}-1\right)=0$$

(b) $\oint_C x\,dx+y\,dy=\iint_D \left[\frac{\partial}{\partial x}(y)-\frac{\partial}{\partial y}(x)\right]dA=\iint_D 0\,dA=0$

5. We can parametrize C as $x=\cos\theta$, $y=\sin\theta$, $0\le\theta\le 2\pi$. Then the line integral is

$$\oint_C P\,dx+Q\,dy=\int_0^{2\pi} \cos^4\theta\sin^5\theta\,(-\sin\theta)d\theta+\int_0^{2\pi}(-\cos^7\theta\sin^6\theta)\cos\theta\,d\theta=-\frac{29\pi}{1024}$$, according to a CAS.

The double integral is

$$\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(-7x^6y^6-5x^4y^4)\,dy\,dx=-\frac{29\pi}{1024}$$,

verifying Green's Theorem in this case.

6.

Since $y=x^2$ along the first part of C and $y=x$ along the second part, the line integral is

$$\oint_C P\,dx+Q\,dy=\int_0^1\left[x^4\sin x+x^2\sin(x^2)(2x)\right]dx+\int_1^0(x^2\sin x+x^2\sin x)dx$$

$$=-16\cos 1-23\sin 1+28$$

according to a CAS. The double integral is

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_0^1 \int_{x^2}^{x} (2x\sin y - 2y\sin x)\,dy\,dx = -16\cos 1 - 23\sin 1 + 28$$

7. The region D enclosed by C is $[0, 1] \times [0, 1]$, so

$$\int_C e^y dx + 2xe^y dy = \iint_D \left[\frac{\partial}{\partial x}(2xe^y) - \frac{\partial}{\partial y}(e^y) \right] dA = \int_0^1 \int_0^1 (2e^y - e^y)\,dy\,dx$$

$$= \int_0^1 dx \int_0^1 e^y dy = (1)(e^1 - e^0) = e - 1$$

8. The region D enclosed by C is given by $\{ (x, y) \mid 0 \le x \le 1,\ 3x \le y \le 3 \}$, so

$$\int_C x^2y^2 dx + 4xy^3 dy = \iint_D \left[\frac{\partial}{\partial x}(4xy^3) - \frac{\partial}{\partial y}(x^2y^2) \right] dA = \int_0^1 \int_{3x}^{3} (4y^3 - 2x^2y)\,dy\,dx$$

$$= \int_0^1 \left[y^4 - x^2y^2 \right]_{y=3x}^{y=3} dx = \int_0^1 (81 - 9x^2 - 72x^4)\,dx = 81 - 3 - \frac{72}{5} = \frac{318}{5}$$

9.

$$\int_C (y + e^{\sqrt{x}})dx + (2x + \cos y^2)dy = \iint_D \left[\frac{\partial}{\partial x}(2x + \cos y^2) - \frac{\partial}{\partial y}(y + e^{\sqrt{x}}) \right] dA$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (2 - 1)\,dx\,dy = \int_0^1 (y^{1/2} - y^2)\,dy = \frac{1}{3}$$

10.

$$\int_C xe^{-2x} dx + (x^4 + 2x^2y^2)\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^4 + 2x^2y^2) - \frac{\partial}{\partial y}(xe^{-2x}) \right] dA = \iint_D (4x^3 + 4xy^2 - 0)\,dA$$

$$= 4\iint_D x(x^2 + y^2)\,dA = 4\int_0^{2\pi} \int_1^2 (r\cos\theta)(r^2)\,r\,dr\,d\theta$$

$$= 4\int_0^{2\pi} \cos\theta\,d\theta \int_1^2 r^4 dr = 4\left[\sin\theta\right]_0^{2\pi} \left[\frac{1}{5} r^5 \right]_1^2 = 0$$

11.

$$\int_C y^3 dx - x^3 dy = \iint_D \left[\frac{\partial}{\partial x}(-x^3) - \frac{\partial}{\partial y}(y^3) \right] dA = \iint_D (-3x^2 - 3y^2) dA = \int_0^{2\pi} \int_0^2 (-3r^2) r\,dr\,d\theta$$

$$= -3\int_0^{2\pi} d\theta \int_0^2 r^3 dr = -3(2\pi)(4) = -24\pi$$

12. $\int_C \sin y\,dx + x\cos y\,dy = \iint_D \left[\frac{\partial}{\partial x}(x\cos y) - \frac{\partial}{\partial y}(\sin y) \right] dA = \iint_D (\cos y - \cos y) dA = \iint_D 0\,dA = 0$

13. $\boldsymbol{F}(x, y) = \left\langle \sqrt{x} + y^3, x^2 + \sqrt{y} \right\rangle$ and the region D enclosed by C is given by $\{ (x, y) \mid 0 \le x \le \pi, 0 \le y \le \sin x \}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_{-C} \left(\sqrt{x} + y^3\right) dx + \left(x^2 + \sqrt{y}\right) dy = -\iint_D \left[\frac{\partial}{\partial x}\left(x^2 + \sqrt{y}\right) - \frac{\partial}{\partial y}\left(\sqrt{x} + y^3\right) \right] dA$$

$$= -\int_0^{\pi} \int_0^{\sin x} (2x - 3y^2) dy\,dx = -\int_0^{\pi} \left[2xy - y^3 \right]_{y=0}^{y=\sin x} dx$$

$$= -\int_0^{\pi} (2x\sin x - \sin^3 x) dx = -\int_0^{\pi} (2x\sin x - (1 - \cos^2 x)\sin x) dx$$

$$= -\left[2\sin x - 2x\cos x + \cos x - \frac{1}{3}\cos^3 x \right]_0^{\pi}$$

$$= -\left(2\pi - 2 + \frac{2}{3} \right) = \frac{4}{3} - 2\pi$$

14. $\boldsymbol{F}(x, y) = \left\langle y^2 \cos x, x^2 + 2y\sin x \right\rangle$ and the region D enclosed by C is given by $\{ (x, y) \mid 0 \le x \le 2, 0 \le y \le 3x \}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_{-C} (y^2 \cos x) dx + (x^2 + 2y\sin x) dy$$

$$= -\iint_D \left[\frac{\partial}{\partial x}(x^2 + 2y\sin x) - \frac{\partial}{\partial y}(y^2 \cos x) \right] dA$$

$$= -\iint_D (2x + 2y\cos x - 2y\cos x) dA = -\int_0^2 \int_0^{3x} 2x\,dy\,dx$$

$$=-\int_0^2 2x[y]_{y=0}^{y=3x}dx=-\int_0^2 6x^2dx=-2x^3\Big]_0^2=-16$$

15. $\boldsymbol{F}(x, y)=\left\langle e^x+x^2y, e^y-xy^2\right\rangle$ and the region D enclosed by C is the disk $x^2+y^2\leq 25$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=-\int_{-C}(e^x+x^2y)dx+(e^y-xy^2)dy$$

$$=-\int\int_D\left[\frac{\partial}{\partial x}(e^y-xy^2)-\frac{\partial}{\partial y}(e^x+x^2y)\right]dA=-\int\int_D(-y^2-x^2)dA$$

$$=\int\int_D(x^2+y^2)dA=\int_0^{2\pi}\int_0^5(r^2)r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_0^5 r^3dr=2\pi\left[\frac{1}{4}r^4\right]_0^5=\frac{625}{2}\pi$$

16. $\boldsymbol{F}(x, y)=\left\langle y-\ln(x^2+y^2), 2\tan^{-1}\left(\frac{y}{x}\right)\right\rangle$ and the region D enclosed by C is the disk with radius 1 centered at (2, 3).C is oriented positively, so

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_C(y-\ln(x^2+y^2))dx+\left(2\tan^{-1}\left(\frac{y}{x}\right)\right)dy$$

$$=\int\int_D\left[\frac{\partial}{\partial x}\left(2\tan^{-1}\left(\frac{y}{x}\right)\right)-\frac{\partial}{\partial y}(y-\ln(x^2+y^2))\right]dA$$

$$=\int\int_D\left[2\left(\frac{-yx^{-2}}{1+(y/x)^2}\right)-\left(1-\frac{2y}{x^2+y^2}\right)\right]dA=\int\int_D\left[-\frac{2y}{x^2+y^2}-1+\frac{2y}{x^2+y^2}\right]dA$$

$$=-\int\int_D dA=-(\text{ area of } D)=-\pi$$

17. By Green's Theorem, $W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_C x(x+y)dx+xy^2dy=\int\int_D(y^2-x)dy\,dx$ where C is the path described in the question and D is the triangle bounded by C . So

$$W=\int_0^1\int_0^{1-x}(y^2-x)dy\,dx=\int_0^1\left[\frac{1}{3}y^3-xy\right]_{y=0}^{y=1-x}dx=\int_0^1\left(\frac{1}{3}(1-x)^3-x(1-x)\right)dx$$

$$= \left[-\frac{1}{12}(1-x)^4 - \frac{1}{2}x^2 + \frac{1}{3}x^3 \right]_0^1 = \left(-\frac{1}{2} + \frac{1}{3} \right) - \left(-\frac{1}{12} \right) = -\frac{1}{12}$$

18. By Green's Theorem, $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C x\,dx + (x^3 + 3xy^2)\,dy = \iint_D (3x^2 + 3y^2 - 0)\,dA$, where D is the semicircular region bounded by C . Converting to polar coordinates, we have

$$W = 3\int_0^2 \int_0^\pi r^2 \cdot r\,d\theta\,dr = 3\pi \left[\frac{1}{4} r^4 \right]_0^2 = 12\pi \ .$$

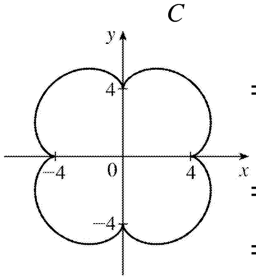
19. Let C_1 be the arch of the cycloid from $(0, 0)$ to $(2\pi, 0)$, which corresponds to $0 \le t \le 2\pi$, and let C_2 be the segment from $(2\pi, 0)$ to $(0, 0)$, so C_2 is given by $x = 2\pi - t$, $y = 0$, $0 \le t \le 2\pi$. Then $C = C_1 \cup C_2$ is traversed
clockwise, so $-C$ is oriented positively. Thus $-C$ encloses the area under one arch of the cycloid and from (5) we have

$$A = -\oint_{-C} y\,dx = \int_{C_1} y\,dx + \int_{C_2} y\,dx = \int_0^{2\pi} (1 - \cos t)(1 - \cos t)\,dt + \int_0^{2\pi} 0(-dt)$$

$$= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt + 0 = \left[t - 2\sin t + \frac{1}{2}t + \frac{1}{4}\sin 2t \right]_0^{2\pi} = 3\pi$$

20.

$$A = \oint_C x\,dy = \int_0^{2\pi} (5\cos t - \cos 5t)(5\cos t - 5\cos 5t)\,dt$$



$$= \int_0^{2\pi} (25\cos^2 t - 30\cos t\cos 5t + 5\cos^2 5t)\,dt$$

$$= \left[25\left(\frac{1}{2}t + \frac{1}{4}\sin 2t \right) - 30\left(\frac{1}{8}\sin 4t + \frac{1}{12}\sin 6t \right) + 5\left(\frac{1}{2}t + \frac{1}{20}\sin 10t \right) \right]_0^{2\pi}$$

$$= 30\pi$$

21. **(a)** Using Equation 17.2.8 , we write parametric equations of the line segment as $x = (1-t)x_1 + tx_2$, $y = (1-t)y_1 + ty_2$, $0 \le t \le 1$. Then $dx = (x_2 - x_1)dt$ and $dy = (y_2 - y_1)dt$, so

$$\int_C x\,dy - y\,dx = \int_0^1 [(1-t)x_1 + tx_2](y_2 - y_1)dt + [(1-t)y_1 + ty_2](x_2 - x_1)dt$$

$$= \int_0^1 (x_1(y_2-y_1)-y_1(x_2-x_1)+t[(y_2-y_1)(x_2-x_1)-(x_2-x_1)(y_2-y_1)])dt$$

$$= \int_0^1 (x_1y_2-x_2y_1)dt=x_1y_2-x_2y_1$$

(b) We apply Green's Theorem to the path $C=C_1\cup C_2\cup\cdots\cup C_n$, where C_i is the line segment that joins (x_i, y_i) to (x_{i+1}, y_{i+1}) for $i=1, 2, \ldots, n-1$, and C_n is the line segment that joins (x_n, y_n) to (x_1, y_1). From (5), $\frac{1}{2}\int_C xdy-ydx=\iint_D dA$, where D is the polygon bounded by C. Therefore

$$\text{area of polygon } =A(D)=\iint_D dA=\frac{1}{2}\int_C xdy-ydx$$

$$=\frac{1}{2}\left(\int_{C_1} xdy-ydx+\int_{C_2} xdy-ydx+\ldots+\int_{C_{n-1}} xdy-ydx+\int_{C_n} xdy-ydx\right)$$

To evaluate these integrals we use the formula from (a) to get
$A(D)=[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+\ldots+(x_{n-1}y_n-x_ny_{n-1})+(x_ny_1-x_1y_n)]$.

(c)

$$A=\frac{1}{2}[(0\cdot 1-2\cdot 0)+(2\cdot 3-1\cdot 1)+(1\cdot 2-0\cdot 3)+(0\cdot 1-(-1)\cdot 2)+(-1\cdot 0-0\cdot 1)]$$

$$=\frac{1}{2}(0+5+2+2)=\frac{9}{2}$$

22. By Green's Theorem, $\frac{1}{2A}\oint_C x^2dy=\frac{1}{2A}\iint_D 2xdA=\frac{1}{A}\iint_D xdA=\bar{x}$ and $-\frac{1}{2A}\oint_C y^2dx=-\frac{1}{2A}\iint_D(-2y)dA=\frac{1}{A}\iint_D ydA=\bar{y}$.

23. Here $A=\frac{1}{2}(1)(1)=\frac{1}{2}$ and $C=C_1+C_2+C_3$, where $C_1: x=x, y=0, 0\le x\le 1$; $C_2: x=x, y=1-x, x=1$ to $x=0$; and $C_3: x=0, y=1$ to $y=0$. Then $\bar{x}=\frac{1}{2A}\int_C x^2dy=\int_{C_1} x^2dy+\int_{C_2} x^2dy+\int_{C_3} x^2dy=0+\int_1^0 (x^2)(-dx)+0=\frac{1}{3}$. Similarly,

$$\bar{y}=-\frac{1}{2A}\int_C y^2dx=\int_{C_1} y^2dx+\int_{C_2} y^2dx+\int_{C_3} y^2dx=0+\int_1^0 (1-x)^2(-dx)+0=\frac{1}{3}.$$

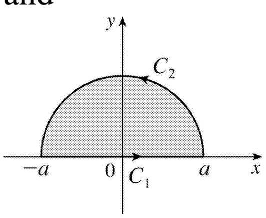
Therefore $(\bar{x},\bar{y})=\left(\frac{1}{3},\frac{1}{3}\right)$.

24. $A=\frac{\pi a^2}{2}$ so $\bar{x}=\frac{1}{\pi a^2}\oint_C x^2dy$ and $\bar{y}=-\frac{1}{\pi a^2}\oint_C y^2dx$.

Orienting the semicircular region as in the figure,

$$\bar{x}=\frac{1}{\pi a^2}\oint_{C_1+C_2} x^2dy$$
$$=\frac{1}{\pi a^2}\left[0+\int_0^\pi (a^2\cos^2 t)(a\cos t)dt\right]=0$$

and



$$\bar{y}=-\frac{1}{\pi a^2}\left[\int_{-a}^{a} 0dx+\int_0^\pi (a^2\sin^2 t)(-a\sin t)dt\right]=\frac{a}{\pi}\int_0^\pi \sin^3 tdt=\frac{a}{\pi}\left[-\cos t+\frac{1}{3}(\cos^3 t)\right]_0^\pi=\frac{4a}{3\pi}.$$

Thus $(\bar{x},\bar{y})=\left(0,\frac{4a}{3\pi}\right)$.

25. By Green's Theorem, $-\frac{1}{3}\rho\oint_C y^3dx=-\frac{1}{3}\rho\iint_D(-3y^2)dA=\iint_D y^2\rho dA=I_x$ and

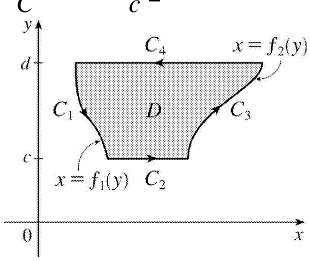
$$\frac{1}{3}\rho\oint_C x^3dy=\frac{1}{3}\rho\iint_D(3x^2)dA=\iint_D x^2\rho dA=I_y.$$

26. By symmetry the moments of inertia about any two diameters are equal. Centering the disk at the origin, the moment of inertia about a diameter equals

$$I_y = \frac{1}{3}\rho\oint_C x^3 dy = \frac{1}{3}\rho\int_0^{2\pi}(a^4\cos^4 t)dt = \frac{1}{3}a^4\rho\int_0^{2\pi}\left[\frac{3}{8}+\frac{1}{2}\cos 2t+\frac{1}{8}\cos 4t\right]dt$$
$$= \frac{1}{3}a^4\rho\cdot\frac{3(2\pi)}{8} = \frac{1}{4}\pi a^4\rho$$

27. Since C is a simple closed path which doesn't pass through or enclose the origin, there exists an open region that doesn't contain the origin but does contain D . Thus $P=-y/(x^2+y^2)$ and $Q=x/(x^2+y^2)$ have continuous partial derivatives on this open region containing D and we can apply Green's Theorem. But by Exercise 17.3.33(a) , $\partial P/\partial y=\partial Q/\partial x$, so $\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}=\iint_D 0dA=0$.

28. We express D as a type II region: $D=\{ (x, y) \mid f_1(y)\le x\le f_2(y), c\le y\le d \}$ where f_1 and f_2 are continuous functions. Then $\iint_D \frac{\partial Q}{\partial x}\,dA=\int_c^d\int_{f_1(y)}^{f_2(y)}\frac{\partial Q}{\partial x}\,dx\,dy=\int_c^d\left[Q(f_2(y), y)-Q(f_1(y), y)\right]dy$ by the Fundamental Theorem of Calculus. But referring to the figure, $\oint_C Qdy=\oint_{C_1+C_2+C_3+C_4} Qdy$. Then $\int_{C_1} Qdy=\int_d^c Q(f_1(y), y)dy$, $\int_{C_2} Qdy=\int_{C_4} Qdy=0$, and $\int_{C_3} Qdy=\int_c^d Q(f_2(y), y)dy$. Hence $\oint_C Qdy=\int_c^d\left[Q(f_2(y), y)-Q(f_1(y), y)\right]dy=\iint_D(\partial Q/\partial x)dA.$



29. Using the first part of (5), we have that $\iint_R dx\,dy=A(R)=\int_{\partial R} xdy$. But $x=g(u, v)$, and $dy=\frac{\partial h}{\partial u}\,du+\frac{\partial h}{\partial v}\,dv$, and we orient ∂S by taking the positive direction to be that which corresponds, under the mapping, to the positive direction along ∂R , so

$$\int_{\partial R} x\,dy = \int_{\partial S} g(u,v)\left(\frac{\partial h}{\partial u}\,du+\frac{\partial h}{\partial v}\,dv\right)=\int_{\partial S} g(u,v)\,\frac{\partial h}{\partial u}\,du+g(u,v)\,\frac{\partial h}{\partial v}\,dv$$

$=\pm\int\int_S \left[\frac{\partial}{\partial u}\left(g(u,v)\,\frac{\partial h}{\partial v}\right)-\frac{\partial}{\partial v}\left(g(u,v)\,\frac{\partial h}{\partial u}\right)\right]dA$ [using Green's Theorem in the uv – plane]

$$=\pm\int\int_S \left(\frac{\partial g}{\partial u}\,\frac{\partial h}{\partial v}+g(u,v)\,\frac{\partial^2 h}{\partial u\,\partial v}-\frac{\partial g}{\partial v}\,\frac{\partial h}{\partial u}-g(u,v)\,\frac{\partial^2 h}{\partial v\,\partial u}\right)dA$$

$=\pm\int\int_S \left(\frac{\partial x}{\partial u}\,\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\,\frac{\partial y}{\partial u}\right)dA$ [by the equality of mixed partials] $=\pm\int\int_S \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$

The sign is chosen to be positive if the orientation that we gave to ∂S corresponds to the usual positive orientation, and it is negative otherwise. In either case, since $A(R)$ is positive, the sign chosen must be the same as the sign of $\frac{\partial(x,y)}{\partial(u,v)}$. Therefore $A(R)=\int\int_R dx\,dy=\int\int_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv.$

1. **(a)**

$$\text{curl}\,\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ xyz & 0 & -x^2y\end{vmatrix}=(-x^2-0)\boldsymbol{i}-(-2xy-xy)\boldsymbol{j}+(0-xz)\boldsymbol{k}$$

$$=-x^2\boldsymbol{i}+3xy\,\boldsymbol{j}-xz\boldsymbol{k}$$

(b) $\text{div}\,\boldsymbol{F}=\nabla\cdot\boldsymbol{F}=\frac{\partial}{\partial x}(xyz)+\frac{\partial}{\partial y}(0)+\frac{\partial}{\partial z}(-x^2y)=yz+0+0=yz$

2. **(a)**

$$\text{curl}\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x^2yz & xy^2z & xyz^2\end{vmatrix}=(xz^2-xy^2)\boldsymbol{i}-(yz^2-x^2y)\boldsymbol{j}+(y^2z-x^2z)\boldsymbol{k}$$

$$=x(z^2-y^2)\boldsymbol{i}+y(x^2-z^2)\boldsymbol{j}+z(y^2-x^2)\boldsymbol{k}$$

(b) $\text{div}\,\boldsymbol{F}=\nabla\cdot\boldsymbol{F}=\frac{\partial}{\partial x}(x^2yz)+\frac{\partial}{\partial y}(xy^2z)+\frac{\partial}{\partial z}(xyz^2)=2xyz+2xyz+2xyz=6xyz$

3. **(a)**

$$\text{curl}\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 1 & x+yz & xy-\sqrt{z}\end{vmatrix}=(x-y)\boldsymbol{i}-(y-0)\boldsymbol{j}+(1-0)\boldsymbol{k}$$

$$=(x-y)\boldsymbol{i}-y\,\boldsymbol{j}+\boldsymbol{k}$$

(b) $\text{div}\,\boldsymbol{F}=\nabla\cdot\boldsymbol{F}=\frac{\partial}{\partial x}(1)+\frac{\partial}{\partial y}(x+yz)+\frac{\partial}{\partial z}(xy-\sqrt{z})=z-\frac{1}{2\sqrt{z}}$

4. **(a)**

$$\text{curl}\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 0 & \cos xz & -\sin xy\end{vmatrix}$$

$$=(-x\cos xy+x\sin xz)\boldsymbol{i}-(-y\cos xy-0)\boldsymbol{j}+(-z\sin xz-0)\boldsymbol{k}$$

$$=x(\sin xz-\cos xy)\boldsymbol{i}+y\cos xy\,\boldsymbol{j}-z\sin xz\,\boldsymbol{k}$$

(b) $\text{div}\,\boldsymbol{F}=\nabla\cdot\boldsymbol{F}=\frac{\partial}{\partial x}(0)+\frac{\partial}{\partial y}(\cos xz)+\frac{\partial}{\partial z}(-\sin xy)=0+0+0=0$

5. **(a)** curl $\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ e^x\sin y & e^x\cos y & z\end{vmatrix}=(0-0)\boldsymbol{i}-(0-0)\boldsymbol{j}+(e^x\cos y-e^x\cos y)\boldsymbol{k}=\boldsymbol{0}$

(b) div $\boldsymbol{F}=\nabla\cdot\boldsymbol{F}=\frac{\partial}{\partial x}(e^x\sin y)+\frac{\partial}{\partial y}(e^x\cos y)+\frac{\partial}{\partial z}(z)=e^x\sin y-e^x\sin y+1=1$

6. **(a)**

$$\text{curl } \boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \frac{x}{x^2+y^2+z^2} & \frac{y}{x^2+y^2+z^2} & \frac{z}{x^2+y^2+z^2}\end{vmatrix}$$

$$=\frac{1}{(x^2+y^2+z^2)^2}[(-2yz+2yz)\boldsymbol{i}-(-2xz+2xz)\boldsymbol{j}+(-2xy+2xy)\boldsymbol{k}]=\boldsymbol{0}$$

(b)

$$\text{div } \boldsymbol{F}=\nabla\cdot\boldsymbol{F}=\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2+z^2}\right)+\frac{\partial}{\partial z}\left(\frac{z}{x^2+y^2+z^2}\right)$$

$$=\frac{x^2+y^2+z^2-2x^2}{(x^2+y^2+z^2)^2}+\frac{x^2+y^2+z^2-2y^2}{(x^2+y^2+z^2)^2}+\frac{x^2+y^2+z^2-2z^2}{(x^2+y^2+z^2)^2}$$

$$=\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2}=\frac{1}{x^2+y^2+z^2}$$

7. **(a)**

$$\text{curl}\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \ln x & \ln(xy) & \ln(xyz)\end{vmatrix}=\left(\frac{xz}{xyz}-0\right)\boldsymbol{i}-\left(\frac{yz}{xyz}-0\right)\boldsymbol{j}+\left(\frac{y}{xy}-0\right)\boldsymbol{k}$$

$$=\left\langle\frac{1}{y},-\frac{1}{x},\frac{1}{x}\right\rangle$$

(b) div $\boldsymbol{F}=\nabla\cdot\boldsymbol{F}=\frac{\partial}{\partial x}(\ln x)+\frac{\partial}{\partial y}(\ln(xy))+\frac{\partial}{\partial z}(\ln(xyz))=\frac{1}{x}+\frac{x}{xy}+\frac{xy}{xyz}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$

8. **(a)**

$$\text{curl}\boldsymbol{F} = \nabla \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xe^{-y} & xz & ze^{y} \end{vmatrix} = (ze^{y} - x)\boldsymbol{i} - (0-0)\boldsymbol{j} + (z - xe^{-y}(-1))\boldsymbol{k}$$

$$= \left\langle ze^{y} - x,\ 0,\ z + xe^{-y} \right\rangle$$

(b) div $\boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(xe^{-y}) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(ze^{y}) = e^{-y} + 0 + e^{y} = e^{y} + e^{-y}$

9. If the vector field is $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$, then we know $R=0$. In addition, the x –component of each vector of $\boldsymbol{F}$ is 0 , so $P=0$, hence $\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = \frac{\partial R}{\partial y} = \frac{\partial R}{\partial z} = 0$. Q decreases as y increases, so $\frac{\partial Q}{\partial y} < 0$, but Q doesn't change in the x – or z –directions, so $\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial z} = 0$.

(a) div$\boldsymbol{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0 + \frac{\partial Q}{\partial y} + 0 < 0$

(b) curl$\boldsymbol{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k} = (0-0)\boldsymbol{i} + (0-0)\boldsymbol{j} + (0-0)\boldsymbol{k} = \boldsymbol{0}$

10. If the vector field is $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$, then we know $R=0$. In addition, P and Q don't vary in the z –direction, so $\frac{\partial R}{\partial x} = \frac{\partial R}{\partial y} = \frac{\partial R}{\partial z} = \frac{\partial P}{\partial z} = \frac{\partial Q}{\partial z} = 0$. As x increases, the x –component of each vector of $\boldsymbol{F}$ increases while the y –component remains constant, so $\frac{\partial P}{\partial x} > 0$ and $\frac{\partial Q}{\partial x} = 0$. Similarly, as y increases, the y –component of each vector increases while the x –component remains constant, so $\frac{\partial Q}{\partial y} > 0$ and $\frac{\partial P}{\partial y} = 0$.

(a) div$\boldsymbol{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + 0 > 0$

(b) curl$\boldsymbol{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k} = (0-0)\boldsymbol{i} + (0-0)\boldsymbol{j} + (0-0)\boldsymbol{k} = \boldsymbol{0}$

11. If the vector field is $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$, then we know $R=0$. In addition, the y –component of each vector of $\boldsymbol{F}$ is 0 , so $Q=0$, hence $\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y} = \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial x} = \frac{\partial R}{\partial y} = \frac{\partial R}{\partial z} = 0$. P increases as y increases, so $\frac{\partial P}{\partial y} > 0$, but P doesn't change in the x – or z –directions, so $\frac{\partial P}{\partial x} = \frac{\partial P}{\partial z} = 0$.

(a) $\operatorname{div}\boldsymbol{F}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0+0+0=0$

(b)

$$\operatorname{curl}\boldsymbol{F}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\boldsymbol{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\boldsymbol{k}$$

$$=(0-0)\boldsymbol{i}+(0-0)\boldsymbol{j}+\left(0-\frac{\partial P}{\partial y}\right)\boldsymbol{k}=-\frac{\partial P}{\partial y}\boldsymbol{k}$$

Since $\frac{\partial P}{\partial y}>0$, $-\frac{\partial P}{\partial y}\boldsymbol{k}$ is a vector pointing in the negative z –direction.

12. **(a)** $\operatorname{curl} f=\nabla\times f$ is meaningless because f is a scalar field.
(b) $\operatorname{grad} f$ is a vector field.
(c) $\operatorname{div}\boldsymbol{F}$ is a scalar field.
(d) $\operatorname{curl}(\operatorname{grad} f)$ is a vector field.
(e) $\operatorname{grad}\boldsymbol{F}$ is meaningless because $\boldsymbol{F}$ is not a scalar field.
(f) $\operatorname{grad}(\operatorname{div}\boldsymbol{F})$ is a vector field.
(g) $\operatorname{div}(\operatorname{grad} f)$ is a scalar field.
(h) $\operatorname{grad}(\operatorname{div} f)$ is meaningless because f is a scalar field.
(i) $\operatorname{curl}(\operatorname{curl}\boldsymbol{F})$ is a vector field.
(j) $\operatorname{div}(\operatorname{div}\boldsymbol{F})$ is meaningless because $\operatorname{div}\boldsymbol{F}$ is a scalar field.
(k) $(\operatorname{grad} f)\times(\operatorname{div}\boldsymbol{F})$ is meaningless because $\operatorname{div}\boldsymbol{F}$ is a scalar field.
(l) $\operatorname{div}(\operatorname{curl}(\operatorname{grad} f))$ is a scalar field.

13. $\operatorname{curl}\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ yz & xz & xy\end{vmatrix}=(x-x)\boldsymbol{i}-(y-y)\boldsymbol{j}+(z-z)\boldsymbol{k}=\boldsymbol{0}$

and $\boldsymbol{F}$ is defined on all of R^3 with component functions which have continuous partial derivatives, so by Theorem 4, $\boldsymbol{F}$ is conservative. Thus, there exists a function f such that $\boldsymbol{F}=\nabla f$. Then $f_x(x, y, z)=yz$ implies $f(x, y, z)=xyz+g(y, z)$ and $f_y(x, y, z)=xz+g_y(y, z)$. But $f_y(x, y, z)=xz$, so $g(y, z)=h(z)$ and $f(x, y, z)=xyz+h(z)$. Thus $f_z(x, y, z)=xy+h'(z)$ but $f_z(x, y, z)=xy$ so $h(z)=K$, a constant. Hence a potential function for $\boldsymbol{F}$ is $f(x, y, z)=xyz+K$.

14. $\operatorname{curl}\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 3z^2 & \cos y & 2xz\end{vmatrix}=(0-0)\boldsymbol{i}-(2z-6z)\boldsymbol{j}+(0-0)\boldsymbol{k}=4z\boldsymbol{j}\neq\boldsymbol{0}$,

so $\boldsymbol{F}$ is not conservative.

15. curl $\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 2xy & x^2+2yz & y^2\end{vmatrix}=(2y-2y)\boldsymbol{i}-(0-0)\boldsymbol{j}+(2x-2x)\boldsymbol{k}=\boldsymbol{0}$, $\boldsymbol{F}$ is defined on all of R^3 , and the partial derivatives of the component functions are continuous, so $\boldsymbol{F}$ is conservative. Thus there exists a function f such that $\nabla f=\boldsymbol{F}$. Then $f_x(x, y, z)=2xy$ implies $f(x, y, z)=x^2y+g(y, z)$ and $f_y(x, y, z)=x^2+g_y(y, z)$. But $f_y(x, y, z)=x^2+2yz$, so $g(y, z)=y^2z+h(z)$ and $f(x, y, z)=x^2y+y^2z+h(z)$. Thus $f_z(x, y, z)=y^2+h'(z)$ but $f_z(x, y, z)=y^2$ so $h(z)=K$ and $f(x, y, z)=x^2y+y^2z+K$.

16. curl $\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ e^z & 1 & xe^z\end{vmatrix}=(0-0)\boldsymbol{i}-(e^z-e^z)\boldsymbol{j}+(0-0)\boldsymbol{k}=\boldsymbol{0}$ and $\boldsymbol{F}$ is defined on all of R^3 with component functions that have continuous partial deriatives, so $\boldsymbol{F}$ is conservative. Thus there exists a function f such that $\nabla f=\boldsymbol{F}$. Then $f_x(x, y, z)=e^z$ implies $f(x, y, z)=xe^z+g(y, z)\Rightarrow$ $f_y(x, y, z)=g_y(y, z)$. But $f_y(x, y, z)=1$, so $g(y, z)=y+h(z)$ and $f(x, y, z)=xe^z+y+h(z)$. Thus $f_z(x, y, z)=xe^z+h'(z)$ but $f_z(x, y, z)=xe^z$, so $h(z)=K$, a constant. Hence a potential function for $\boldsymbol{F}$ is $f(x, y, z)=xe^z+y+K$.

17. curl $\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ ye^{-x} & \mathrm{e}^{-x} & 2z\end{vmatrix}=(0-0)\boldsymbol{i}-(0-0)\boldsymbol{j}+(-e^{-x}-e^{-x})\boldsymbol{k}=-2e^{-x}\boldsymbol{k}\neq\boldsymbol{0}$, so $\boldsymbol{F}$ is not conservative.

18.

$$\text{curl}\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ y\cos xy & x\cos xy & -\sin z\end{vmatrix}$$
$$=(0-0)\boldsymbol{i}-(0-0)\boldsymbol{j}+[(-xy\sin xy+\cos xy)-(-xy\sin xy+\cos xy)]\boldsymbol{k}=0$$

$\boldsymbol{F}$ is defined on all of R^3 , and the partial derivatives of the component functions are continuous, so $\boldsymbol{F}$ is conservative. Thus there exists a function f such that $\nabla f=\boldsymbol{F}$. Then $f_x(x, y, z)=y\cos xy$ implies $f(x, y, z)=\sin xy+g(y, z)\Rightarrow f_y(x, y, z)=x\cos xy+g_y(y, z)$. But $f_y(x, y, z)=x\cos xy$, so $g(y, z)=h(z)$ and $f(x, y, z)=\sin xy+h(z)$. Thus $f_z(x, y, z)=h'(z)$ but $f_z(x, y, z)=-\sin z$ so $h(z)=\cos z+K$ and a potential function for $\boldsymbol{F}$ is $f(x, y, z)=\sin xy+\cos z+K$.

19. No. Assume there is such a $\boldsymbol{G}$. Then $\operatorname{div}(\operatorname{curl}\boldsymbol{G})=y^2+z^2+x^2\neq 0$, which contradicts Theorem 11.

20. No. Assume there is such a $\boldsymbol{G}$. Then $\operatorname{div}(\operatorname{curl}\boldsymbol{G})=xz\neq 0$ which contradicts Theorem 11.

21. $\operatorname{curl}\boldsymbol{F}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x) & g(y) & h(z) \end{vmatrix}=(0-0)\boldsymbol{i}+(0-0)\boldsymbol{j}+(0-0)\boldsymbol{k}=\boldsymbol{0}$.
Hence $\boldsymbol{F}=f(x)\boldsymbol{i}+g(y)\boldsymbol{j}+h(z)\boldsymbol{k}$ is irrotational.

22. $\operatorname{div}\boldsymbol{F}=\dfrac{\partial(f(y, z))}{\partial x}+\dfrac{\partial(g(x, z))}{\partial y}+\dfrac{\partial(h(x, y))}{\partial z}=0$ so $\boldsymbol{F}$ is incompressible.

23.

$$\begin{aligned}\operatorname{div}(\boldsymbol{F}+\boldsymbol{G})&=\frac{\partial(P_1+P_2)}{\partial x}+\frac{\partial(Q_1+Q_2)}{\partial y}+\frac{\partial(R_1+R_2)}{\partial z}\\&=\left(\frac{\partial P_1}{\partial x}+\frac{\partial Q_1}{\partial y}+\frac{\partial R_1}{\partial z}\right)+\left(\frac{\partial P_2}{\partial x}+\frac{\partial Q_2}{\partial y}+\frac{\partial R_3}{\partial z}\right)=\operatorname{div}\boldsymbol{F}+\operatorname{div}\boldsymbol{G}\end{aligned}$$

24.

$$\begin{aligned}\operatorname{curl}\boldsymbol{F}+\operatorname{curl}\boldsymbol{G}&=\left[\left(\frac{\partial R_1}{\partial y}-\frac{\partial Q_1}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial P_1}{\partial z}-\frac{\partial R_1}{\partial x}\right)\boldsymbol{j}+\left(\frac{\partial Q_1}{\partial x}-\frac{\partial P_1}{\partial y}\right)\boldsymbol{k}\right]\\&\quad+\left[\left(\frac{\partial R_2}{\partial y}-\frac{\partial Q_2}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial P_2}{\partial z}-\frac{\partial R_2}{\partial x}\right)\boldsymbol{j}+\left(\frac{\partial Q_2}{\partial x}-\frac{\partial P_2}{\partial y}\right)\boldsymbol{k}\right]\\&=\left[\frac{\partial(R_1+R_2)}{\partial y}-\frac{\partial(Q_1+Q_2)}{\partial z}\right]\boldsymbol{i}+\left[\frac{\partial(P_1+P_2)}{\partial z}-\frac{\partial(R_1+R_2)}{\partial x}\right]\boldsymbol{j}\\&\quad+\left[\frac{\partial(Q_1+Q_2)}{\partial x}-\frac{\partial(P_1+P_2)}{\partial y}\right]\boldsymbol{k}=\operatorname{curl}(\boldsymbol{F}+\boldsymbol{G})\end{aligned}$$

25.

$$\begin{aligned}\operatorname{div}(f\boldsymbol{F})&=\frac{\partial(fP_1)}{\partial x}+\frac{\partial(fQ_1)}{\partial y}+\frac{\partial(fR_1)}{\partial z}\\&=\left(f\frac{\partial P_1}{\partial x}+P_1\frac{\partial f}{\partial x}\right)+\left(f\frac{\partial Q_1}{\partial y}+Q_1\frac{\partial f}{\partial y}\right)+\left(f\frac{\partial R_1}{\partial z}+R_1\frac{\partial f}{\partial z}\right)\end{aligned}$$

$$=f\left(\frac{\partial P_1}{\partial x}+\frac{\partial Q_1}{\partial y}+\frac{\partial R_1}{\partial z}\right)+\left\langle P_1, Q_1, R_1\right\rangle\cdot\left\langle\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right\rangle=f\operatorname{div}\mathbf{F}+\mathbf{F}\cdot\nabla f$$

26.

$$\operatorname{curl}(f\mathbf{F})=\left[\frac{\partial(fR_1)}{\partial y}-\frac{\partial(fQ_1)}{\partial z}\right]\mathbf{i}+\left[\frac{\partial(fP_1)}{\partial z}-\frac{\partial(fR_1)}{\partial x}\right]\mathbf{j}+\left[\frac{\partial(fQ_1)}{\partial x}-\frac{\partial(fP_1)}{\partial y}\right]\mathbf{k}$$

$$=\left[f\frac{\partial R_1}{\partial y}+R_1\frac{\partial f}{\partial y}-f\frac{\partial Q_1}{\partial z}-Q_1\frac{\partial f}{\partial z}\right]\mathbf{i}+\left[f\frac{\partial P_1}{\partial z}+P_1\frac{\partial f}{\partial z}-f\frac{\partial R_1}{\partial x}-R_1\frac{\partial f}{\partial x}\right]\mathbf{j}$$

$$+\left[f\frac{\partial Q_1}{\partial x}+Q_1\frac{\partial f}{\partial x}-f\frac{\partial P_1}{\partial y}-P_1\frac{\partial f}{\partial y}\right]\mathbf{k}$$

$$=f\left[\frac{\partial R_1}{\partial y}-\frac{\partial Q_1}{\partial z}\right]\mathbf{i}+f\left[\frac{\partial P_1}{\partial z}-\frac{\partial R_1}{\partial x}\right]\mathbf{j}+f\left[\frac{\partial Q_1}{\partial x}-\frac{\partial P_1}{\partial y}\right]\mathbf{k}$$

$$+\left[R_1\frac{\partial f}{\partial y}-Q_1\frac{\partial f}{\partial z}\right]\mathbf{i}+\left[P_1\frac{\partial f}{\partial z}-R_1\frac{\partial f}{\partial x}\right]\mathbf{j}+\left[Q_1\frac{\partial f}{\partial x}-P_1\frac{\partial f}{\partial y}\right]\mathbf{k}$$

$$=f\operatorname{curl}\mathbf{F}+(\nabla f)\times\mathbf{F}$$

27.

$$\operatorname{div}(\mathbf{F})\times\mathbf{G}=\nabla\cdot(\mathbf{F}\times\mathbf{G})=\begin{vmatrix}\partial/\partial x & \partial/\partial y & \partial/\partial z\\ P_1 & Q_1 & R_1\\ P_2 & Q_2 & R_2\end{vmatrix}=\frac{\partial}{\partial x}\begin{vmatrix}Q_1 & R_1\\ Q_2 & R_2\end{vmatrix}-\frac{\partial}{\partial y}\begin{vmatrix}P_1 & R_1\\ P_2 & R_2\end{vmatrix}+\frac{\partial}{\partial z}\begin{vmatrix}P_1 & Q_1\\ P_2 & Q_2\end{vmatrix}$$

$$=\left[Q_1\frac{\partial R_2}{\partial x}+R_2\frac{\partial Q_1}{\partial x}-Q_2\frac{\partial R_1}{\partial x}-R_1\frac{\partial Q_2}{\partial x}\right]$$

$$-\left[P_1\frac{\partial R_2}{\partial y}+R_2\frac{\partial P_1}{\partial y}-P_2\frac{\partial R_1}{\partial y}-R_1\frac{\partial P_2}{\partial y}\right]$$

$$+\left[P_1\frac{\partial Q_2}{\partial z}+Q_2\frac{\partial P_1}{\partial z}-P_2\frac{\partial Q_1}{\partial z}-Q_1\frac{\partial P_2}{\partial z}\right]$$

$$=\left[P_2\left(\frac{\partial R_1}{\partial y}-\frac{\partial Q_1}{\partial z}\right)+Q_2\left(\frac{\partial P_1}{\partial z}-\frac{\partial R_1}{\partial x}\right)+R_2\left(\frac{\partial Q_1}{\partial x}-\frac{\partial P_1}{\partial y}\right)\right]$$

$$-\left[P_1\left(\frac{\partial R_2}{\partial y}-\frac{\partial Q_2}{\partial z}\right)+Q_1\left(\frac{\partial P_2}{\partial z}-\frac{\partial R_2}{\partial x}\right)+R_1\left(\frac{\partial Q_2}{\partial x}-\frac{\partial P_2}{\partial y}\right)\right]$$

$$=\mathbf{G}\cdot\operatorname{curl}\mathbf{F}-\mathbf{F}\cdot\operatorname{curl}\mathbf{G}$$

28. $\operatorname{div}(\nabla f\times\nabla g)=\nabla g\cdot\operatorname{curl}(\nabla f)-\nabla f\cdot\operatorname{curl}(\nabla g)=0$ (by Theorem 3)

29.

$$\operatorname{curl}\operatorname{curl}\boldsymbol{F}=\nabla\times(\nabla\times\boldsymbol{F})=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \partial R_1/\partial y-\partial Q_1/\partial z & \partial P_1/\partial z-\partial R_1/\partial x & \partial Q_1/\partial x-\partial P_1/\partial y\end{vmatrix}$$

$$=\left(\frac{\partial^2 Q_1}{\partial y\partial x}-\frac{\partial^2 P_1}{\partial y^2}-\frac{\partial^2 P_1}{\partial z^2}+\frac{\partial^2 R_1}{\partial z\partial x}\right)\boldsymbol{i}+\left(\frac{\partial^2 R_1}{\partial z\partial y}-\frac{\partial^2 Q_1}{\partial z^2}-\frac{\partial^2 Q_1}{\partial x^2}+\frac{\partial^2 P_1}{\partial x\partial y}\right)\boldsymbol{j}$$
$$+\left(\frac{\partial^2 P_1}{\partial x\partial z}-\frac{\partial^2 R_1}{\partial x^2}-\frac{\partial^2 R_1}{\partial y^2}+\frac{\partial^2 Q_1}{\partial y\partial z}\right)\boldsymbol{k}$$

Now let’s consider grad div$\boldsymbol{F}-\nabla^2\boldsymbol{F}$ and compare with the above.
(Note that $\nabla^2\boldsymbol{F}$ is defined on page 1130).

$$\operatorname{grad}\operatorname{div}\boldsymbol{F}-\nabla^2\boldsymbol{F}=\left[\left(\frac{\partial^2 P_1}{\partial x^2}+\frac{\partial^2 Q_1}{\partial x\partial y}+\frac{\partial^2 R_1}{\partial x\partial z}\right)\boldsymbol{i}+\left(\frac{\partial^2 P_1}{\partial y\partial x}+\frac{\partial^2 Q_1}{\partial y^2}+\frac{\partial^2 R_1}{\partial y\partial z}\right)\boldsymbol{j}\right.$$
$$\left.+\left(\frac{\partial^2 P_1}{\partial z\partial x}+\frac{\partial^2 Q_1}{\partial z\partial y}+\frac{\partial^2 R_1}{\partial z^2}\right)\boldsymbol{k}\right]$$
$$-\left[\left(\frac{\partial^2 P_1}{\partial x^2}+\frac{\partial^2 P_1}{\partial y^2}+\frac{\partial^2 P_1}{\partial z^2}\right)\boldsymbol{i}+\left(\frac{\partial^2 Q_1}{\partial x^2}+\frac{\partial^2 Q_1}{\partial y^2}+\frac{\partial^2 Q_1}{\partial z^2}\right)\boldsymbol{j}\right.$$
$$\left.+\left(\frac{\partial^2 R_1}{\partial x^2}+\frac{\partial^2 R_1}{\partial y^2}+\frac{\partial^2 R_1}{\partial z^2}\right)\boldsymbol{k}\right]$$
$$=\left(\frac{\partial^2 Q_1}{\partial x\partial y}+\frac{\partial^2 R_1}{\partial x\partial z}-\frac{\partial^2 P_1}{\partial y^2}-\frac{\partial^2 P_1}{\partial z^2}\right)\boldsymbol{i}+\left(\frac{\partial^2 P_1}{\partial y\partial x}+\frac{\partial^2 R_1}{\partial y\partial z}-\frac{\partial^2 Q_1}{\partial x^2}-\frac{\partial^2 Q_1}{\partial z^2}\right)\boldsymbol{j}$$
$$+\left(\frac{\partial^2 P_1}{\partial z\partial x}+\frac{\partial^2 Q_1}{\partial z\partial y}-\frac{\partial^2 R_1}{\partial x^2}-\frac{\partial^2 R_2}{\partial y^2}\right)\boldsymbol{k}$$

Then applying Clairaut’s Theorem to reverse the order of differentiation in the second partial derivatives as needed and comparing, we have curl curl$\boldsymbol{F}$=grad div$\boldsymbol{F}-\nabla^2\boldsymbol{F}$ as desired.

30. **(a)** $\nabla\cdot\boldsymbol{r}=\left(\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k}\right)\cdot(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})=1+1+1=3$

(b)

$$\begin{aligned}\nabla\cdot(r\boldsymbol{r})&=\nabla\cdot\sqrt{x^2+y^2+z^2}\,(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\\&=\left(\frac{x^2}{\sqrt{x^2+y^2+z^2}}+\sqrt{x^2+y^2+z^2}\right)+\left(\frac{y^2}{\sqrt{x^2+y^2+z^2}}+\sqrt{x^2+y^2+z^2}\right)\\&\quad+\left(\frac{z^2}{\sqrt{x^2+y^2+z^2}}+\sqrt{x^2+y^2+z^2}\right)\\&=\frac{1}{\sqrt{x^2+y^2+z^2}}(4x^2+4y^2+4z^2)=4\sqrt{x^2+y^2+z^2}=4r\end{aligned}$$

Another method:

By Exercise 25, $\nabla\cdot(r\boldsymbol{r})=\text{div}(r\boldsymbol{r})=r\,\text{div}\boldsymbol{r}+\boldsymbol{r}\cdot\nabla r=3r+\boldsymbol{r}\cdot\frac{\boldsymbol{r}}{r}=4r$.

(c)

$$\begin{aligned}\nabla^2r^3&=\nabla^2(x^2+y^2+z^2)^{3/2}\\&=\frac{\partial}{\partial x}\left[\frac{3}{2}(x^2+y^2+z^2)^{1/2}(2x)\right]+\frac{\partial}{\partial y}\left[\frac{3}{2}(x^2+y^2+z^2)^{1/2}(2y)\right]\\&\quad+\frac{\partial}{\partial z}\left[\frac{3}{2}(x^2+y^2+z^2)^{1/2}(2z)\right]\\&=3\left[\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)(x)+(x^2+y^2+z^2)^{1/2}\right]\\&\quad+3\left[\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y)(y)+(x^2+y^2+z^2)^{1/2}\right]\\&\quad+3\left[\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z)(z)+(x^2+y^2+z^2)^{1/2}\right]\\&=3(x^2+y^2+z^2)^{-1/2}(4x^2+4y^2+4z^2)=12(x^2+y^2+z^2)^{1/2}\\&=12r\end{aligned}$$

Another method: $\frac{\partial}{\partial x}(x^2+y^2+z^2)^{3/2}=3x\sqrt{x^2+y^2+z^2}\Rightarrow\nabla r^3=3r(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})=3r\boldsymbol{r}$, so $\nabla^2r^3=\nabla\cdot\nabla r^3=\nabla\cdot(3r\boldsymbol{r})=3(4r)=12r$ by part (b).

31. **(a)**

$$\nabla r = \nabla \sqrt{x^2+y^2+z^2} = \frac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k}$$

$$= \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \frac{\mathbf{r}}{r}$$

(b)

$$\nabla\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(z)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]\mathbf{k} = \mathbf{0}$$

(c)

$$\nabla\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{-\frac{1}{2\sqrt{x^2+y^2+z^2}}(2x)}{x^2+y^2+z^2}\,\mathbf{i} - \frac{\frac{1}{2\sqrt{x^2+y^2+z^2}}(2y)}{x^2+y^2+z^2}\,\mathbf{j} - \frac{\frac{1}{2\sqrt{x^2+y^2+z^2}}(2z)}{x^2+y^2+z^2}\,\mathbf{k}$$

$$= -\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\mathbf{r}}{r^3}$$

(d)

$$\nabla \ln r = \nabla \ln\left(x^2+y^2+z^2\right)^{1/2} = \frac{1}{2}\nabla \ln\left(x^2+y^2+z^2\right)$$

$$= \frac{x}{x^2+y^2+z^2}\,\mathbf{i} + \frac{y}{x^2+y^2+z^2}\,\mathbf{j} + \frac{z}{x^2+y^2+z^2}\,\mathbf{k} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{x^2+y^2+z^2} = \frac{\mathbf{r}}{r^2}$$

32. $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k} \Rightarrow r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$, so

$$\mathbf{F} = \frac{\mathbf{r}}{r^p} = \frac{x}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{i} + \frac{y}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{j} + \frac{z}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{k}$$

Then $\frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{p/2}}=\frac{(x^2+y^2+z^2)-px^2}{(x^2+y^2+z^2)^{1+p/2}}=\frac{r^2-px^2}{r^{p+2}}$. Similarly,

$\frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{p/2}}=\frac{r^2-py^2}{r^{p+2}}$ and $\frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{p/2}}=\frac{r^2-pz^2}{r^{p+2}}$. Thus

$$\text{div}\boldsymbol{F}=\nabla\cdot\boldsymbol{F}=\frac{r^2-px^2}{r^{p+2}}+\frac{r^2-py^2}{r^{p+2}}+\frac{r^2-pz^2}{r^{p+2}}=\frac{3r^2-px^2-py^2-pz^2}{r^{p+2}}$$

$$=\frac{3r^2-p(x^2+y^2+z^2)}{r^{p+2}}=\frac{3r^2-pr^2}{r^{p+2}}=\frac{3-p}{r^p}$$

Consequently, if $p=3$ we have $\text{div}\boldsymbol{F}=0$.

33. By (13), $\oint_C f(\nabla g)\cdot\boldsymbol{n}ds=\iint_D \text{div}\,(f\nabla g)dA=\iint_D[f\text{div}(\nabla g)+\nabla g\cdot\nabla f]dA$ by Exercise 25. But

$\text{div}(\nabla g)=\nabla^2 g$. Hence $\iint_D f\nabla^2 g dA=\oint_C f(\nabla g)\cdot\boldsymbol{n}ds-\iint_D\nabla g\cdot\nabla f dA.$

34. By Exercise 33, $\iint_D f\nabla^2 g dA=\oint_C f(\nabla g)\cdot\boldsymbol{n}ds-\iint_D\nabla g\cdot\nabla f dA$ and

$\iint_D g\nabla^2 f dA=\oint_C g(\nabla f)\cdot\boldsymbol{n}ds-\iint_D\nabla f\cdot\nabla g dA.$ Hence

$$\iint_D(f\nabla^2 g-g\nabla^2 f)dA=\oint_C[f(\nabla g)\cdot\boldsymbol{n}-g(\nabla f)\cdot\boldsymbol{n}]ds+\iint_D(\nabla f\cdot\nabla g-\nabla g\cdot\nabla f)dA$$

$$=\oint_C[f\nabla g-g\nabla f]\cdot\boldsymbol{n}ds$$

35. **(a)** We know that $\omega=v/d$, and from the diagram $\sin\theta=d/r\Rightarrow v=d\omega=(\sin\theta)r\omega=|\boldsymbol{w}\times\boldsymbol{r}|$. But $\boldsymbol{v}$ is perpendicular to both $\boldsymbol{w}$ and $\boldsymbol{r}$, so that $\boldsymbol{v}=\boldsymbol{w}\times\boldsymbol{r}$.

(b) From (a), $\boldsymbol{v}=\boldsymbol{w}\times\boldsymbol{r}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\0&0&\omega\\x&y&z\end{vmatrix}=(0\cdot z-\omega y)\boldsymbol{i}+(\omega x-0\cdot z)\boldsymbol{j}+(0\cdot y-x\cdot 0)\boldsymbol{k}=-\omega y\boldsymbol{i}+\omega x\boldsymbol{j}$

(c)

$$\text{curl}\boldsymbol{v}=\nabla\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\partial/\partial x&\partial/\partial y&\partial/\partial y\\-\omega y&\omega x&0\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(0)-\frac{\partial}{\partial z}(\omega x)\right]\boldsymbol{i}+\left[\frac{\partial}{\partial z}(-\omega y)-\frac{\partial}{\partial x}(0)\right]\boldsymbol{j}+\left[\frac{\partial}{\partial x}(\omega x)-\frac{\partial}{\partial y}(-\omega y)\right]\boldsymbol{k}$$
$$=[\omega-(-\omega)]\boldsymbol{k}=2\omega\,\boldsymbol{k}=2\boldsymbol{w}$$

36. Let $\boldsymbol{H}=\left\langle h_1, h_2, h_3\right\rangle$ and $\boldsymbol{E}=\left\langle E_1, E_2, E_3\right\rangle$.

(a)

$$\nabla\times(\nabla\times\boldsymbol{E})=\nabla\times(\text{curl}\boldsymbol{E})=\nabla\times\left(-\frac{1}{c}\frac{\partial\boldsymbol{H}}{\partial t}\right)=-\frac{1}{c}\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \partial h_1/\partial t & \partial h_2/\partial t & \partial h_3/\partial t\end{vmatrix}$$
$$=-\frac{1}{c}\left[\left(\frac{\partial^2 h_3}{\partial y\partial t}-\frac{\partial^2 h_2}{\partial z\partial t}\right)\boldsymbol{i}+\left(\frac{\partial^2 h_1}{\partial z\partial t}-\frac{\partial^2 h_3}{\partial x\partial t}\right)\boldsymbol{j}+\left(\frac{\partial^2 h_2}{\partial x\partial t}-\frac{\partial^2 h_1}{\partial y\partial t}\right)\boldsymbol{k}\right]$$
$$=-\frac{1}{c}\frac{\partial}{\partial t}\left[\left(\frac{\partial h_3}{\partial y}-\frac{\partial h_2}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial h_1}{\partial z}-\frac{\partial h_3}{\partial x}\right)\boldsymbol{j}+\left(\frac{\partial h_2}{\partial x}-\frac{\partial h_1}{\partial y}\right)\boldsymbol{k}\right]$$

[assuming that the partial derivatives are continuous so that the order of differentiation does not matter]

$$=-\frac{1}{c}\frac{\partial}{\partial t}\text{curl}\boldsymbol{H}=-\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}\right)=-\frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2}$$

(b)

$$\nabla\times(\nabla\times\boldsymbol{H})=\nabla\times(\text{curl}\boldsymbol{H})=\nabla\times\left(\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}\right)=\frac{1}{c}\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \partial E_1/\partial t & \partial E_2/\partial t & \partial E_3/\partial t\end{vmatrix}$$
$$=\frac{1}{c}\left[\left(\frac{\partial^2 E_3}{\partial y\partial t}-\frac{\partial^2 E_2}{\partial z\partial t}\right)\boldsymbol{i}+\left(\frac{\partial^2 E_1}{\partial z\partial t}-\frac{\partial^2 E_3}{\partial x\partial t}\right)\boldsymbol{j}+\left(\frac{\partial^2 E_2}{\partial x\partial t}-\frac{\partial^2 E_1}{\partial y\partial t}\right)\boldsymbol{k}\right]$$
$$=\frac{1}{c}\frac{\partial}{\partial t}\left[\left(\frac{\partial E_3}{\partial y}-\frac{\partial E_2}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial E_1}{\partial z}-\frac{\partial E_3}{\partial x}\right)\boldsymbol{j}+\left(\frac{\partial E_2}{\partial x}-\frac{\partial E_1}{\partial y}\right)\boldsymbol{k}\right]$$

[assuming that the partial derivatives are continuous so that the order of differentiation does not matter]

$$=\frac{1}{c}\frac{\partial}{\partial t}\text{curl}\boldsymbol{E}=\frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial\boldsymbol{H}}{\partial t}\right)=-\frac{1}{c^2}\frac{\partial^2\boldsymbol{H}}{\partial t^2}$$

(c) Using Exercise 29, we have that curl curl$\boldsymbol{E}$=grad div$\boldsymbol{E}-\nabla^2\boldsymbol{E}\Rightarrow$

$\nabla^2 \boldsymbol{E}=\text{grad div}\boldsymbol{E}-\text{curl curl}\boldsymbol{E}=\text{grad}\,0+\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2}=\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2}$.

(d) As in part (c), $\nabla^2 \boldsymbol{H}=\text{grad div}\boldsymbol{H}-\text{curl curl}\boldsymbol{H}=\text{grad}\,0+\frac{1}{c^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2}=\frac{1}{c^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2}$.

37. For any continuous function f on R^3, define a vector field $\boldsymbol{G}(x, y, z)=\langle g(x, y, z), 0, 0\rangle$ where $g(x, y, z)=\int_0^x f(t, y, z)\,dt$. Then

div $\boldsymbol{G}=\frac{\partial}{\partial x}(g(x, y, z))+\frac{\partial}{\partial y}(0)+\frac{\partial}{\partial z}(0)=\frac{\partial}{\partial x}\int_0^x f(t, y, z)dt=f(x, y, z)$ by the Fundamental Theorem of Calculus. Thus every continuous function f on R^3 is the divergence of some vector field.

1. $\boldsymbol{r}(u, v)=u\cos v\,\boldsymbol{i}+u\sin v\,\boldsymbol{j}+u^2\boldsymbol{k}$, so the corresponding parametric equations for the surface are $x=u\cos v$, $y=u\sin v$, $z=u^2$. For any point (x, y, z) on the surface, we have $x^2+y^2=u^2\cos^2 v+u^2\sin^2 v=u^2=z$. Since no restrictions are placed on the parameters, the surface is $z=x^2+y^2$, which we recognize as a circular paraboloid opening upward whose axis is the z –axis.

2. $\boldsymbol{r}(u, v)=(1+2u)\boldsymbol{i}+(-u+3v)\boldsymbol{j}+(2+4u+5v)\boldsymbol{k}=\langle 1, 0, 2\rangle+u\langle 2,-1,4\rangle+v\langle 0, 3, 5\rangle$. From Example 3, we recognize this as a vector equation of a plane through the point (1, 0, 2) and containing vectors $\boldsymbol{a}=\langle 2, -1, 4\rangle$ and $\boldsymbol{b}=\langle 0, 3, 5\rangle$. If we wish to find a more conventional equation for the plane, a normal vector to the plane is

$$\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -1 & 4 \\ 0 & 3 & 5 \end{vmatrix}=-17\boldsymbol{i}-10\boldsymbol{j}+6\boldsymbol{k}$$

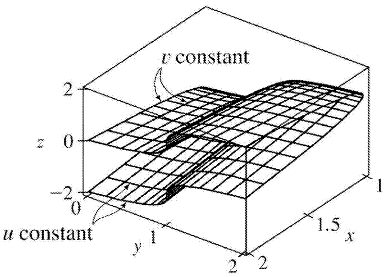
and an equation of the plane is $-17(x-1)-10(y-0)+6(z-2)=0$ or $-17x-10y+6z=-5$.

3. $\boldsymbol{r}(x, \theta)=\langle x, \cos\theta, \sin\theta\rangle$, so the corresponding parametric equations for the surface are $x=x$, $y=\cos\theta$, $z=\sin\theta$. For any point (x, y, z) on the surface, we have $y^2+z^2=\cos^2\theta+\sin^2\theta=1$, so any vertical trace in $x=k$ is the circle $y^2+z^2=1$, $x=k$. Since $x=x$ with no restriction, the surface is a circular cylinder with radius 1 whose axis is the x –axis.

4. $\boldsymbol{r}(x, \theta)=\langle x, x\cos\theta, x\sin\theta\rangle$, so the corresponding parametric equations for the surface are $x=x$, $y=x\cos\theta$, $z=x\sin\theta$. For any point (x, y, z) on the surface, we have $y^2+z^2=x^2\cos^2\theta+x^2\sin^2\theta=x^2$. With $x=x$ and no restrictions on the parameters, the surface is $x^2=y^2+z^2$, which we recognize as a circular cone whose axis is the x –axis.

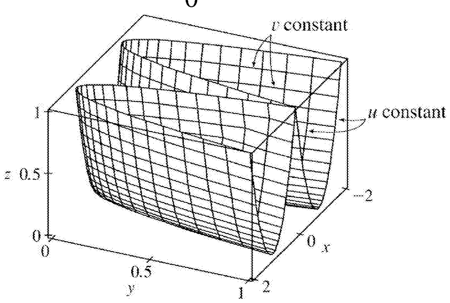
5. $\boldsymbol{r}(u, v)=\left\langle u^2+1, v^3+1, u+v\right\rangle$, $-1\leq u\leq 1$, $-1\leq v\leq 1$.

The surface has parametric equations $x=u^2+1$, $y=v^3+1$, $z=u+v$, $-1\leq u\leq 1$, $-1\leq v\leq 1$. If we keep u constant at u_0 , $x=u_0^2+1$, a constant, so the corresponding grid curves must be the curves parallel to the yz –plane. If v is constant, we have $y=v_0^3+1$, a constant, so these grid curves are the curves parallel to the xz –plane.

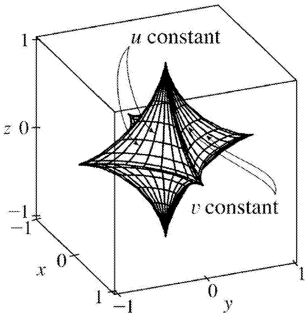


6. $\boldsymbol{r}(u, v)=\left\langle u+v, u^2, v^2\right\rangle$, $-1\leq u\leq 1$, $-1\leq v\leq 1$.

The surface has parametric equations $x=u+v$, $y=u^2$, $z=v^2$, $-1\leq u\leq 1$, $-1\leq v\leq 1$. If $u=u_0$ is constant, $y=u_0^2=$ constant, so the corresponding grid curves are the curves parallel to the xz –plane. If $v=v_0$ is constant, $z=v_0^2=$ constant, so the corresponding grid curves are the curves parallel to the xy –plane.



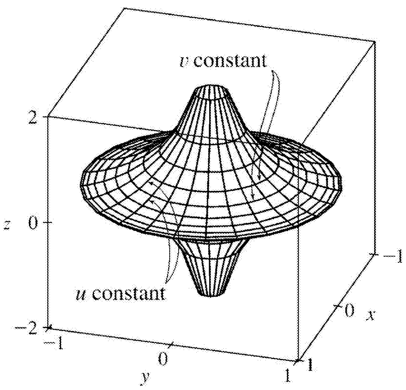
7. $\boldsymbol{r}(u, v)=\left\langle \cos^3 u\cos^3 v, \sin^3 u\cos^3 v, \sin^3 v\right\rangle$. The surface has parametric equations $x=\cos^3 u\cos^3 v$, $y=\sin^3 u\cos^3 v$, $z=\sin^3 v$, $0\leq u\leq \pi$, $0\leq v\leq 2\pi$. Note that if $v=v_0$ is constant then $z=\sin^3 v_0$ is constant, so the corresponding grid curves must be the curves parallel to the xy –plane. The vertically oriented grid curves, then, correspond to $u=u_0$ being held constant, giving $x=\cos^3 u_0\cos^3 v$, $y=\sin^3 u_0\cos^3 v$, $z=\sin^3 v$. These curves lie in vertical planes that contain the z –axis.



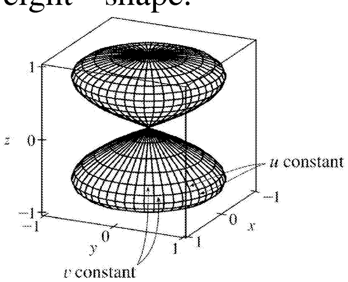
8. $\boldsymbol{r}(u, v)=\left\langle \cos u\sin v, \sin u\sin v, \cos v+\ln\tan(v/2)\right\rangle$.

The surface has parametric equations $x=\cos u\sin v$, $y=\sin u\sin v$, $z=\cos v+\ln\tan(v/2)$, $0\leq u\leq 2\pi$,

$0.1 \le v \le 6.2$. Note that if $v = v_0$ is constant, the parametric equations become $x = \cos u \sin v_0$, $y = \sin u \sin v_0$, $z = \cos v_0 + \ln \tan (v_0/2)$ which represent a circle of radius $\sin v_0$ in the plane $z = \cos v_0 + \ln \tan (v_0/2)$. So the circular grid curves we see lying horizontally are the grid curves with v constant. The vertically oriented grid curves correspond to $u = u_0$ being held constant, giving $x = \cos u_0 \sin v$, $y = \sin u_0 \sin v$, $z = \cos v + \ln \tan (v/2)$. These curves lie in vertical planes that contain the z –axis.

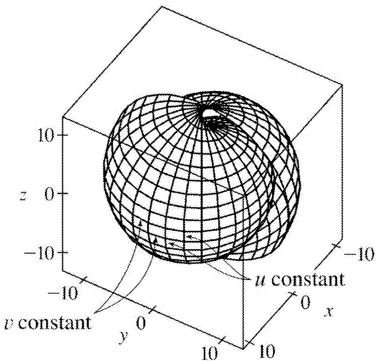


9. $x = \cos u \sin 2v$, $y = \sin u \sin 2v$, $z = \sin v$.
The complete graph of the surface is given by the parametric domain $0 \le u \le \pi, 0 \le v \le 2\pi$. Note that if $v = v_0$ is constant, the parametric equations become $x = \cos u \sin 2v_0$, $y = \sin u \sin 2v_0$, $z = \sin v_0$ which represent a circle of radius $\sin 2v_0$ in the plane $z = \sin v_0$. So the circular grid curves we see lying horizontally are the grid curves which have v constant. The vertical grid curves, then, correspond to $u = u_0$ being held constant, giving $x = \cos u_0 \sin 2v$ and $y = \sin u_0 \sin 2v$ with $z = \sin v$ which has a ''figure–eight'' shape.



10. $x = u \sin u \cos v$, $y = u \cos u \cos v$, $z = u \sin v$.
We graph the portion of the surface with parametric domain $0 \le u \le 4\pi, 0 \le v \le 2\pi$. Note that if $v = v_0$ is constant, the parametric equations become $x = u \sin u \cos v_0$, $y = u \cos u \cos v_0$, $z = u \sin v_0$. The equations for x and y show that the projections onto the xy –plane give a spiral shape, so the corresponding grid curves are the almost–horizontal spiral curves we see. The vertical grid curves, which look approximately circular, correspond to $u = u_0$ being held constant, giving $x = u_0 \sin u_0 \cos v$,

$y=u_0\cos u_0\cos v$, $z=u_0\sin v$.



11. $\boldsymbol{r}(u, v)=\cos v\boldsymbol{i}+\sin v\boldsymbol{j}+u\boldsymbol{k}$. The parametric equations for the surface are $x=\cos v$, $y=\sin v$, $z=u$. Then $x^2+y^2=\cos^2 v+\sin^2 v=1$ and $z=u$ with no restriction on u , so we have a circular cylinder, graph IV. The grid curves with u constant are the horizontal circles we see in the plane $z=u$. If v is constant, both x and y are constant with z free to vary, so the corresponding grid curves are the lines on the cylinder parallel to the z –axis.

12. $\boldsymbol{r}(u, v)=u\cos v\boldsymbol{i}+u\sin v\boldsymbol{j}+u\boldsymbol{k}$. The parametric equations for the surface are $x=u\cos v$, $y=u\sin v$, $z=u$. Then $x^2+y^2=u^2\cos^2 v+u^2\sin^2 v=u^2=z^2$, which represents the equation of a cone with axis the z –axis, graph V. The grid curves with u constant are the horizontal circles we see, corresponding to the equations $x^2+y^2=u^2$ in the plane $z=u$. If v is constant, x, y, z are each scalar multiples of u , corresponding to the straight line grid curves through the origin.

13. $\boldsymbol{r}(u, v)=u\cos v\boldsymbol{i}+u\sin v\boldsymbol{j}+v\boldsymbol{k}$. The parametric equations for the surface are $x=u\cos v, y=u\sin v, z=v$. We look at the grid curves first; if we fix v , then x and y parametrize a straight line in the plane $z=v$ which intersects the z –axis. If u is held constant, the projection onto the xy –plane is circular; with $z=v$, each grid curve is a helix. The surface is a spiraling ramp, graph I.

14. $x=u^3$, $y=u\sin v$, $z=u\cos v$. Then $y^2+z^2=u^2\sin^2 v+u^2\cos^2 v=u^2$, so if u is held constant, each grid curve is a circle of radius u in the plane $x=u^3$. The graph then must be graph III. If v is held constant, so $v=v_0$, we have $y=u\sin v_0$ and $z=u\cos v_0$. Then $y=(\tan v_0)z$, so the grid curves we see running lengthwise along the surface in the planes $y=kz$ correspond to keeping v constant.

15. $x=(u-\sin u)\cos v$, $y=(1-\cos u)\sin v$, $z=u$. If u is held constant, x and y give an equation of an ellipse in the plane $z=u$, thus the grid curves are horizontally oriented ellipses. Note that when $u=0$, the ‘‘ellipse’’ is the single point $(0, 0, 0)$, and when $u=\pi$, we have $y=0$ while x ranges from $-\pi$ to π , a line segment parallel to the x –axis in the plane $z=\pi$. This is the upper ‘‘seam’’ we see in graph II. When v is held constant, $z=u$ is free to vary, so the corresponding grid curves are the curves we see running up and down along the surface.

16. $x=(1-u)(3+\cos v)\cos 4\pi u$, $y=(1-u)(3+\cos v)\sin 4\pi u$, $z=3u+(1-u)\sin v$. These equations correspond to graph VI: when $u=0$, then $x=3+\cos v$, $y=0$, and $z=\sin v$, which are equations of a circle with radius 1 in the xz –plane centered at (3, 0, 0) . When $u=\frac{1}{2}$, then $x=\frac{3}{2}+\frac{1}{2}\cos v$, $y=0$, and $z=\frac{3}{2}+\frac{1}{2}\sin v$, which are equations of a circle with radius $\frac{1}{2}$ in the xz –plane centered at $\left(\frac{3}{2}, 0, \frac{3}{2}\right)$. When $u=1$, then $x=y=0$ and $z=3$, giving the topmost point shown in the graph. This suggests that the grid curves with u constant are the vertically oriented circles visible on the surface. The spiralling grid curves correspond to keeping v constant.

17. From Example 3, parametric equations for the plane through the point (1, 2, –3) that contains the vectors $\boldsymbol{a}=\langle 1, 1, -1\rangle$ and $\boldsymbol{b}=\langle 1, -1, 1\rangle$ are $x=1+u(1)+v(1)=1+u+v$, $y=2+u(1)+v(-1)=2+u-v$, $z=-3+u(-1)+v(1)=-3-u+v$.

18. Solving the equation for z gives $z^2=1-2x^2-4y^2 \Rightarrow z=-\sqrt{1-2x^2-4y^2}$ (since we want the lower half of the ellipsoid). If we let x and y be the parameters, parametric equations are $x=x, y=y, z=-\sqrt{1-2x^2-4y^2}$.
Alternate solution : The equation can be rewritten as $\frac{x^2}{(1/\sqrt{2})^2}+\frac{y^2}{(1/2)^2}+z^2=1$, and if we let $x=\frac{1}{\sqrt{2}}u\cos v$ and $y=\frac{1}{2}u\sin v$, then $z=-\sqrt{1-2x^2-4y^2}=-\sqrt{1-u^2\cos^2 v-u^2\sin^2 v}=-\sqrt{1-u^2}$, where $0\le u\le 1$ and $0\le v\le 2\pi$.

19. Solving the equation for y gives $y^2=1-x^2+z^2 \Rightarrow y=\sqrt{1-x^2+z^2}$. (We choose the positive root since we want the part of the hyperboloid that corresponds to $y\ge 0$). If we let x and z be the parameters, parametric equations are $x=x$, $z=z$, $y=\sqrt{1-x^2+z^2}$.

20. $x=4-y^2-2z^2$, $y=y$, $z=z$ where $y^2+2z^2\le 4$ since $x\ge 0$. Then the associated vector equation is $\boldsymbol{r}(y, z)=(4-y^2-2z^2)\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$.

21. Since the cone intersects the sphere in the circle $x^2+y^2=2$, $z=\sqrt{2}$ and we want the portion of the sphere above this, we can parametrize the surface as $x=x$, $y=y$, $z=\sqrt{4-x^2-y^2}$ where $x^2+y^2\le 2$.
Alternate solution: Using spherical coordinates, $x=2\sin\phi\cos\theta$, $y=2\sin\phi\sin\theta$, $z=2\cos\phi$ where $0\le\phi\le\frac{\pi}{4}$ and $0\le\theta\le 2\pi$.

22. In spherical coordinates, parametric equations are $x=4\sin\phi\cos\theta$, $y=4\sin\phi\sin\theta$, $z=4\cos\phi$. The intersection of the sphere with the plane $z=2$ corresponds to $z=4\cos\phi=2\Rightarrow\cos\phi=\frac{1}{2}\Rightarrow\phi=\frac{\pi}{3}$. By symmetry, the intersection of the sphere with the plane $z=-2$ corresponds to $\phi=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$. Thus the surface is described by $0\le\theta\le 2\pi$, $\frac{\pi}{3}\le\phi\le\frac{2\pi}{3}$.

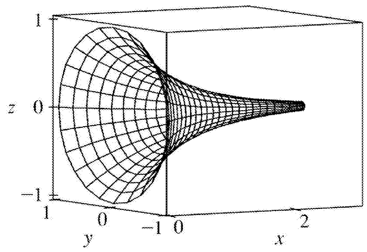
23. Parametric equations are $x=x$, $y=4\cos\theta$, $z=4\sin\theta$, $0\le x\le 5$, $0\le\theta\le 2\pi$.

24. Using x and y as the parameters, $x=x$, $y=y$, $z=x+3$ where $0\le x^2+y^2\le 1$. Also, since the plane intersects the cylinder in an ellipse, the surface is a planar ellipse in the plane $z=x+3$. Thus, parametrizing with respect to s and θ , we have $x=s\cos\theta$, $y=s\sin\theta$, $z=3+s\cos\theta$ where $0\le s\le 1$ and $0\le\theta\le 2\pi$.

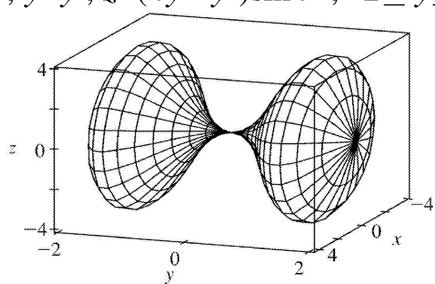
25. The surface appears to be a portion of a circular cylinder of radius 3 with axis the x -axis. An equation of the cylinder is $y^2+z^2=9$, and we can impose the restrictions $0\le x\le 5$, $y\le 0$ to obtain the portion shown.
To graph the surface on a CAS, we can use parametric equations $x=u$, $y=3\cos v$, $z=3\sin v$ with the parameter
domain $0\le u\le 5$, $\frac{\pi}{2}\le v\le\frac{3\pi}{2}$. Alternatively, we can regard x and z as parameters. Then parametric equations are $x=x$, $z=z$, $y=-\sqrt{9-z^2}$, where $0\le x\le 5$ and $-3\le z\le 3$.

26. The surface appears to be a portion of a sphere of radius 1 centered at the origin. In spherical coordinates, the sphere has equation $\rho=1$, and imposing the restrictions $\frac{\pi}{2}\le\theta\le 2\pi$, $\frac{\pi}{4}\le\phi\le\pi$ will give only the portion of the sphere shown. Thus, to graph the surface on a CAS we can either use spherical coordinates with the stated restrictions, or we can use parametric equations: $x=\sin\phi\cos\theta$, $y=\sin\phi\sin\theta$, $z=\cos\phi$, $\frac{\pi}{2}\le\theta\le 2\pi$, $\frac{\pi}{4}\le\phi\le\pi$.

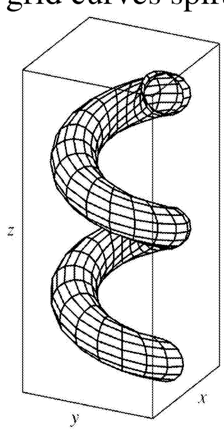
27. Using Equations 3, we have the parametrization $x=x$, $y=e^{-x}\cos\theta$, $z=e^{-x}\sin\theta$, $0\le x\le 3$, $0\le\theta\le 2\pi$.



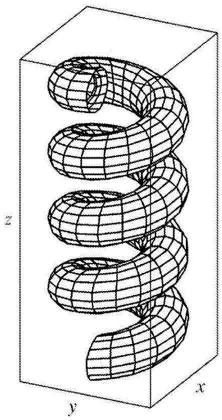
28. Letting θ be the angle of rotation about the y –axis, we have the parametrization $x=(4y^2-y^4)\cos\theta$, $y=y$, $z=(4y^2-y^4)\sin\theta$, $-2\le y\le 2$, $0\le\theta\le 2\pi$.



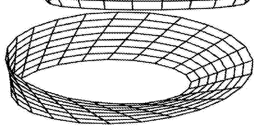
29. **(a)** Replacing cos u by sin u and sin u by cos u gives parametric equations $x=(2+\sin v)\sin u$, $y=(2+\sin v)\cos u$, $z=u+\cos v$. From the graph, it appears that the direction of the spiral is reversed. We can verify this observation by noting that the projection of the spiral grid curves onto the xy –plane, given by $x=(2+\sin v)\sin u$, $y=(2+\sin v)\cos u$, $z=0$, draws a circle in the clockwise direction for each value of v . The original equations, on the other hand, give circular projections drawn in the counterclockwise direction. The equation for z is identical in both surfaces, so as z increases, these grid curves spiral up in opposite directions for the two surfaces.

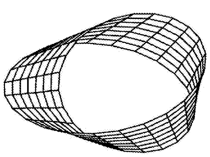


(b) Replacing cos u by cos $2u$ and sin u by sin $2u$ gives parametric equations $x=(2+\sin v)\cos 2u$, $y=(2+\sin v)\sin 2u$, $z=u+\cos v$. From the graph, it appears that the number of coils in the surface doubles within the same parametric domain. We can verify this observation by noting that the projection of the spiral grid curves onto the xy –plane, given by $x=(2+\sin v)\cos 2u$, $y=(2+\sin v)\sin 2u$, $z=0$ (where v is constant), complete circular revolutions for $0\le u\le\pi$ while the original surface requires $0\le u\le 2\pi$ for a complete revolution. Thus, the new surface winds around twice as fast as the original surface, and since the equation for z is identical in both surfaces, we observe twice as many circular coils in the same z –interval.



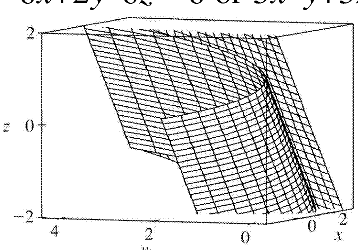
30. First we graph the surface as viewed from the front, then from two additional viewpoints.





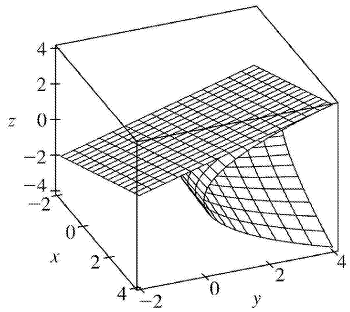
The surface appears as a twisted sheet, and is unusual because it has only one side. (The Mobius strip is discussed in more detail in Section 17.7).

31. $\boldsymbol{r}(u, v)=(u+v)\boldsymbol{i}+3u^2\boldsymbol{j}+(u-v)\boldsymbol{k}$.
$\boldsymbol{r}_u=\boldsymbol{i}+6u\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{r}_v=\boldsymbol{i}-\boldsymbol{k}$, so $\boldsymbol{r}_u\times\boldsymbol{r}_v=-6u\boldsymbol{i}+2\boldsymbol{j}-6u\boldsymbol{k}$. Since the point (2, 3, 0) corresponds to $u=1$, $v=1$, a normal vector to the surface at (2, 3, 0) is $-6\boldsymbol{i}+2\boldsymbol{j}-6\boldsymbol{k}$, and an equation of the tangent plane is $-6x+2y-6z=-6$ or $3x-y+3z=3$.

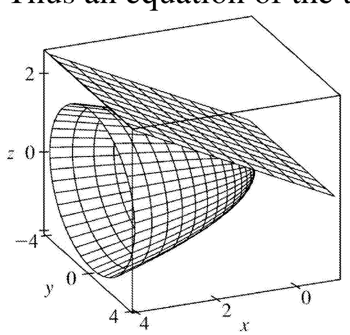


32. $\boldsymbol{r}(u, v)=u^2\boldsymbol{i}+v^2\boldsymbol{j}+uv\boldsymbol{k}\Rightarrow\boldsymbol{r}(1, 1)=(1, 1, 1)$.
$\boldsymbol{r}_u=2u\boldsymbol{i}+v\boldsymbol{k}$ and $\boldsymbol{r}_v=2v\boldsymbol{j}+u\boldsymbol{k}$, so a normal vector to the surface at the point (1, 1, 1) is

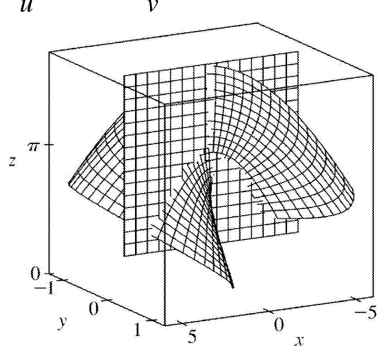
$\mathbf{r}_u(1,1)\times\mathbf{r}_v(1,1)=(2\mathbf{i}+\mathbf{k})\times(2\mathbf{j}+\mathbf{k})=-2\mathbf{i}-2\mathbf{j}+4\mathbf{k}$. Thus an equation of the tangent plane at the point (1, 1, 1) is $-2(x-1)-2(y-1)+4(z-1)=0$ or $x+y-2z=0$.



33. $\mathbf{r}(u,v)=u^2\mathbf{i}+2u\sin v\,\mathbf{j}+u\cos v\,\mathbf{k}\Rightarrow\mathbf{r}(1,0)=(1,0,1)$.
$\mathbf{r}_u=2u\mathbf{i}+2\sin v\,\mathbf{j}+\cos v\,\mathbf{k}$ and $\mathbf{r}_v=2u\cos v\,\mathbf{j}-u\sin v\,\mathbf{k}$, so a normal vector to the surface at the point (1, 0, 1) is
$\mathbf{r}_u(1,0)\times\mathbf{r}_v(1,0)=(2\mathbf{i}+\mathbf{k})\times(2\mathbf{j})=-2\mathbf{i}+4\mathbf{k}$.
Thus an equation of the tangent plane at (1, 0, 1) is $-2(x-1)+0(y-0)+4(z-1)=0$ or $-x+2z=1$.



34. $\mathbf{r}(u,v)=uv\mathbf{i}+u\sin v\,\mathbf{j}+v\cos u\,\mathbf{k}\Rightarrow\mathbf{r}(0,\pi)=(0,0,\pi)$
$\mathbf{r}_u=v\mathbf{i}+\sin v\,\mathbf{j}-v\sin u\,\mathbf{k}$ and $\mathbf{r}_v=u\mathbf{i}+u\cos v\,\mathbf{j}+\cos u\,\mathbf{k}$, so a normal vector to the surface at the point $(0,0,\pi)$ is
$\mathbf{r}_u(0,\pi)\times\mathbf{r}_v(0,\pi)=(\pi\mathbf{i})\times(\mathbf{k})=-\pi\mathbf{j}$. Thus an equation of the tangent plane is $-\pi(y-0)=0$ or $y=0$.



35. Here $z=f(x,y)=4-x-2y$ and D is the disk $x^2+y^2\le 4$. Thus, by Formula 9,

$$A(S)=\iint_D \sqrt{1+(-1)^2+(-2)^2}\,dA=\sqrt{6}\iint_D dA=\sqrt{6}\,A(D)=4\sqrt{6}\,\pi$$

36. $\mathbf{r}_u=\langle 0, 1, -5\rangle$, $\mathbf{r}_v=\langle 1, -2, 1\rangle$, and $\mathbf{r}_u\times\mathbf{r}_v=\langle -9, -5, -1\rangle$. Then by Definition 6, $A(S)=\iint_D |\mathbf{r}_u\times\mathbf{r}_v|\,dA =$

$$\int_0^1\int_0^1 \left|\langle -9, -5, -1\rangle\right| du\,dv=\sqrt{107}\int_0^1 du\int_0^1 dv=\sqrt{107}$$

37. $z=f(x, y)=xy$ with $0\le x^2+y^2\le 1$, so $f_x=y$, $f_y=x\Rightarrow$

$$A(S) = \iint_D \sqrt{1+y^2+x^2}\,dA = \int_0^{2\pi}\int_0^1 \sqrt{r^2+1}\,r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{3}(r^2+1)^{3/2}\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi}\frac{1}{3}(2\sqrt{2}-1)d\theta=\frac{2\pi}{3}(2\sqrt{2}-1)$$

38. $z=f(\mathrm{x,y})=1+3x+2y^2$ with $0\le x\le 2y$, $0\le y\le 1$. Thus, by Formula 9,

$$A(S) =\iint_D \sqrt{1+3^2+(4y)^2}\,dA = \int_0^1\int_0^{2y} \sqrt{10+16y^2}\,dx\,dy = \int_0^1 2y\sqrt{10+16y^2}\,dy$$

$$= \frac{1}{16}\cdot\frac{2}{3}(10+16y^2)^{3/2}\Big]_0^1 = \frac{1}{24}(26^{3/2}-10^{3/2})$$

39. $z=f(x, y)=y^2-x^2$ with $1\le x^2+y^2\le 4$. Then

$$A(S) = \iint_D \sqrt{1+4x^2+4y^2}\,dA = \int_0^{2\pi}\int_1^2 \sqrt{1+4r^2}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta\int_1^2 \sqrt{1+4r^2}\,r\,dr$$

$$= [\theta]_0^{2\pi}\left[\frac{1}{12}(1+4r^2)^{3/2}\right]_1^2=\frac{\pi}{6}(17\sqrt{17}-5\sqrt{5})$$

40. A parametric representation of the surface is $x=y^2+z^2$, $y=y$, $z=z$ with $0\le y^2+z^2\le 9$. Hence $\mathbf{r}_y\times\mathbf{r}_z=(2y\mathbf{i}+\mathbf{j})\times(2z\mathbf{i}+\mathbf{k})=\mathbf{i}-2y\,\mathbf{j}-2z\,\mathbf{k}$.

Note: In general, if $x=f(y, z)$ then $\mathbf{r}_y\times\mathbf{r}_z=\mathbf{i}-\frac{\partial f}{\partial y}\,\mathbf{j}-\frac{\partial f}{\partial z}\,\mathbf{k}$, and $A(S)=\iint_D \sqrt{1+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2}\,dA.$

Then

$$A(S) = \iint_{0\le y^2+z^2\le 9} \sqrt{1+4y^2+4z^2}\,dA = \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\,r\,dr\,d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^3 r\sqrt{1+4r^2}\,dr = 2\pi\left[\frac{1}{12}(1+4r^2)^{3/2}\right]_0^3 = \frac{\pi}{6}\left(37\sqrt{37}-1\right)$$

41. A parametric representation of the surface is $x=x$, $y=4x+z^2$, $z=z$ with $0\le x\le 1$, $0\le z\le 1$. Hence $\boldsymbol{r}_x\times\boldsymbol{r}_z=(\boldsymbol{i}+4\boldsymbol{j})\times(2z\boldsymbol{j}+\boldsymbol{k})=4\boldsymbol{i}-\boldsymbol{j}+2z\boldsymbol{k}$.

Note: In general, if $y=f(x, z)$ then $\boldsymbol{r}_x\times\boldsymbol{r}_z=\frac{\partial f}{\partial x}\boldsymbol{i}-\boldsymbol{j}+\frac{\partial f}{\partial z}\boldsymbol{k}$ and $A(S)=\iint_D\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2}\,dA$. Then

$$A(S) = \int_0^1\int_0^1\sqrt{17+4z^2}\,dx\,dz=\int_0^1\sqrt{17+4z^2}\,dz$$
$$= \frac{1}{2}\left(z\sqrt{17+4z^2}+\frac{17}{2}\ln\left|2z+\sqrt{4z^2+17}\right|\right)\Big]_0^1=\frac{\sqrt{21}}{2}+\frac{17}{4}\left[\ln\left(2+\sqrt{21}\right)-\ln\sqrt{17}\right]$$

42. Let S_1 be that portion of the surface which lies above the plane $z=0$. Then $A(S)=2A(S_1)$ by symmetry. On S_1 , $z=\sqrt{a^2-x^2}$ so $|\boldsymbol{r}_x\times\boldsymbol{r}_y|=\sqrt{1+\frac{x^2}{a^2-x^2}}=\frac{a}{\sqrt{a^2-x^2}}$. Hence

$$A(S_1)=\iint_{0\le x^2+y^2\le a^2}\frac{a}{\sqrt{a^2-x^2}}\,dA=\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\frac{a}{\sqrt{a^2-x^2}}\,dy\,dx=\int_{-a}^{a}2a\,dx=4a^2.$$

Thus $A(S)=8a^2$.

Alternate solution: If $A(S_2)$ is the surface area in the first octant, then $A(S)=8A(S_2)$. A parametric representation of the surface in the first octant is $x=a\sin\theta$, $y=y$, $z=a\cos\theta$ (θ being the angle in the xz –plane measured from the positive z –axis), where $0\le\theta\le\frac{\pi}{2}$ and $0\le y\le a\cos\theta$. The restrictions on y follow from: $x^2+y^2\le a^2$ or $a^2\sin^2\theta+y^2\le a^2$ so $y^2\le a^2(1-\sin^2\theta)$; thus in the first octant $0\le y\le a\cos\theta$. Then $\boldsymbol{r}_y\times\boldsymbol{r}_\theta=\langle -a\sin\theta, 0, -a\cos\theta\rangle$ and $A(S_2)=\int_0^{\pi/2}\int_0^{a\cos\theta}a\,dy\,d\theta=\int_0^{\pi/2}a^2\cos\theta\,d\theta=a^2$.

Hence $A(S)=8a^2$.

43. Let $A(S_1)$ be the surface area of that portion of the surface which lies above the plane $z=0$. Then $A(S)=2A(S_1)$. Following Example 10, a parametric representation of S_1 is $x=a\sin\phi\cos\theta$, $y=a\sin\phi\sin\theta$,
$z=a\cos\phi$ and $|\mathbf{r}_\phi\times\mathbf{r}_\theta|=a^2\sin\phi$. For D, $0\le\phi\le\frac{\pi}{2}$ and for each fixed ϕ,
$\left(x-\frac{1}{2}a\right)^2+y^2\le\left(\frac{1}{2}a\right)^2$ or $\left[a\sin\phi\cos\theta-\frac{1}{2}a\right]^2+a^2\sin^2\phi\sin^2\theta\le(a/2)^2$ implies
$a^2\sin^2\phi-a^2\sin\phi\cos\theta\le 0$ or $\sin\phi(\sin\phi-\cos\theta)\le 0$. But $0\le\phi\le\frac{\pi}{2}$, so $\cos\theta\ge\sin\phi$ or
$\sin\left(\frac{\pi}{2}+\theta\right)\ge\sin\phi$ or $\phi-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}-\phi$. Hence $D=\left\{(\phi,\theta)\mid 0\le\phi\le\frac{\pi}{2},\ \phi-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}-\phi\right\}$.
Then

$$A(S_1)=\int_0^{\pi/2}\int_{\phi-(\pi/2)}^{(\pi/2)-\phi}a^2\sin\phi\,d\theta\,d\phi=a^2\int_0^{\pi/2}(\pi-2\phi)\sin\phi\,d\phi$$

$$=a^2\left[(-\pi\cos\phi)-2(-\phi\cos\phi+\sin\phi)\right]_0^{\pi/2}=a^2(\pi-2)$$

Thus $A(S)=2a^2(\pi-2)$.

Alternate solution: Working on S_1 we could parametrize the portion of the sphere by $x=x$, $y=y$,
$z=\sqrt{a^2-x^2-y^2}$. Then $|\mathbf{r}_x\times\mathbf{r}_y|=\sqrt{1+\dfrac{x^2}{a^2-x^2-y^2}+\dfrac{y^2}{a^2-x^2-y^2}}=\dfrac{a}{\sqrt{a^2-x^2-y^2}}$ and

$$A(S_1)=\iint_{0\le(x-(a/2))^2+y^2\le(a/2)^2}\frac{a}{\sqrt{a^2-x^2-y^2}}\,dA=\int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta}\frac{a}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta$$

$$\int_{-\pi/2}^{\pi/2}-a(a^2-r^2)^{1/2}\Big]_{r=0}^{r=a\cos\theta}d\theta=\int_{-\pi/2}^{\pi/2}a^2\left[1-(1-\cos^2\theta)^{1/2}\right]d\theta$$

$$\int_{-\pi/2}^{\pi/2}a^2(1-|\sin\theta|)d\theta=2a^2\int_0^{\pi/2}(1-\sin\theta)d\theta=2a^2\left(\frac{\pi}{2}-1\right)$$

Thus $A(S)=4a^2\left(\frac{\pi}{2}-1\right)=2a^2(\pi-2)$.

Notes:

(1) Perhaps working in spherical coordinates is the most obvious approach here. However, you must be careful in setting up D.

(2) In the alternate solution, you can avoid having to use $|\sin\theta|$ by working in the first octant and

then multiplying by 4 . However, if you set up S_1 as above and arrived at $A(S_1)=a^2\pi$, you now see your error.

44. $\boldsymbol{r}_u=\langle \cos v, \sin v, 0\rangle, \boldsymbol{r}_v=\langle -u\sin v, u\cos v, 1\rangle$, and $\boldsymbol{r}_u\times\boldsymbol{r}_v=\langle \sin v, -\cos v, u\rangle$. Then

$$A(S) = \int_0^\pi\int_0^1 \sqrt{1+u^2}\,du\,dv=\int_0^\pi dv\int_0^1\sqrt{1+u^2}\,du$$

$$= \pi\left[\frac{u}{2}\sqrt{u^2+1}+\frac{1}{2}\ln\left|u+\sqrt{u^2+1}\right|\right]_0^1=\frac{\pi}{2}\left[\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]$$

45. $\boldsymbol{r}_u=\langle v, 1, 1\rangle$, $\boldsymbol{r}_v=\langle u, 1, -1\rangle$ and $\boldsymbol{r}_u\times\boldsymbol{r}_v=\langle -2, u+v, v-u\rangle$. Then

$$A(S) = \iint_{u^2+v^2\le 1}\sqrt{4+2u^2+2v^2}\,dA=\int_0^{2\pi}\int_0^1 r\sqrt{4+2r^2}\,dr\,d\theta=\int_0^{2\pi}d\theta\int_0^1 r\sqrt{4+2r^2}\,dr$$

$$= 2\pi\left[\frac{1}{6}\left(4+2r^2\right)^{3/2}\right]_0^1=\frac{\pi}{3}\left(6\sqrt{6}-8\right)=\pi\left(2\sqrt{6}-\frac{8}{3}\right)$$

46. $z=f(x, y)=\cos\left(x^2+y^2\right)$ with $x^2+y^2\le 1$.

$$A(S)=\iint_D\sqrt{1+\left(-2x\sin\left(x^2+y^2\right)\right)^2+\left(-2y\sin\left(x^2+y^2\right)\right)^2}\,dA$$

$$=\iint_D\sqrt{1+4x^2\sin^2\left(x^2+y^2\right)+4y^2\sin^2\left(x^2+y^2\right)}\,dA$$

$$=\iint_D\sqrt{1+4\left(x^2+y^2\right)\sin^2\left(x^2+y^2\right)}\,dA$$

$$=\int_0^{2\pi}\int_0^1\sqrt{1+4r^2\sin^2\left(r^2\right)}\,r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_0^1 r\sqrt{1+4r^2\sin^2\left(r^2\right)}\,dr$$

$$=2\pi\int_0^1 r\sqrt{1+4r^2\sin^2\left(r^2\right)}\,dr\approx 4.1073$$

47. $z=f(x, y)=e^{-x^2-y^2}$ with $x^2+y^2\le 4$.

$$A(S)=\iint_D \sqrt{1+\left(-2xe^{-x^2-y^2}\right)^2+\left(-2ye^{-x^2-y^2}\right)^2}\,dA$$

$$=\iint_D \sqrt{1+4(x^2+y^2)e^{-2(x^2+y^2)}}\,dA$$

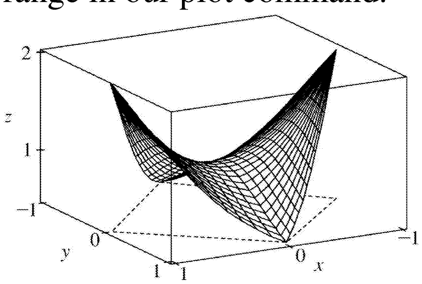
$$=\int_0^{2\pi}\int_0^2 \sqrt{1+4r^2e^{-2r^2}}\,r\,dr\,d\theta=\int_0^{2\pi} d\theta\int_0^2 r\sqrt{1+4r^2e^{-2r^2}}\,dr$$

$$=2\pi\int_0^2 r\sqrt{1+4r^2e^{-2r^2}}\,dr\approx 13.9783$$

48. Let $f(x, y)=\dfrac{1+x^2}{1+y^2}$. Then $f_x=\dfrac{2x}{1+y^2}$, $f_y=(1+x^2)\left[-\dfrac{2y}{(1+y^2)^2}\right]=-\dfrac{2y(1+x^2)}{(1+y^2)^2}$.

We use a CAS to estimate $\int_{-1}^{1}\int_{-(1-|x|)}^{1-|x|}\sqrt{1+f_x^2+f_y^2}\,dy\,dx\approx 2.6959$.

In order to graph only the part of the surface above the square, we use $-(1-|x|)\le y\le 1-|x|$ as the y – range in our plot command.



49. **(a)** The midpoints of the four squares are $\left(\frac{1}{4}, \frac{1}{4}\right)$, $\left(\frac{1}{4}, \frac{3}{4}\right)$, $\left(\frac{3}{4}, \frac{1}{4}\right)$, and $\left(\frac{3}{4}, \frac{3}{4}\right)$; the derivatives of the function $f(x, y)=x^2+y^2$ are $f_x(x, y)=2x$ and $f_y(x, y)=2y$, so the Midpoint Rule gives

$$A(S)=\int_0^1\int_0^1\sqrt{[f_x(x, y)]^2+[f_y(x, y)]^2+1}\,dy\,dx$$

$$\approx \frac{1}{4}\left(\sqrt{\left[2\left(\frac{1}{4}\right)\right]^2+\left[2\left(\frac{1}{4}\right)\right]^2+1}+\sqrt{\left[2\left(\frac{1}{4}\right)\right]^2+\left[2\left(\frac{3}{4}\right)\right]^2+1}\right.$$

$$+\sqrt{\left[2\left(\frac{3}{4}\right)\right]^2+\left[2\left(\frac{1}{4}\right)\right]^2+1}+\sqrt{\left[2\left(\frac{3}{4}\right)\right]^2+\left[2\left(\frac{3}{4}\right)\right]^2+1}\;\Bigg)$$

$$=\frac{1}{4}\left(\sqrt{\frac{3}{2}}+2\sqrt{\frac{7}{2}}+\sqrt{\frac{11}{2}}\right)\approx 1.8279$$

(b) A CAS estimates the integral to be $A(S)=\int_0^1\int_0^1\sqrt{f_x^2+f_y^2+1}\,dy\,dx=\int_0^1\int_0^1\sqrt{4x^2+4y^2+1}\,dy\,dx\approx 1.8616$. This agrees with the Midpoint estimate only in the first decimal place.

50. $\mathbf{r}(u,v)=\left\langle \cos^3 u\cos^3 v,\ \sin^3 u\cos^3 v,\ \sin^3 v\right\rangle$, so $\mathbf{r}_u=\left\langle -3\cos^2 u\sin u\cos^3 v,\ 3\sin^2 u\cos u\cos^3 v,\ 0\right\rangle$,

$\mathbf{r}_v=\left\langle -3\cos^3 u\cos^2 v\sin v,\ -3\sin^3 u\cos^2 v\sin v,\ 3\sin^2 v\cos v\right\rangle$, and

$\mathbf{r}_u\times\mathbf{r}_v=\left\langle 9\cos u\sin^2 u\cos^4 v\sin^2 v,\ 9\cos^2 u\sin u\cos^4 v\sin^2 v,\ 9\cos^2 u\sin^2 u\cos^5 v\sin v\right\rangle$. Then

$$|\mathbf{r}_u\times\mathbf{r}_v|=9\sqrt{\cos^2 u\sin^4 u\cos^8 v\sin^4 v+\cos^4 u\sin^2 u\cos^8 v\sin^4 v+\cos^4 u\sin^4 u\cos^{10} v\sin^2 v}$$
$$=9\sqrt{\cos^2 u\sin^2 u\cos^8 v\sin^2 v\,(\sin^2 v+\cos^2 u\sin^2 u\cos^2 v)}$$
$$=9\cos^4 v|\cos u\sin u\sin v|\sqrt{\sin^2 v+\cos^2 u\sin^2 u\cos^2 v}$$

Using a CAS, we have $A(S)=\int_0^{\pi}\int_0^{2\pi}9\cos^4 v|\cos u\sin u\sin v|\sqrt{\sin^2 v+\cos^2 u\sin^2 u\cos^2 v}\,dv\,du\approx 4.4506$.

51. $z=1+2x+3y+4y^2$, so

$$A(S)=\iint_D\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA=\int_1^4\int_0^1\sqrt{1+4+(3+8y)^2}\,dy\,dx$$
$$=\int_1^4\int_0^1\sqrt{14+48y+64y^2}\,dy\,dx.$$

Using a CAS, we have

$$\int_1^4\int_0^1\sqrt{14+48y+64y^2}\,dy\,dx=\frac{45}{8}\sqrt{14}+\frac{15}{16}\ln\left(11\sqrt{5}+3\sqrt{14}\sqrt{5}\right)-\frac{15}{16}\ln\left(3\sqrt{5}+\sqrt{14}\sqrt{5}\right)\text{ or}$$

$$\frac{45}{8}\sqrt{14}+\frac{15}{16}\ln\frac{11\sqrt{5}+3\sqrt{70}}{3\sqrt{5}+\sqrt{70}}\ .$$

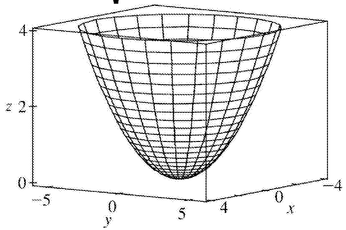
52. **(a)**

$\mathbf{r}_u = a\cos v\,\mathbf{i} + b\sin v\,\mathbf{j} + 2u\,\mathbf{k}$, $\mathbf{r}_v = -au\sin v\,\mathbf{i} + bu\cos v\,\mathbf{j} + 0\,\mathbf{k}$, and $\mathbf{r}_u \times \mathbf{r}_v = -2bu^2\cos v\,\mathbf{i} - 2au^2\sin v\,\mathbf{j} + abu\,\mathbf{k}$.

$$A(S) = \int_0^{2\pi}\int_0^2 \left|\mathbf{r}_u \times \mathbf{r}_v\right| du\,dv = \int_0^{2\pi}\int_0^2 \sqrt{4b^2u^4\cos^2 v + 4a^2u^4\sin^2 v + a^2b^2u^2}\,du\,dv$$

(b) $x^2 = a^2u^2\cos^2 v$, $y^2 = b^2u^2\sin^2 v$, $z = u^2 \Rightarrow x^2/a^2 + y^2/b^2 = u^2 = z$ which is an elliptic paraboloid. To find D , notice that $0 \le u \le 2 \Rightarrow 0 \le z \le 4 \Rightarrow 0 \le x^2/a^2 + y^2/b^2 \le 4$. Therefore, using Formula 9, we have

$$A(S) = \int_{-2a}^{2a}\int_{-b\sqrt{4-(x^2/a^2)}}^{b\sqrt{4-(x^2/a^2)}} \sqrt{1 + (2x/a^2)^2 + (2y/b^2)^2}\,dy\,dx.$$



(c)

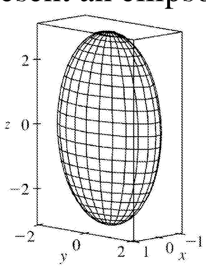
(d) We substitute $a=2$, $b=3$ in the integral in part (a) to get

$A(S) = \int_0^{2\pi}\int_0^2 2u\sqrt{9u^2\cos^2 v + 4u^2\sin^2 v + 9}\,du\,dv$. We use a CAS to estimate the integral accurate to four decimal places. To speed up the calculation, we can set Digits:=7; (in Maple) or use the approximation command N (in Mathematica). We find that $A(S) \approx 115.6596$.

53. **(a)** $x = a\sin u\cos v$, $y = b\sin u\sin v$, $z = c\cos u \Rightarrow$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = (\sin u\cos v)^2 + (\sin u\sin v)^2 + (\cos u)^2$$
$$= \sin^2 u + \cos^2 u = 1$$

and since the ranges of u and v are sufficient to generate the entire graph, the parametric equations represent an ellipsoid.



(b)

(c) From the parametric equations (with $a=1$, $b=2$, and $c=3$),
we calculate $\mathbf{r}_u = \cos u\cos v\,\mathbf{i} + 2\cos u\sin v\,\mathbf{j} - 3\sin u\,\mathbf{k}$ and

$\boldsymbol{r}_v = -\sin u \sin v\,\boldsymbol{i} + 2\sin u \cos v\,\boldsymbol{j}$. So $\boldsymbol{r}_u \times \boldsymbol{r}_v = 6\sin^2 u \cos v\,\boldsymbol{i} + 3\sin^2 u \sin v\,\boldsymbol{j} + 2\sin u \cos u\,\boldsymbol{k}$, and the surface area is given by

$$A(S) = \int_0^{2\pi}\int_0^{\pi} \left|\boldsymbol{r}_u \times \boldsymbol{r}_v\right| du\,dv$$

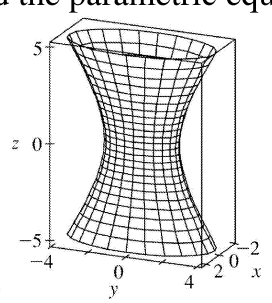
$$= \int_0^{2\pi}\int_0^{\pi} \sqrt{36\sin^4 u \cos^2 v + 9\sin^4 u \sin^2 v + 4\cos^{2u} \sin^2 u}\; du\,dv$$

54. **(a)** $x = a\cosh u \cos v$, $y = b\cosh u \sin v$, $z = c\sinh u \Rightarrow$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \cosh^2 u \cos^2 v + \cosh^2 u \sin^2 v - \sinh^2 u$$

$$= \cosh^2 u - \sinh^2 u = 1$$

and the parametric equations represent a hyperboloid of one sheet.

(b) 

(c)

$\boldsymbol{r}_u = \sinh u \cos v\,\boldsymbol{i} + 2\sinh u \sin v\,\boldsymbol{j} + 3\cosh u\,\boldsymbol{k}$ and $\boldsymbol{r}_v = -\cosh u \sin v\,\boldsymbol{i} + 2\cosh u \cos v\,\boldsymbol{j}$, so

$\boldsymbol{r}_u \times \boldsymbol{r}_v = -6\cosh^2 u \cos v\,\boldsymbol{i} - 3\cosh^2 u \sin v\,\boldsymbol{j} + 2\cosh u \sinh u\,\boldsymbol{k}$. We integrate between

$u = \sinh^{-1}(-1) = -\ln(1+\sqrt{2})$ and $u = \sinh^{-1} 1 = \ln(1+\sqrt{2})$, since then z varies between -3 and 3, as desired. So the surface area is

$$A(S) = \int_0^{2\pi}\int_{-\ln(1+\sqrt{2})}^{\ln(1+\sqrt{2})} \left|\boldsymbol{r}_u \times \boldsymbol{r}_v\right| du\,dv$$

$$= \int_0^{2\pi}\int_{-\ln(1+\sqrt{2})}^{\ln(1+\sqrt{2})} \sqrt{36\cosh^4 u \cos^2 v + 9\cosh^4 u \sin^2 v + 4\cosh^2 u \sinh^2 u}\; du\,dv$$

55. $\boldsymbol{r}(u, v) = \left\langle \cos^3 u \cos^3 v, \sin^3 u \cos^3 v, \sin^3 v \right\rangle$, so $\boldsymbol{r}_u = \left\langle -3\cos^2 u \sin u \cos^3 v, 3\sin^2 u \cos u \cos^3 v, 0 \right\rangle$,

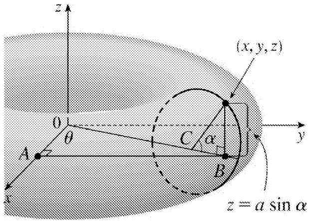
$\mathbf{r}_v = \left\langle -3\cos^3 u\cos^2 v\sin v,\ -3\sin^3 u\cos^2 v\sin v,\ 3\sin^2 v\cos v \right\rangle$, and

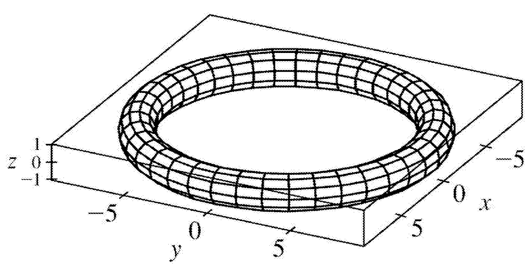
$\mathbf{r}_u \times \mathbf{r}_v = \left\langle 9\cos u\sin^2 u\cos^4 v\sin^2 v,\ 9\cos^2 u\sin u\cos^4 v\sin^2 v,\ 9\cos^2 u\sin^2 u\cos^5 v\sin v \right\rangle$. Then

$$\begin{aligned} \left|\mathbf{r}_u \times \mathbf{r}_v\right| &= 9\sqrt{\cos^2 u\sin^4 u\cos^8 v\sin^4 v+\cos^4 u\sin^2 u\cos^8 v\sin^4 v+\cos^4 u\sin^4 u\cos^{10} v\sin^2 v} \\ &= 9\sqrt{\cos^2 u\sin^2 u\cos^8 v\sin^2 v(\sin^2 v+\cos^2 u\sin^2 u\cos^2 v)} \\ &= 9\cos^4 v\,|\cos u\sin u\sin v|\sqrt{\sin^2 v+\cos^2 u\sin^2 u\cos^2 v} \end{aligned}$$

Using a CAS, we have $A(S)=\int_0^{\pi}\int_0^{2\pi} 9\cos^4 v\,|\cos u\sin u\sin v|\sqrt{\sin^2 v+\cos^2 u\sin^2 u\cos^2 v}\,dv\,du \approx 4.4506$.

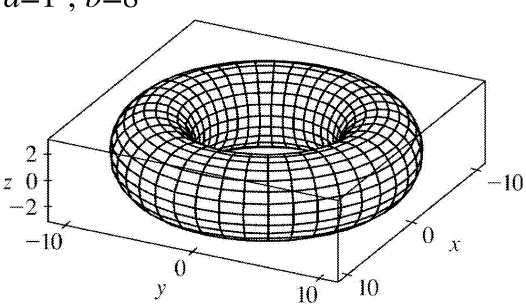
56. **(a)** Here $z=a\sin\alpha$, $y=|AB|$, and $x=|OA|$. But $|OB|=|OC|+|CB|=b+a\cos\alpha$ and $\sin\theta=\dfrac{|AB|}{|OB|}$ so that $y=|OB|\sin\theta=(b+a\cos\alpha)\sin\theta$. Similarly $\cos\theta=\dfrac{|OA|}{|OB|}$ so $x=(b+a\cos\alpha)\cos\theta$. Hence a parametric representation for the torus is $x=b\cos\theta+a\cos\alpha\cos\theta$, $y=b\sin\theta+a\cos\alpha\sin\theta$, $z=a\sin\alpha$, where $0\le\alpha\le 2\pi$, $0\le\theta\le 2\pi$.



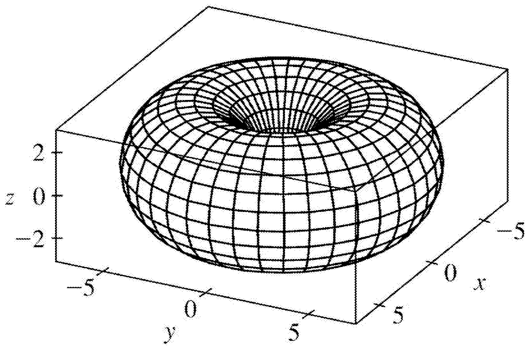


(b)

$a=1$, $b=8$



$a=3$, $b=8$



$a=3$, $b=4$

(c) $x=b\cos\theta+a\cos\alpha\cos\theta$, $y=b\sin\theta+a\cos\alpha\sin\theta$, $z=a\sin\alpha$, so
$\mathbf{r}_\alpha=\langle -a\sin\alpha\cos\theta, -a\sin\alpha\sin\theta, a\cos\alpha\rangle$, $\mathbf{r}_\theta=\langle -(b+a\cos\alpha)\sin\theta, (b+a\cos\alpha)\cos\theta, 0\rangle$ and

$$\begin{aligned}\mathbf{r}_\alpha\times\mathbf{r}_\theta &= (-ab\cos\alpha\cos\theta-a^2\cos\alpha\cos^2\theta)\mathbf{i}+(-ab\sin\alpha\cos\theta-a^2\sin\alpha\cos^2\theta)\mathbf{j}\\ &\quad+(-ab\cos^2\alpha\sin\theta-a^2\cos^2\alpha\sin\theta\cos\theta-ab\sin^2\alpha\sin\theta-a^2\sin^2\alpha\sin\theta\cos\theta)\mathbf{k}\\ &=-a(b+a\cos\alpha)[(\cos\theta\cos\alpha)\mathbf{i}+(\sin\theta\cos\alpha)\mathbf{j}+(\sin\alpha)\mathbf{k}]\end{aligned}$$

Then $|\mathbf{r}_\alpha\times\mathbf{r}_\theta|=a(b+a\cos\alpha)\sqrt{\cos^2\theta\cos^2\alpha+\sin^2\theta\cos^2\alpha+\sin^2\alpha}=a(b+a\cos\alpha)$.
Note: $b>a$, $-1\le\cos\alpha\le 1$ so $|b+a\cos\alpha|=b+a\cos\alpha$. Hence

$$A(S)=\int_0^{2\pi}\int_0^{2\pi} a(b+a\cos\alpha)\,d\alpha\,d\theta=2\pi\left[ab\alpha+a^2\sin\alpha\right]_0^{2\pi}=4\pi^2ab .$$

1. Each face of the cube has surface area $2^2=4$, and the points P_{ij}^* are the points where the cube intersects the coordinate axes. Here, $f(x, y, z)=\sqrt{x^2+2y^2+3z^2}$, so by Definition 1,

$$\iint_S f(x, y, z)dS \approx [f(1, 0, 0)](4)+[f(-1, 0, 0)](4)+[f(0, 1, 0)](4)+[f(0, -1, 0)](4)$$
$$+[f(0, 0, 1)](4)+[f(0, 0, -1)](4)$$
$$= 4(1+1+2\sqrt{2}+2\sqrt{3})=8(1+\sqrt{2}+\sqrt{3})\approx 33.170$$

2. Each quarter–cylinder has surface area $\frac{1}{4}[2\pi(1)(2)]=\pi$, and the top and bottom disks have surface area $\pi(1)^2=\pi$. We can take (0, 0, 1) as a sample point in the top disk, (0, 0, −1) in the bottom disk, and $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ in the four quarter–cylinders. Then $\iint_S f(x, y, z)dS$ can be approximated by the Riemann sum
$f(1, 0, 0)(\pi)+f(-1, 0, 0)(\pi)+f(0, 1, 0)(\pi)+f(0, -1, 0)(\pi)+f(0, 0, 1)(\pi)+f(0, 0, -1)(\pi)$
$=(2+2+3+3+4+4)\pi=18\pi\approx 56.5$.

3. We can use the xz – and yz –planes to divide H into four patches of equal size, each with surface area equal to $\frac{1}{8}$ the surface area of a sphere with radius $\sqrt{50}$, so $\Delta S=\frac{1}{8}(4)\pi(\sqrt{50})^2=25\pi$. Then $(\pm 3, \pm 4, 5)$ are sample points in the four patches, and using a Riemann sum as in Definition 1, we have

$$\iint_H f(x, y, z)dS \approx f(3, 4, 5)\Delta S+f(3, -4, 5)\Delta S+f(-3, 4, 5)\Delta S+f(-3,-4,5)\Delta S$$
$$= (7+8+9+12)(25\pi)=900\pi\approx 2827$$

4. On the surface, $f(x, y, z)=g\left(\sqrt{x^2+y^2+z^2}\right)=g(2)=-5$. So since the area of a sphere is $4\pi r^2$,

$$\iint_S f(x, y, z)dS=\iint_S g(2)dS=-5\iint_S dS=-5[4\pi(2)^2]=-80\pi.$$

5. $z=1+2x+3y$ so $\frac{\partial z}{\partial x}=2$ and $\frac{\partial z}{\partial y}=3$. Then by Formula 2,

$$\iint_S x^2yz\,dS = \iint_D x^2yz\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\,dA$$

$$= \int_0^3 \int_0^2 x^2 y(1+2x+3y)\sqrt{4+9+1}\, dy\, dx$$

$$= \sqrt{14}\int_0^3 \int_0^2 (x^2y+2x^3y+3x^2y^2)dy\, dx$$

$$= \sqrt{14}\int_0^3 \left[\frac{1}{2}x^2y^2+x^3y^2+x^2y^3 \right]_{y=0}^{y=2} dx$$

$$= \sqrt{14}\int_0^3 (10x^2+4x^3)dx=\sqrt{14}\left[\frac{10}{3}x^3+x^4 \right]_0^3=171\sqrt{14}$$

6. S is the region in the plane $2x+y+z=2$ or $z=2-2x-y$ over $D=\{(x, y)|0\le x\le 1, 0\le y\le 2-2x\}$. Thus

$$\iint_S xy dS = \iint_D xy\sqrt{(-2)^2+(-1)^2+1}\, dA$$

$$= \sqrt{6}\int_0^1 \int_0^{2-2x} xy dy dx=\sqrt{6}\int_0^1 \left[\frac{1}{2}xy^2 \right]_{y=0}^{y=2-2x} dx$$

$$= \frac{\sqrt{6}}{2}\int_0^1 (4x-8x^2+4x^3)dx= \frac{\sqrt{6}}{2}\left(2-\frac{8}{3}+1\right)= \frac{\sqrt{6}}{6}$$

7. S is the part of the plane $z=1-x-y$ over the region $D=\{(x, y)|0\le x\le 1,\ 0\le y\le 1-x\}$. Thus

$$\iint_S yz dS = \iint_D y(1-x-y)\sqrt{(-1)^2+(-1)^2+1}\, dA$$

$$= \sqrt{3}\int_0^1 \int_0^{1-x} (y-xy-y^2)dy dx=\sqrt{3}\int_0^1 \left[\frac{1}{2}y^2-\frac{1}{2}xy^2-\frac{1}{3}y^3 \right]_{y=0}^{y=1-x} dx$$

$$= \sqrt{3}\int_0^1 \frac{1}{6}(1-x)^3 dx= -\frac{\sqrt{3}}{24}(1-x)^4\Big]_0^1= \frac{\sqrt{3}}{24}$$

8. $z=\frac{2}{3}(x^{3/2}+y^{3/2})$ and

$$\iint_S y dS = \iint_D y\sqrt{(\sqrt{x})^2+(\sqrt{y})^2+1}\, dA=\int_0^1 \int_0^1 y\sqrt{x+y+1}\, dx dy$$

$$= \int_0^1 y\left[\frac{2}{3}(x+y+1)^{3/2} \right]_{x=0}^{x=1} dy=\int_0^1 \frac{2}{3}y\left[(y+2)^{3/2}-(y+1)^{3/2}\right]dy$$

Substituting $u=y+2$ in the first term and t=y+1 in the second, we have

$$\iint_S y\,dS = \frac{2}{3}\int_2^3 (u-2)u^{3/2}\,du - \frac{2}{3}\int_1^2 (t-1)t^{3/2}\,dt$$

$$= \frac{2}{3}\left[\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2}\right]_2^3 - \frac{2}{3}\left[\frac{2}{7}t^{7/2} - \frac{2}{5}t^{5/2}\right]_1^2$$

$$= \frac{2}{3}\left[\frac{2}{7}(3^{7/2}-2^{7/2}) - \frac{4}{5}(3^{5/2}-2^{5/2}) - \frac{2}{7}(2^{7/2}-1) + \frac{2}{5}(2^{5/2}-1)\right]$$

$$= \frac{2}{3}\left(\frac{18}{35}\sqrt{3} + \frac{8}{35}\sqrt{2} - \frac{4}{35}\right) = \frac{4}{105}(9\sqrt{3}+4\sqrt{2}-2)$$

9. S is the portion of the cone $z^2=x^2+y^2$ for $1\le z\le 3$, or equivalently, S is the part of the surface $z=\sqrt{x^2+y^2}$ over the region $D=\{(x, y)\,|\,1\le x^2+y^2\le 9\}$. Thus

$$\iint_S x^2z^2\,dS = \iint_D x^2(x^2+y^2)\sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + 1}\,dA$$

$$= \iint_D x^2(x^2+y^2)\sqrt{\frac{x^2+y^2}{x^2+y^2}+1}\,dA = \iint_D \sqrt{2}\,x^2(x^2+y^2)\,dA$$

$$= \sqrt{2}\int_0^{2\pi}\int_1^3 (r\cos\theta)^2(r^2)\,r\,dr\,d\theta = \sqrt{2}\int_0^{2\pi}\cos^2\theta\,d\theta\int_1^3 r^5\,dr$$

$$= \sqrt{2}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\frac{1}{6}r^6\right]_1^3 = \sqrt{2}(\pi)\cdot\frac{1}{6}(3^6-1) = \frac{364\sqrt{2}}{3}\pi$$

10. Using y and z as parameters, we have $\mathbf{r}(y, z)=(y+2z^2)\mathbf{i}+y\,\mathbf{j}+z\mathbf{k}$, $0\le y\le 1$, $0\le z\le 1$.

Then $\mathbf{r}_y\times\mathbf{r}_z=(\mathbf{i}+\mathbf{j})\times(4z\mathbf{i}+\mathbf{k})=\mathbf{i}-\mathbf{j}-4z\mathbf{k}$ and $|\mathbf{r}_y\times\mathbf{r}_z|=\sqrt{2+16z^2}$. Thus

$$\iint_S z\,dS = \int_0^1\int_0^1 z\sqrt{2+16z^2}\,dy\,dz = \int_0^1 z\sqrt{2+16z^2}\,dz = \left[\frac{1}{32}\cdot\frac{2}{3}(2+16z^2)^{3/2}\right]_0^1$$

$$= \frac{1}{48}(18^{3/2}-2^{3/2}) = \frac{13}{12}\sqrt{2}$$

11. Using x and z as parameters, we have $\mathbf{r}(x, z)=x\mathbf{i}+(x^2+z^2)\,\mathbf{j}+z\mathbf{k}$, $x^2+z^2\le 4$. Then

$\mathbf{r}_x\times\mathbf{r}_z=(\mathbf{i}+2x\,\mathbf{j})\times(2z\,\mathbf{j}+\mathbf{k})=2x\mathbf{i}-\mathbf{j}+2z\mathbf{k}$ and $|\mathbf{r}_x\times\mathbf{r}_z|=\sqrt{4x^2+1+4z^2}=\sqrt{1+4(x^2+z^2)}$. Thus

$$\iint_S y\,dS = \iint_{x^2+z^2\le 4} (x^2+z^2)\sqrt{1+4(x^2+z^2)}\,dA = \int_0^{2\pi}\int_0^2 r^2\sqrt{1+4r^2}\,r\,dr\,d\theta$$

$$=\int_0^{2\pi} d\theta \int_0^2 r^2\sqrt{1+4r^2}\,r\,dr = 2\pi\int_0^2 r^2\sqrt{1+4r^2}\,r\,dr$$

$$\left[\text{ let } u=1+4r^2 \Rightarrow r^2=\frac{1}{4}(u-1) \text{ and } \frac{1}{8}du=r\,dr\ \right]$$

$$=2\pi\int_1^{17}\frac{1}{4}(u-1)\sqrt{u}\cdot\frac{1}{8}\,du=\frac{1}{16}\pi\int_1^{17}(u^{3/2}-u^{1/2})du$$

$$=\frac{1}{16}\pi\left[\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right]_1^{17}=\frac{1}{16}\pi\left[\frac{2}{5}(17)^{5/2}-\frac{2}{3}(17)^{3/2}-\frac{2}{5}+\frac{2}{3}\right]=\frac{\pi}{60}(391\sqrt{17}+1)$$

12. Here S consists of three surfaces: S_1, the lateral surface of the cylinder; S_2, the front formed by the plane $x+y=2$; and the back, S_3, in the plane $y=0$. On S_1 : using cylindrical coordinates, $\boldsymbol{r}(\theta, y)=\sin\theta\,\boldsymbol{i}+y\,\boldsymbol{j}+\cos\theta\,\boldsymbol{k}$, $0\le\theta\le 2\pi$, $0\le y\le 2-\sin\theta$, $|\boldsymbol{r}_\theta\times\boldsymbol{r}_y|=1$ and

$$\iint_{S_1} xy\,dS=\int_0^{2\pi}\int_0^{2-\sin\theta}(\sin\theta)y\,dy\,d\theta=\int_0^{2\pi}\left[2\sin\theta-2\sin^2\theta+\frac{1}{2}\sin^3\theta\right]d\theta=-2\pi\ .$$

On S_2 : $\boldsymbol{r}(x, z)=x\boldsymbol{i}+(2-x)\boldsymbol{j}+z\boldsymbol{k}$ and $|\boldsymbol{r}_x\times\boldsymbol{r}_z|=|-\boldsymbol{i}-\boldsymbol{j}|=\sqrt{2}$, where $x^2+z^2\le 1$ and

$$\iint_{S_2} xy\,dS = \iint_{x^2+z^2\le 1} x(2-x)\sqrt{2}\,dA=\int_0^{2\pi}\int_0^1\sqrt{2}\,(2r\sin\theta-r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$=\sqrt{2}\int_0^{2\pi}\left[\frac{2}{3}\sin\theta-\frac{1}{4}\sin^2\theta\right]d\theta=-\frac{\sqrt{2}}{4}\pi$$

On S_3 : $y=0$ so $\iint_{S_3} xy\,dS=0$. Hence $\int_S xy\,dS=-2\pi-\frac{\sqrt{2}}{4}\pi=-\frac{1}{4}(8+\sqrt{2})\pi$.

13. Using spherical coordinates and Example 17.6.10 we have $\boldsymbol{r}(\phi,\theta)=2\sin\phi\cos\theta\,\boldsymbol{i}+2\sin\phi\sin\theta\,\boldsymbol{j}+2\cos\phi\,\boldsymbol{k}$ and $|\boldsymbol{r}_\phi\times\boldsymbol{r}_\theta|=4\sin\phi$. Then

$$\iint_S (x^2z+y^2z)dS=\int_0^{2\pi}\int_0^{\pi/2}(4\sin^2\phi)(2\cos\phi)(4\sin\phi)d\phi\,d\theta=16\pi\sin^4\phi\Big]_0^{\pi/2}=16\pi.$$

14. Using spherical coordinates, $\boldsymbol{r}(\phi,\theta)=\sin\phi\cos\theta\,\boldsymbol{i}+\sin\phi\sin\theta\,\boldsymbol{j}+\cos\phi\,\boldsymbol{k}$,

$0\leq\phi\leq\frac{\pi}{4}$, $0\leq\theta\leq 2\pi$, and $|\boldsymbol{r}_\phi\times\boldsymbol{r}_\theta|=\sin\phi$ (see Example 17.6.10). Then

$$\iint_S xyz\,dS=\int_0^{2\pi}\int_0^{\pi/4}(\sin^3\phi\cos\phi\cos\theta\sin\theta)d\phi\,d\theta=0 \text{ since } \int_0^{2\pi}\cos\theta\sin\theta\,d\theta=0 .$$

15. Using cylindrical coordinates, we have $\boldsymbol{r}(\theta, z)=3\cos\theta\,\boldsymbol{i}+3\sin\theta\,\boldsymbol{j}+z\boldsymbol{k}$, $0\leq\theta\leq 2\pi$, $0\leq z\leq 2$, and $|\boldsymbol{r}_\theta\times\boldsymbol{r}_z|=3$.

$$\iint_S (x^2y+z^2)dS=\int_0^{2\pi}\int_0^2(27\cos^2\theta\sin\theta+z^2)3dz\,d\theta=\int_0^{2\pi}(162\cos^2\theta\sin\theta+8)d\theta=16\pi$$

16. Let S_1 be the lateral surface, S_2 the top disk, and S_3 the bottom disk.

On S_1 : $\boldsymbol{r}(\theta, z)=3\cos\theta\,\boldsymbol{i}+3\sin\theta\,\boldsymbol{j}+z\boldsymbol{k}$, $0\leq\theta\leq 2\pi$, $0\leq z\leq 2$, $|\boldsymbol{r}_\theta\times\boldsymbol{r}_z|=3$,

$$\iint_{S_1}(x^2+y^2+z^2)dS=\int_0^{2\pi}\int_0^2(9+z^2)3dz\,d\theta=2\pi(54+8)=124\pi .$$

On S_2 : $\boldsymbol{r}(\theta, r)=r\cos\theta\,\boldsymbol{i}+r\sin\theta\,\boldsymbol{j}+2\boldsymbol{k}$, $0\leq r\leq 3$, $0\leq\theta\leq 2\pi$, $|\boldsymbol{r}_\theta\times\boldsymbol{r}_r|=r$,

$$\iint_{S_2}(x^2+y^2+z^2)dS=\int_0^{2\pi}\int_0^3(r^2+4)r\,dr\,d\theta=2\pi\left(\frac{81}{4}+18\right)=\frac{153}{2}\pi .$$

On S_3 : $\boldsymbol{r}(\theta, r)=r\cos\theta\,\boldsymbol{i}+r\sin\theta\,\boldsymbol{j}$, $0\leq r\leq 3$, $0\leq\theta\leq 2\pi$, $|\boldsymbol{r}_\theta\times\boldsymbol{r}_r|=r$,

$$\iint_{S_3}(x^2+y^2+z^2)dS=\int_0^{2\pi}\int_0^3(r^2+0)r\,dr\,d\theta=2\pi\left(\frac{81}{4}\right)=\frac{81}{2}\pi .$$

Hence $\iint_S(x^2+y^2+z^2)dS=124\pi+\frac{153}{2}\pi+\frac{81}{2}\pi=241\pi$.

17. $\boldsymbol{r}(u, v)=u^2\boldsymbol{i}+u\sin v\,\boldsymbol{j}+u\cos v\,\boldsymbol{k}$, $0\leq u\leq 1$, $0\leq v\leq\pi/2$ and

$\boldsymbol{r}_u\times\boldsymbol{r}_v=(2u\boldsymbol{i}+\sin v\,\boldsymbol{j}+\cos v\,\boldsymbol{k})\times(u\cos v\,\boldsymbol{j}-u\sin v\,\boldsymbol{k})=-u\boldsymbol{i}+2u^2\sin v\,\boldsymbol{j}+2u^2\cos v\,\boldsymbol{k}$ and

$|\boldsymbol{r}_u\times\boldsymbol{r}_v|=\sqrt{u^2+4u^4\sin^2v+4u^4\cos^2v}=\sqrt{u^2+4u^4(\sin^2v+\cos^2v)}=u\sqrt{1+4u^2}$ (since $u\geq 0$). Then

$$\iint_S yz\,dS=\int_0^{\pi/2}\int_0^1(u\sin v)(u\cos v)\cdot u\sqrt{1+4u^2}\,du\,dv=\int_0^1 u^3\sqrt{1+4u^2}\,du\int_0^{\pi/2}\sin v\cos v\,dv$$

$$\left[\text{ let } t=1+4u^2 \Rightarrow u^2=\frac{1}{4}(t-1) \text{ and } \frac{1}{8}dt=u\,du\right]$$

$$= \int_1^5 \frac{1}{8} \cdot \frac{1}{4} (t-1)\sqrt{t}\, dt \int_0^{\pi/2} \sin v \cos v dv = \frac{1}{32} \int_1^5 \left(t^{3/2} - \sqrt{t} \right) dt \int_0^{\pi/2} \sin v \cos v dv$$

$$= \frac{1}{32} \left[\frac{2}{5} t^{5/2} - \frac{2}{3} t^{3/2} \right]_1^5 \left[\frac{1}{2} \sin^2 v \right]_0^{\pi/2} = \frac{1}{32} \left(\frac{2}{5} (5)^{5/2} - \frac{2}{3} (5)^{3/2} - \frac{2}{5} + \frac{2}{3} \right) \cdot \frac{1}{2} (1-0)$$

$$= \frac{5}{48} \sqrt{5} + \frac{1}{240}$$

18. $\mathbf{r}_u = \cos v \mathbf{i} + \sin v \mathbf{j}$, $\mathbf{r}_v = -u \sin v \mathbf{i} + u \cos v \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}_u \times \mathbf{r}_v = \sin v \mathbf{i} - \cos v \mathbf{j} + u \mathbf{k} \Rightarrow |\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{1+u^2}$, so

$$\iint_S \sqrt{1+x^2+y^2}\, dS = \int_0^{\pi} \int_0^1 \sqrt{1+u^2} \sqrt{1+u^2}\, du\, dv = \frac{4}{3} \pi \ .$$

19. $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$, $z = g(x, y) = 4 - x^2 - y^2$, and D is the square $[0, 1] \times [0, 1]$, so by Equation 8

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D [-xy(-2x) - yz(-2y) + zx] dA$$

$$= \int_0^1 \int_0^1 [2x^2 y + 2y^2 (4 - x^2 - y^2) + x(4 - x^2 - y^2)] dy\, dx$$

$$= \int_0^1 \left(\frac{1}{3} x^2 + \frac{11}{3} x - x^3 + \frac{34}{15} \right) dx = \frac{713}{180}$$

20. $\mathbf{F}(x, y, z) = xy\mathbf{i} + 4x^2 \mathbf{j} + yz\mathbf{k}$, $z = g(x, y) = xe^y$, and D is the square $[0, 1] \times [0, 1]$, so by Equation 8

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D [-xy(e^y) - 4x^2 (xe^y) + yz] dA = \int_0^1 \int_0^1 (-xye^y - 4x^3 e^y + xye^y) dy\, dx$$

$$= \int_0^1 \left[-4x^3 e^y \right]_{y=0}^{y=1} dx = (e-1) \int_0^1 (-4x^3) dx = 1 - e$$

21. $\mathbf{F}(x, y, z) = xze^y \mathbf{i} - xze^y \mathbf{j} + z\mathbf{k}$, $z = g(x, y) = 1 - x - y$, and $D = \{ (x, y) | 0 \le x \le 1, 0 \le y \le 1 - x \}$. Since S has downward orientation, we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -\iint_D \left[-xze^y (-1) - (-xze^y)(-1) + z \right] dA = -\int_0^1 \int_0^{1-x} (1 - x - y) dy\, dx$$

$$= -\int_0^1 \left(\frac{1}{2} x^2 - x + \frac{1}{2} \right) dx = -\frac{1}{6}$$

22. $\boldsymbol{F}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + z^4\boldsymbol{k}$, $z = g(x, y) = \sqrt{x^2+y^2}$, and D is the disk $\{(x, y) \mid x^2+y^2 \le 1\}$. Since S has downward orientation, we have

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = -\int_D \left[-x\left(\frac{x}{\sqrt{x^2+y^2}}\right) - y\left(\frac{y}{\sqrt{x^2+y^2}}\right) + z^4 \right] dA$$

$$= -\iint_D \left[\frac{-x^2-y^2}{\sqrt{x^2+y^2}} + \left(\sqrt{x^2+y^2}\right)^4 \right] dA = -\int_0^{2\pi}\int_0^1 \left(\frac{-r^2}{r} + r^4 \right) r\,dr\,d\theta$$

$$= -\int_0^{2\pi} d\theta \int_0^1 (r^5 - r^2)\,dr = -2\pi\left(\frac{1}{6} - \frac{1}{3}\right) = \frac{\pi}{3}$$

23. $\boldsymbol{F}(x, y, z) = x\boldsymbol{i} - z\boldsymbol{j} + y\boldsymbol{k}$, $z = g(x, y) = \sqrt{4-x^2-y^2}$ and D is the quarter disk $\{(x, y) \mid 0 \le x \le 2, 0 \le y \le \sqrt{4-x^2}\}$. S has downward orientation, so by Formula 8,

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = -\iint_D \left[-x\cdot\frac{1}{2}(4-x^2-y^2)^{-1/2}(-2x) - (-z)\cdot\frac{1}{2}(4-x^2-y^2)^{-1/2}(-2y) + y \right] dA$$

$$= -\iint_D \left(\frac{x^2}{\sqrt{4-x^2-y^2}} - \sqrt{4-x^2-y^2}\cdot\frac{y}{\sqrt{4-x^2-y^2}} + y \right) dA$$

$$= -\iint_D x^2(4-(x^2+y^2))^{-1/2}\,dA = -\int_0^{\pi/2}\int_0^2 (r\cos\theta)^2(4-r^2)^{-1/2}\,r\,dr\,d\theta$$

$$= -\int_0^{\pi/2} \cos^2\theta\,d\theta \int_0^2 r^3(4-r^2)^{-1/2}\,dr$$

$$\left[\text{ let } u = 4-r^2 \Rightarrow r^2 = 4-u \text{ and } -\frac{1}{2}du = r\,dr \right]$$

$$= -\int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta \right) d\theta \int_4^0 -\frac{1}{2}(4-u)(u)^{-1/2}\,du$$

$$= -\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \right]_0^{\pi/2} \left(-\frac{1}{2}\right) \left[8\sqrt{u} - \frac{2}{3}u^{3/2} \right]_4^0$$

$$= -\frac{\pi}{4}\left(-\frac{1}{2}\right)\left(-16 + \frac{16}{3}\right) = -\frac{4}{3}\pi$$

24. $\boldsymbol{F}(x, y, z) = xz\boldsymbol{i} + x\boldsymbol{j} + y\boldsymbol{k}$

Using spherical coordinates, S is given by $x = 5\sin\phi\cos\theta$, $y = 5\sin\phi\sin\theta$, $z = 5\cos\phi$, $0 \le \theta \le \pi$, $0 \le \phi \le \pi$. $\boldsymbol{F}(\boldsymbol{r}(\phi, \theta)) = (5\sin\phi\cos\theta)(5\cos\phi)\boldsymbol{i} + (5\sin\phi\cos\theta)\boldsymbol{j} + (5\sin\phi\sin\theta)\boldsymbol{k}$ and

$\mathbf{r}_\phi \times \mathbf{r}_\theta = 25\sin^2\phi\cos\theta\,\mathbf{i} + 25\sin^2\phi\sin\theta\,\mathbf{j} + 25\cos\phi\sin\phi\,\mathbf{k}$, so

$\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta) = 625\sin^3\phi\cos\phi\cos^2\theta + 125\sin^3\phi\cos\theta\sin\theta + 125\sin^2\phi\cos\phi\sin\theta$

Then

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \left[\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)\right] dA$$

$$= \int_0^\pi \int_0^\pi (625\sin^3\phi\cos\phi\cos^2\theta + 125\sin^3\phi\cos\theta\sin\theta + 125\sin^2\phi\cos\phi\sin\theta)\,d\theta\,d\phi$$

$$= 125\int_0^\pi \left[5\sin^3\phi\cos\phi\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right) + \sin^3\phi\left(\frac{1}{2}\sin^2\theta\right) + \sin^2\phi\cos\phi\,(-\cos\theta)\right]_{\theta=0}^{\theta=\pi} d\phi$$

$$= 125\int_0^\pi \left(\frac{5}{2}\pi\sin^3\phi\cos\phi + 2\sin^2\phi\cos\phi\right) d\phi$$

$$= 125\left[\frac{5}{2}\pi\cdot\frac{1}{4}\sin^4\phi + 2\cdot\frac{1}{3}\sin^3\phi\right]_0^\pi = 0$$

25. Let S_1 be the paraboloid $y = x^2 + z^2$, $0 \le y \le 1$ and S_2 the disk $x^2 + z^2 \le 1$, $y = 1$. Since S is a closed surface, we use the outward orientation. On S_1 : $\mathbf{F}(\mathbf{r}(x, z)) = (x^2 + z^2)\,\mathbf{j} - z\mathbf{k}$ and $\mathbf{r}_x \times \mathbf{r}_z = 2x\mathbf{i} - \mathbf{j} + 2z\mathbf{k}$ (since the **j** –component must be negative on S_1). Then

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+z^2\le 1} [-(x^2+z^2) - 2z^2]\,dA = -\int_0^{2\pi}\int_0^1 (r^2 + 2r^2\cos^2\theta)\,r\,dr\,d\theta$$

$$= -\int_0^{2\pi} \frac{1}{4}(1 + 2\cos^2\theta)\,d\theta = -\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = -\pi$$

On S_2 : $\mathbf{F}(\mathbf{r}(x, z)) = \mathbf{j} - z\mathbf{k}$ and $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{j}$. Then $\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+z^2\le 1} (1)\,dA = \pi$. Hence $\iint_S \mathbf{F}\cdot d\mathbf{S} = -\pi + \pi = 0$.

26. Here S consists of three surfaces: S_1 , the lateral surface of the cylinder; S_2 , the front formed by the plane $x + y = 2$; and the back, S_3 , in the plane $y = 0$.

On S_1 : $\mathbf{F}(\mathbf{r}(\theta, y)) = \sin\theta\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$ and $\mathbf{r}_\theta \times \mathbf{r}_y = \sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{k} \Rightarrow$

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{2-\sin\theta} (\sin^2\theta + 5\cos\theta)\,dy\,d\theta$$

$$= \int_0^{2\pi} (2\sin^2\theta + 10\cos\theta - \sin^3\theta - 5\sin\theta\cos\theta)\,d\theta = 2\pi$$

On S_2 : $\boldsymbol{F}(\boldsymbol{r}(x, z)) = x\boldsymbol{i} + (2-x)\,\boldsymbol{j} + 5\,\boldsymbol{k}$ and $\boldsymbol{r}_z \times \boldsymbol{r}_x = \boldsymbol{i} + \boldsymbol{j}$.

$$\iint_{S_2} \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_{x^2+z^2 \le 1} [\,x + (2-x)\,]\,dA = 2\pi$$

On S_3 : $\boldsymbol{F}(\boldsymbol{r}(x, z)) = x\boldsymbol{i} + 5\,\boldsymbol{k}$ and $\boldsymbol{r}_x \times \boldsymbol{r}_z = -\boldsymbol{j}$ so $\iint_{S_3} \boldsymbol{F}\cdot d\boldsymbol{S} = 0$. Hence $\iint_{S} \boldsymbol{F}\cdot d\boldsymbol{S} = 4\pi$.

27. Here S consists of the six faces of the cube as labeled in the figure. On S_1 : $\boldsymbol{F} = \boldsymbol{i} + 2y\,\boldsymbol{j} + 3z\,\boldsymbol{k}$,

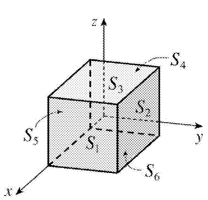
$\boldsymbol{r}_y \times \boldsymbol{r}_z = \boldsymbol{i}$ and $\iint_{S_1} \boldsymbol{F}\cdot d\boldsymbol{S} = \int_{-1}^{1}\int_{-1}^{1} dy\,dz = 4$;

S_2 : $\boldsymbol{F} = x\boldsymbol{i} + 2\,\boldsymbol{j} + 3z\,\boldsymbol{k}$, $\boldsymbol{r}_z \times \boldsymbol{r}_x = \boldsymbol{j}$ and $\iint_{S_2} \boldsymbol{F}\cdot d\boldsymbol{S} = \int_{-1}^{1}\int_{-1}^{1} 2\,dx\,dz = 8$;

S_3 : $\boldsymbol{F} = x\boldsymbol{i} + 2y\,\boldsymbol{j} + 3\,\boldsymbol{k}$, $\boldsymbol{r}_x \times \boldsymbol{r}_y = \boldsymbol{k}$ and $\iint_{S_3} \boldsymbol{F}\cdot d\boldsymbol{S} = \int_{-1}^{1}\int_{-1}^{1} 3\,dx\,dy = 12$;

S_4 : $\boldsymbol{F} = -\boldsymbol{i} + 2y\,\boldsymbol{j} + 3z\,\boldsymbol{k}$, $\boldsymbol{r}_z \times \boldsymbol{r}_y = -\boldsymbol{i}$ and $\iint_{S_4} \boldsymbol{F}\cdot d\boldsymbol{S} = 4$;

S_5 : $\boldsymbol{F} = x\boldsymbol{i} - 2\,\boldsymbol{j} + 3z\,\boldsymbol{k}$, $\boldsymbol{r}_x \times \boldsymbol{r}_z = -\boldsymbol{j}$ and $\iint_{S_5} \boldsymbol{F}\cdot d\boldsymbol{S} = 8$;



S_6 : $\boldsymbol{F} = x\boldsymbol{i} + 2y\,\boldsymbol{j} - 3\,\boldsymbol{k}$, $\boldsymbol{r}_y \times \boldsymbol{r}_x = -\boldsymbol{k}$ and $\iint_{S_6} \boldsymbol{F}\cdot d\boldsymbol{S} = \int_{-1}^{1}\int_{-1}^{1} 3\,dx\,dy = 12$. Hence $\iint_{S} \boldsymbol{F}\cdot d\boldsymbol{S} = \sum_{i=1}^{6} \iint_{S_i} \boldsymbol{F}\cdot d\boldsymbol{S} = 48$.

28. $\boldsymbol{r}_u = \cos v\,\boldsymbol{i} + \sin v\,\boldsymbol{j}$, $\boldsymbol{r}_v = -u\sin v\,\boldsymbol{i} + u\cos v\,\boldsymbol{j} + \boldsymbol{k} \Rightarrow \boldsymbol{r}_u \times \boldsymbol{r}_v = \sin v\,\boldsymbol{i} - \cos v\,\boldsymbol{j} + u\,\boldsymbol{k}$ and

$\boldsymbol{F}(\boldsymbol{r}(u, v)) = u\sin v\,\boldsymbol{i} + u\cos v\,\boldsymbol{j} + v^2\,\boldsymbol{k}$. Then

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\int_0^{\pi}\int_0^1 (u\sin^2 v-u\cos^2 v+uv^2)\,du\,dv=\int_0^{\pi}\int_0^1(-u\cos 2v+uv^2)\,du\,dv$$

$$=\int_0^{\pi}\left[-\frac{1}{2}\cos 2v+\frac{1}{2}v^2\right]dv=\frac{1}{6}\pi^3.$$

29. $z=xy\Rightarrow \partial z/\partial x=y$, $\partial z/\partial y=x$, so by Formula 2, a CAS gives

$$\iint_S xyz\,dS=\int_0^1\int_0^1 xy(xy)\sqrt{y^2+x^2+1}\,dx\,dy\approx 0.1642 .$$

30. As in Exercise 29, we use a CAS to calculate

$$\iint_S x^2yz\,dS=\int_0^1\int_0^1 x^2y(xy)\sqrt{y^2+x^2+1}\,dx\,dy$$

$$=\frac{1}{60}\sqrt{3}-\frac{1}{12}\ln(1+\sqrt{3})-\frac{1}{192}\ln(\sqrt{2}+1)+\frac{317}{2880}\sqrt{2}+\frac{1}{24}\ln 2$$

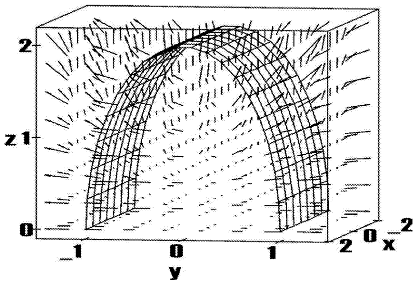
31. We use Formula 2 with $z=3-2x^2-y^2\Rightarrow \partial z/\partial x=-4x$, $\partial z/\partial y=-2y$. The boundaries of the region $3-2x^2-y^2\geq 0$ are $-\sqrt{\frac{3}{2}}\leq x\leq\sqrt{\frac{3}{2}}$ and $-\sqrt{3-2x^2}\leq y\leq\sqrt{3-2x^2}$, so we use a CAS (with precision reduced to seven or fewer digits; otherwise the calculation takes a very long time) to calculate

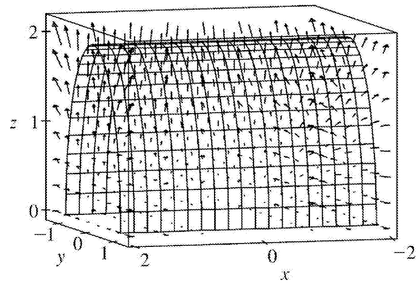
$$\iint_S x^2y^2z^2\,dS=\int_{-\sqrt{3/2}}^{\sqrt{3/2}}\int_{-\sqrt{3-2x^2}}^{\sqrt{3-2x^2}} x^2y^2(3-2x^2-y^2)^2\sqrt{16x^2+4y^2+1}\,dy\,dx\approx 3.4895$$

32. The flux of $\boldsymbol{F}$ across S is given by $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_S \boldsymbol{F}\cdot \boldsymbol{n}\,dS$. Now on S , $z=g(x, y)=2\sqrt{1-y^2}$, so $\partial g/\partial x=0$ and $\partial g/\partial y=-2y(1-y^2)^{-1/2}$. Therefore, by (8),

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\int_{-2}^{2}\int_{-1}^{1}\left(-x^2y\left[-2y(1-y^2)^{-1/2}\right]+\left[2\sqrt{1-y^2}\right]^2e^{x/5}\right)dy\,dx$$

$$=\frac{1}{3}(16\pi+80e^{2/5}-80e^{-2/5})$$





33. If S is given by $y=h(x, z)$, then S is also the level surface $f(x, y, z)=y-h(x, z)=0$.

$\boldsymbol{n}=\dfrac{\nabla f(x, y, z)}{|\nabla f(x, y, z)|}=\dfrac{-h_x\boldsymbol{i}+\boldsymbol{j}-h_z\boldsymbol{k}}{\sqrt{h_x^2+1+h_z^2}}$, and $-\boldsymbol{n}$ is the unit normal that points to the left. Now we proceed as in the derivation of (8), using Formula 2 to evaluate

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_S \boldsymbol{F}\cdot \boldsymbol{n}dS$$

$$=\iint_D(P\boldsymbol{i}+Q\boldsymbol{j}+R\boldsymbol{k})\frac{\frac{\partial h}{\partial x}\boldsymbol{i}-\boldsymbol{j}+\frac{\partial h}{\partial z}\boldsymbol{k}}{\sqrt{\left(\frac{\partial h}{\partial x}\right)^2+1+\left(\frac{\partial h}{\partial z}\right)^2}}\sqrt{\left(\frac{\partial h}{\partial x}\right)^2+1+\left(\frac{\partial h}{\partial z}\right)^2}\,dA$$

where D is the projection of $f(x, y, z)$ onto the xz –plane. Therefore

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_D\left(P\frac{\partial h}{\partial x}-Q+R\frac{\partial h}{\partial z}\right)dA$$

34. If S is given by $x=k(y, z)$, then S is also the level surface $f(x, y, z)=x-k(y, z)=0$.

$\boldsymbol{n}=\dfrac{\nabla f(x, y, z)}{|\nabla f(x, y, z)|}=\dfrac{\boldsymbol{i}-k_y\boldsymbol{j}-k_z\boldsymbol{k}}{\sqrt{1+k_y^2+k_z^2}}$, and since the x –component is positive this is the unit normal that points forward. Now we proceed as in the derivation of (8), using Formula 2 for

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_S \boldsymbol{F}\cdot \boldsymbol{n}dS$$

$$=\iint_D(P\boldsymbol{i}+Q\boldsymbol{j}+R\boldsymbol{k})\frac{\boldsymbol{i}-\frac{\partial k}{\partial y}\boldsymbol{j}-\frac{\partial k}{\partial z}\boldsymbol{k}}{\sqrt{1+\left(\frac{\partial k}{\partial y}\right)^2+\left(\frac{\partial k}{\partial z}\right)^2}}\sqrt{1+\left(\frac{\partial k}{\partial y}\right)^2+\left(\frac{\partial k}{\partial z}\right)^2}\,dA$$

where D is the projection of $f(x, y, z)$ onto the yz –plane. Therefore

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_D \left(P-Q\frac{\partial k}{\partial y}-R\frac{\partial k}{\partial z}\right)dA$$

35. $m=\iint_S KdS=K\cdot 4\pi\left(\frac{1}{2}a^2\right)=2\pi a^2 K$; by symmetry $M_{xz}=M_{yz}=0$, and

$M_{xy}=\iint_S zKdS=K\int_0^{2\pi}\int_0^{\pi/2}(a\cos\phi)(a^2\sin\phi)d\phi\, d\theta=2\pi Ka^3\left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/2}=\pi Ka^3$. Hence

$(\bar{x}, \bar{y}, \bar{z})=\left(0, 0, \frac{1}{2}a\right)$.

36. S is given by $\boldsymbol{r}(x, y)=x\boldsymbol{i}+y\boldsymbol{j}+\sqrt{x^2+y^2}\,\boldsymbol{k}$, $|\boldsymbol{r}_x\times\boldsymbol{r}_y|=\sqrt{1+\frac{x^2+y^2}{x^2+y^2}}=\sqrt{2}$ so

$$m=\iint_S\left(10-\sqrt{x^2+y^2}\right)dS=\iint_{1\le x^2+y^2\le 16}\left(10-\sqrt{x^2+y^2}\right)\sqrt{2}\,dA$$

$$=\int_0^{2\pi}\int_1^4\sqrt{2}\,(10-r)r\,dr\,d\theta=2\pi\sqrt{2}\left[5r^2-\frac{1}{3}r^3\right]_1^4=108\sqrt{2}\,\pi$$

37. **(a)** $I_z=\iint_S(x^2+y^2)\rho(x, y, z)dS$

(b)

$$I_z=\iint_S(x^2+y^2)\left(10-\sqrt{x^2+y^2}\right)dS=\iint_{1\le x^2+y^2\le 16}(x^2+y^2)\left(10-\sqrt{x^2+y^2}\right)\sqrt{2}\,dA$$

$$=\int_0^{2\pi}\int_1^4\sqrt{2}\,(10r^3-r^4)dr\,d\theta=2\sqrt{2}\,\pi\left(\frac{4329}{10}\right)=\frac{4329}{5}\sqrt{2}\,\pi$$

38. S is given by $\boldsymbol{r}(x, y)=x\boldsymbol{i}+y\boldsymbol{j}+\sqrt{x^2+y^2}\,\boldsymbol{k}$ and $|\boldsymbol{r}_x\times\boldsymbol{r}_y|=\sqrt{2}$.

(a) $m=\iint_S kdS=k\int_{0\le x^2+y^2\le a^2}\sqrt{2}\,dS=\sqrt{2}\,a^2k\pi$; by symmetry $M_{xz}=M_{yz}=0$, and

$M_{xy}=\iint_S zk\,dS=k\int_0^{2\pi}\int_0^a \sqrt{2}\,r^2\,dr\,d\theta=\frac{2}{3}\sqrt{2}\,a^3k\pi$. Hence $(\bar{x}, \bar{y}, \bar{z})=(0, 0, \frac{2}{3}a)$.

(b) $I_z=\iint_S (x^2+y^2)k\,dS=\int_0^{2\pi}\int_0^a \sqrt{2}\,kr^3\,dr\,d\theta=2\pi\sqrt{2}\,k\left(\frac{1}{4}a^4\right)=\frac{\sqrt{2}}{2}\pi ka^4$.

39. $\rho(x, y, z)=1200$, $\boldsymbol{V}=y\boldsymbol{i}+\boldsymbol{j}+z\boldsymbol{k}$, $\boldsymbol{F}=\rho\boldsymbol{V}=(1200)(y\boldsymbol{i}+\boldsymbol{j}+z\boldsymbol{k})$.$S$ is given by $\boldsymbol{r}(x, y)=x\boldsymbol{i}+y\boldsymbol{j}+\left[9-\frac{1}{4}(x^2+y^2)\right]\boldsymbol{k}$, $0\le x^2+y^2\le 36$ and $\boldsymbol{r}_x\times\boldsymbol{r}_y=\frac{1}{2}x\boldsymbol{i}+\frac{1}{2}y\boldsymbol{j}+\boldsymbol{k}$.
Thus the rate of flow is given by

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{0\le x^2+y^2\le 36}(1200)\left(\frac{1}{2}xy+\frac{1}{2}y+\left[9-\frac{1}{4}(x^2+y^2)\right]\right)dA$$

$$=1200\int_0^6\int_0^{2\pi}\left[\frac{1}{2}r^2\sin\theta\cos\theta+\frac{1}{2}r\sin\theta+9-\frac{1}{4}r^2\right]r\,d\theta\,dr$$

$$=1200\int_0^6 2\pi\left(9r-\frac{1}{4}r^3\right)dr=(1200)(2\pi)(81)=194{,}400\pi$$

40. $\rho(x, y, z)=1500$, $\boldsymbol{F}=\rho\boldsymbol{V}=(1500)(-y\boldsymbol{i}+x\boldsymbol{j}+2z\boldsymbol{k})$. S is given by $\boldsymbol{r}(\phi,\theta)=5\sin\phi\cos\theta\,\boldsymbol{i}+5\sin\phi\sin\theta\,\boldsymbol{j}+5\cos\phi\,\boldsymbol{k}$, $0\le\phi\le\pi$, $0\le\theta\le 2\pi$, and $\boldsymbol{r}_\phi\times\boldsymbol{r}_\theta=25\sin^2\phi\cos\theta\,\boldsymbol{i}+25\sin^2\phi\sin\theta\,\boldsymbol{j}+25\sin\phi\cos\phi\,\boldsymbol{k}$. Thus the rate of outward flow is

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=1500\int_0^{2\pi}\int_0^{\pi}(-125\sin^3\phi\sin\theta\cos\theta+125\sin^3\phi\sin\theta\cos\theta+250\sin\phi\cos^2\phi)\,d\phi\,d\theta$$

$$=(3000\pi)(250)\left(-\frac{1}{3}\cos^3\phi\right)\Big]_0^{\pi}=500{,}000\pi.$$

41. S consists of the hemisphere S_1 given by $z=\sqrt{a^2-x^2-y^2}$ and the disk S_2 given by $0\le x^2+y^2\le a^2$, $z=0$. On S_1 : $\boldsymbol{E}=a\sin\phi\cos\theta\,\boldsymbol{i}+a\sin\phi\sin\theta\,\boldsymbol{j}+2a\cos\phi\,\boldsymbol{k}$, $\boldsymbol{T}_\phi\times\boldsymbol{T}_\theta=a^2\sin^2\phi\cos\theta\,\boldsymbol{i}+a^2\sin^2\phi\sin\theta\,\boldsymbol{j}+a^2\sin\phi\cos\phi\,\boldsymbol{k}$. Thus

$$\iint_{S_1}\boldsymbol{E}\cdot d\boldsymbol{S}=\int_0^{2\pi}\int_0^{\pi/2}(a^3\sin^3\phi+2a^3\sin\phi\cos^2\phi)\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi/2}(a^3\sin\phi+a^3\sin\phi\cos^2\phi)\,d\phi\,d\theta=(2\pi)a^3\left(1+\frac{1}{3}\right)=\frac{8}{3}\pi a^3$$

On S_2: $\boldsymbol{E}=x\boldsymbol{i}+y\boldsymbol{j}$, and $\boldsymbol{r}_y\times\boldsymbol{r}_x=-\boldsymbol{k}$ so $\iint_{S_2}\boldsymbol{E}\cdot d\boldsymbol{S}=0$.

Hence the total charge is $q=\varepsilon_0\iint_S\boldsymbol{E}\cdot d\boldsymbol{S}=\frac{8}{3}\pi a^3\varepsilon_0$.

42. Referring to the figure in Exercise 27, on

S_1 : $\boldsymbol{E}=\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$, $\boldsymbol{r}_y\times\boldsymbol{r}_z=\boldsymbol{i}$ and $\iint_{S_1}\boldsymbol{E}\cdot d\boldsymbol{S}=\int_{-1}^{1}\int_{-1}^{1}dy\,dz=4$;

S_2 : $\boldsymbol{E}=x\boldsymbol{i}+\boldsymbol{j}+z\boldsymbol{k}$, $\boldsymbol{r}_z\times\boldsymbol{r}_x=\boldsymbol{j}$ and $\iint_{S_2}\boldsymbol{E}\cdot d\boldsymbol{S}=\int_{-1}^{1}\int_{-1}^{1}dx\,dz=4$;

S_3 : $\boldsymbol{E}=x\boldsymbol{i}+y\boldsymbol{j}+\boldsymbol{k}$, $\boldsymbol{r}_x\times\boldsymbol{r}_y=\boldsymbol{k}$ and $\iint_{S_3}\boldsymbol{E}\cdot d\boldsymbol{S}=\int_{-1}^{1}\int_{-1}^{1}dx\,dy=4$;

S_4 : $\boldsymbol{E}=-\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$, $\boldsymbol{r}_z\times\boldsymbol{r}_y=-\boldsymbol{i}$ and $\iint_{S_4}\boldsymbol{E}\cdot d\boldsymbol{S}=4$.

Similarly $\iint_{S_5}\boldsymbol{E}\cdot d\boldsymbol{S}=\iint_{S_6}\boldsymbol{E}\cdot d\boldsymbol{S}=4$. Hence $q=\varepsilon_0\iint_S\boldsymbol{E}\cdot d\boldsymbol{S}=\varepsilon_0\sum_{i=1}^{6}\iint_{S_i}\boldsymbol{E}\cdot d\boldsymbol{S}=24\varepsilon_0$.

43. $K\nabla u=6.5(4y\boldsymbol{j}+4z\boldsymbol{k})$. S is given by $\boldsymbol{r}(x,\theta)=x\boldsymbol{i}+\sqrt{6}\cos\theta\,\boldsymbol{j}+\sqrt{6}\sin\theta\,\boldsymbol{k}$ and since we want the inward heat flow, we use $\boldsymbol{r}_x\times\boldsymbol{r}_\theta=-\sqrt{6}\cos\theta\,\boldsymbol{j}-\sqrt{6}\sin\theta\,\boldsymbol{k}$. Then the rate of heat flow inward is given by

$$\iint_S(-K\nabla u)\cdot d\boldsymbol{S}=\int_0^{2\pi}\int_0^4-(6.5)(-24)dx\,d\theta=(2\pi)(156)(4)=1248\pi.$$

44. $u(x,y,z)=\dfrac{c}{\sqrt{x^2+y^2+z^2}}$,

$$\boldsymbol{F}=-K\nabla u=-K\left[-\frac{cx}{(x^2+y^2+z^2)^{3/2}}\boldsymbol{i}-\frac{cy}{(x^2+y^2+z^2)^{3/2}}\boldsymbol{j}-\frac{cz}{(x^2+y^2+z^2)^{3/2}}\boldsymbol{k}\right]$$

$$=\frac{cK}{(x^2+y^2+z^2)^{3/2}}(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})$$

and the outward unit normal is $\boldsymbol{n}=\frac{1}{a}(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})$.

Thus

$\boldsymbol{F}\cdot\boldsymbol{n}=\dfrac{cK}{a(x^2+y^2+z^2)^{3/2}}\,(x^2+y^2+z^2)$, but on S , $x^2+y^2+z^2=a^2$ so $\boldsymbol{F}\cdot\boldsymbol{n}=\dfrac{cK}{a^2}$. Hence the rate of heat flow across S is $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\dfrac{cK}{a^2}\iint_S dS=\dfrac{cK}{a^2}\,(4\pi a^2)=4\pi Kc$.

1. Both H and P are oriented piecewise–smooth surfaces that are bounded by the simple, closed, smooth curve $x^2+y^2=4$, $z=0$ (which we can take to be oriented positively for both surfaces). Then H and P satisfy the hypotheses of Stokes' Theorem, so by (3) we know

$\iint_H \operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\iint_P \operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}$ (where C is the boundary curve).

2. The plane $z=5$ intersects the paraboloid $z=9-x^2-y^2$ in the circle $x^2+y^2=4$, $z=5$. This boundary curve C is oriented in the counterclockwise direction, so the vector equation is $\boldsymbol{r}(t)=2\cos t\,\boldsymbol{i}+2\sin t\,\boldsymbol{j}+5\,\boldsymbol{k}$, $0\le t\le 2\pi$. Then $\boldsymbol{r}'(t)=-2\sin t\,\boldsymbol{i}+2\cos t\,\boldsymbol{j}$, $\boldsymbol{F}(\boldsymbol{r}(t))=10\sin t\,\boldsymbol{i}+10\cos t\,\boldsymbol{j}+4\cos t\sin t\,\boldsymbol{k}$, and by Stokes' Theorem,

$$\iint_S \operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^{2\pi}\boldsymbol{F}(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)dt=\int_0^{2\pi}(-20\sin^2 t+20\cos^2 t)dt$$
$$=20\int_0^{2\pi}\cos 2t\,dt=0$$

3. The boundary curve C is the circle $x^2+y^2=4$, $z=0$ oriented in the counterclockwise direction. The vector equation is $\boldsymbol{r}(t)=2\cos t\,\boldsymbol{i}+2\sin t\,\boldsymbol{j}$, $0\le t\le 2\pi$, so $\boldsymbol{r}'(t)=-2\sin t\,\boldsymbol{i}+2\cos t\,\boldsymbol{j}$ and $\boldsymbol{F}(\boldsymbol{r}(t))=(2\cos t)^2e^{(2\sin t)(0)}\boldsymbol{i}+(2\sin t)^2e^{(2\cos t)(0)}\boldsymbol{j}+(0)^2e^{(2\cos t)(2\sin t)}\boldsymbol{k}=4\cos^2 t\,\boldsymbol{i}+4\sin^2 t\,\boldsymbol{j}$. Then, by Stokes' Theorem,

$$\iint_S \operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^{2\pi}\boldsymbol{F}(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)\,dt=\int_0^{2\pi}(-8\cos^2 t\sin t+8\sin^2 t\cos t)dt$$
$$=8\left[\frac{1}{3}\cos^3 t+\frac{1}{3}\sin^3 t\right]_0^{2\pi}=0$$

4. The boundary curve C is the circle $x^2+z^2=9$, $y=3$ with vector equation $\boldsymbol{r}(t)=3\sin t\,\boldsymbol{i}+3\,\boldsymbol{j}+3\cos t\,\boldsymbol{k}$, $0\le t\le 2\pi$ which gives the positive orientation. Then

$\boldsymbol{F}(\boldsymbol{r}(t))=729\sin^2 t\cos t\,\boldsymbol{i}+\sin(27\sin t\cos t)\,\boldsymbol{j}+27\sin t\cos t\,\boldsymbol{k}$ and

$\boldsymbol{F}(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)=2187\sin^2 t\cos^2 t-81\sin^2 t\cos t$. Thus

$$\iint_S \operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}=\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^{2\pi}\boldsymbol{F}(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)dt$$
$$=\int_0^{2\pi}(2187\sin^2 t\cos^2 t-81\sin^2 t\cos t)dt=\int_0^{2\pi}\left(2187\left(\frac{1}{2}\sin 2t\right)^2-81\sin^2 t\cos t\right)dt$$

$$=\left[\frac{2187}{4}\left(\frac{1}{2}t-\frac{1}{8}\sin 4t\right)-81\cdot\frac{1}{3}\sin^3 t\right]_0^{2\pi}=\frac{2187}{4}(\pi)-0=\frac{2187}{4}\pi$$

5. C is the square in the plane $z=-1$. By (3), $\iint_{S_1}\text{curl}\,\boldsymbol{F}\cdot d\boldsymbol{S}=\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}=\iint_{S_2}\text{curl}\,\boldsymbol{F}\cdot d\boldsymbol{S}$ where S_1 is the original cube without the bottom and S_2 is the bottom face of the cube. $\text{curl}\,\boldsymbol{F}=x^2z\boldsymbol{i}+(xy-2xyz)\boldsymbol{j}+(y-xz)\boldsymbol{k}$. For S_2, we choose $\boldsymbol{n}=\boldsymbol{k}$ so that C has the same orientation for both surfaces. Then $\text{curl}\,\boldsymbol{F}\cdot\boldsymbol{n}=y-xz=x+y$ on S_2 , where $z=-1$. Thus $\iint_{S_2}\text{curl}\,\boldsymbol{F}\cdot d\boldsymbol{S}=\int_{-1}^{1}\int_{-1}^{1}(x+y)\,dx\,dy=0$ so

$\iint_{S_1}\text{curl}\,\boldsymbol{F}\cdot d\boldsymbol{S}=0$.

6. The boundary curve C is the unit circle in the yz –plane. By Equation 3,

$\iint_{S_1}\text{curl}\,\boldsymbol{F}\cdot d\boldsymbol{S}=\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}=\iint_{S_2}\text{curl}\,\boldsymbol{F}\cdot d\boldsymbol{S}$ where S_1 is the original hemisphere and S_2 is the disk $y^2+z^2\le 1$, $x=0$. $\text{curl}\,\boldsymbol{F}=(x-x^2)\boldsymbol{i}-(y+e^{xy}\sin z)\boldsymbol{j}+(2xz-xe^{xy}\cos z)\boldsymbol{k}$, and for S_2 we choose $\boldsymbol{n}=\boldsymbol{i}$ so that C has the same orientation for both surfaces. Then curl $\boldsymbol{F}\cdot\boldsymbol{n}=x-x^2$ on S_2, where $x=0$. Thus

$$\iint_{S_2}\text{curl}\,\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{y^2+z^2\le 1}(x-x^2)\,dA=\iint_{y^2+z^2\le 1}0\,dA=0.$$

Alternatively, we can evaluate $\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}$.$C$ with positive orientation is given by $\boldsymbol{r}(t)=\langle 0,\cos t,\sin t\rangle$, $0\le t\le 2\pi$, and

$$\begin{aligned}\iint_S\text{curl}\,\boldsymbol{F}\cdot d\boldsymbol{S}&=\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}\\&=\int_0^{2\pi}\left\langle e^{0(\cos t)}\cos(\sin t),(0)^2(\sin t),(0)(\cos t)\right\rangle\cdot\langle 0,-\sin t,\cos t\rangle\,dt\\&=\int_0^{2\pi}0\,dt=0\end{aligned}$$

7. curl $\boldsymbol{F} = -2z\boldsymbol{i} - 2x\boldsymbol{j} - 2y\boldsymbol{k}$ and we take the surface S to be the planar region enclosed by C, so S is the portion of the plane $x+y+z=1$ over $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1-x\}$. Since C is oriented counterclockwise, we orient S upward. Using Equation 17.7.8, we have $z = g(x, y) = 1-x-y$, $P = -2z$, $Q = -2x$, $R = -2y$, and

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S \text{curl}\,\boldsymbol{F} \cdot d\boldsymbol{S} = \iint_D [-(-2z)(-1) - (-2x)(-1) + (-2y)]\, dA$$
$$= \int_0^1 \int_0^{1-x} (-2)\, dy\, dx = -2\int_0^1 (1-x)\, dx = -1$$

8. curl $\boldsymbol{F} = e^x \boldsymbol{k}$ and S is the portion of the plane $2x+y+2z=2$ over $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 2-2x\}$. We orient S upward and use Equation 17.7.8 with $z = g(x, y) = 1 - x - \frac{1}{2}y$:

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S \text{curl}\,\boldsymbol{F} \cdot d\boldsymbol{S} = \iint_D (0+0+e^x)\, dA = \int_0^1 \int_0^{2-2x} e^x\, dy\, dx$$
$$= \int_0^1 (2-2x)e^x\, dx = \Big[(2-2x)e^x + 2e^x\Big]_0^1 \quad \text{[by integrating by parts]}$$
$$= 2e - 4$$

9. curl $\boldsymbol{F} = (xe^{xy} - 2x)\boldsymbol{i} - (ye^{xy} - y)\boldsymbol{j} + (2z - z)\boldsymbol{k}$ and we take S to be the disk $x^2 + y^2 \le 16$, $z = 5$. Since C is oriented counterclockwise (from above), we orient S upward. Then $\boldsymbol{n} = \boldsymbol{k}$ and curl $\boldsymbol{F} \cdot \boldsymbol{n} = 2z - z$ on S, where $z = 5$. Thus

$$\oint \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S \text{curl}\,\boldsymbol{F} \cdot \boldsymbol{n}\, dS = \iint_S (2z - z)\, dS = \iint_S (10-5)\, dS = 5\text{ (area of } S) = 5(\pi \cdot 4^2) = 80\pi$$

10. S is the part of the surface $z = 1 - x^2 - y^2$ in the first octant. curl $\boldsymbol{F} = 2y\boldsymbol{i} - 2x\boldsymbol{j}$.
Using Equation 17.7.8 with $g(x, y) = 1 - x^2 - y^2$, $P = 2y$, $Q = -2x$, we have

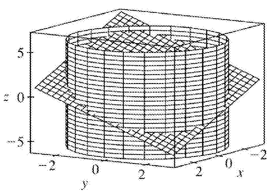
$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S \text{curl}\,\boldsymbol{F} \cdot d\boldsymbol{S} = \iint_D [-2y(-2x) + (2x)(-2y)]\, dA = \iint_D 0\, dA = 0.$$

11. **(a)** The curve of intersection is an ellipse in the plane $x+y+z=1$ with unit normal $\boldsymbol{n} = \frac{1}{\sqrt{3}}(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$,

curl $\boldsymbol{F} = x^2 \boldsymbol{j} + y^2 \boldsymbol{k}$ and curl $\boldsymbol{F} \cdot \boldsymbol{n} = \frac{1}{\sqrt{3}}(x^2 + y^2)$. Then

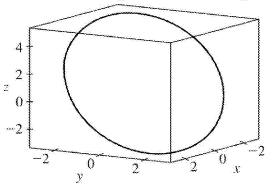
$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_S \frac{1}{\sqrt{3}}\,(x^2+y^2)dS = \iint_{x^2+y^2\le 9} (x^2+y^2)dx\,dy$$

$$= \int_0^{2\pi}\int_0^3 r^3 dr\,d\theta = 2\pi\left(\frac{81}{4}\right) = \frac{81\pi}{2}$$



(b)

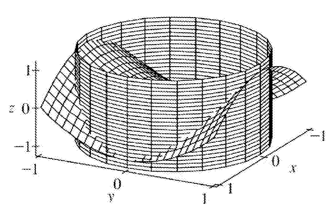
(c) One possible parametrization is $x=3\cos t$, $y=3\sin t$, $z=1-3\cos t-3\sin t$, $0\le t\le 2\pi$.



12. **(a)** S is the part of the surface $z=y^2-x^2$ that lies above the unit disk D .
$\text{curl}\boldsymbol{F}=x\boldsymbol{i}-y\boldsymbol{j}+(x^2-x^2)\boldsymbol{k}=x\boldsymbol{i}-y\boldsymbol{j}$. Using Equation 17.7.8 with $g(x, y)=y^2-x^2$, $P=x$, $Q=-y$, we have

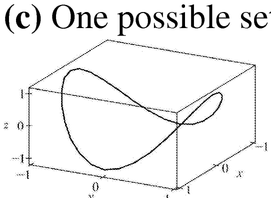
$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_S \text{curl}\boldsymbol{F}\cdot d\boldsymbol{S} = \iint_D [-x(-2x)-(-y)(2y)]dA = 2\iint_D (x^2+y^2)dA$$

$$= 2\int_0^{2\pi}\int_0^1 r^2\, r\,dr\,d\theta = 2(2\pi)\left[\frac{1}{4}r^4\right]_0^1 = \pi$$



(b)

(c) One possible set of parametric equations is $x=\cos t$, $y=\sin t$, $z=\sin^2 t-\cos^2 t$, $0\le t\le 2\pi$.



13. The boundary curve C is the circle $x^2+y^2=1$, $z=1$ oriented in the counterclockwise direction as viewed from above. We can parametrize C by $\boldsymbol{r}(t)=\cos t\,\boldsymbol{i}+\sin t\,\boldsymbol{j}+\boldsymbol{k}$, $0\le t\le 2\pi$, and then

$\mathbf{r}'(t)=-\sin t\,\mathbf{i}+\cos t\,\mathbf{j}$. Thus $\mathbf{F}(\mathbf{r}(t))=\sin^2 t\,\mathbf{i}+\cos t\,\mathbf{j}+\mathbf{k}$, $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)=\cos^2 t-\sin^3 t$, and

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}(\cos^2 t-\sin^3 t)dt=\int_0^{2\pi}\frac{1}{2}(1+\cos 2t)dt-\int_0^{2\pi}(1-\cos^2 t)\sin t\,dt$$

$$=\frac{1}{2}\left[t+\frac{1}{2}\sin 2t\right]_0^{2\pi}-\left[-\cos t+\frac{1}{3}\cos^3 t\right]_0^{2\pi}=\pi$$

Now curl $\mathbf{F}=(1-2y)\mathbf{k}$, and the projection D of S on the xy –plane is the disk $x^2+y^2\le 1$, so by Equation 17.7.8 [ET 16.7.8] with $z=g(x, y)=x^2+y^2$ we have

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\iint_D(1-2y)dA=\int_0^{2\pi}\int_0^1(1-2r\sin\theta)r\,dr\,d\theta=\int_0^{2\pi}\left(\frac{1}{2}-\frac{2}{3}\sin\theta\right)d\theta=\pi$$

14. The plane intersects the coordinate axes at $x=1$, $y=z=2$ so the boundary curve C consists of the three line segments C_1: $\mathbf{r}_1(t)=(1-t)\mathbf{i}+2t\,\mathbf{j}$, $0\le t\le 1$, C_2: $\mathbf{r}_2(t)=(2-2t)\mathbf{j}+2t\,\mathbf{k}$, $0\le t\le 1$, C_3: $\mathbf{r}_3(t)=t\,\mathbf{i}+(2-2t)\mathbf{k}$, $0\le t\le 1$. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\int_0^1[(1-t)\mathbf{i}+2t\,\mathbf{j}]\cdot(-\mathbf{i}+2\mathbf{j})dt+\int_0^1[(2-2t)\mathbf{j}]\cdot(-2\mathbf{j}+2\mathbf{k})dt+\int_0^1(t\,\mathbf{i})\cdot(\mathbf{i}-2\mathbf{k})dt$$

$$=\int_0^1(5t-1)dt+\int_0^1(4t-4)dt+\int_0^1 t\,dt=\frac{3}{2}-2+\frac{1}{2}=0$$

Now curl $\mathbf{F}=xz\mathbf{i}-yz\mathbf{j}$, so by Equation 17.7.8 [ET 16.7.8] with $z=g(x, y)=2-2x-y$ we have

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\iint_D[-x(2-2x-y)(-2)+y(2-2x-y)(-1)]dA$$

$$=\int_0^1\int_0^{2-2x}(4x-4x^2-2y+y^2)dy\,dx$$

$$=\int_0^1\left[4x(2-2x)-4x^2(2-2x)-(2-2x)^2+\frac{1}{3}(2-2x)^3\right]dx$$

$$=\int_0^1\left(\frac{16}{3}x^3-12x^2+8x-\frac{4}{3}\right)dx=\left[\frac{4}{3}x^4-4x^3+4x^2-\frac{4}{3}x\right]_0^1=0$$

15. The boundary curve C is the circle $x^2+z^2=1$, $y=0$ oriented in the counterclockwise direction as viewed from the positive y –axis. Then C can be described by $\mathbf{r}(t)=\cos t\,\mathbf{i}-\sin t\,\mathbf{k}$, $0\le t\le 2\pi$, and $\mathbf{r}'(t)=-\sin t\,\mathbf{i}-\cos t\,\mathbf{k}$. Thus $\mathbf{F}(\mathbf{r}(t))=-\sin t\,\mathbf{j}+\cos t\,\mathbf{k}$, $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)=-\cos^2 t$, and

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^{2\pi} -\cos^2 t\,dt= \left.-\frac{1}{2}t-\frac{1}{4}\sin 2t\right]_0^{2\pi}=-\pi$$

Now curl $\boldsymbol{F}=-\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}$, and S can be parametrized (see Example 17.6.10 [ET 17.6.10]) by $\boldsymbol{r}(\phi,\theta)=\sin\phi\cos\theta\,\boldsymbol{i}+\sin\phi\sin\theta\,\boldsymbol{j}+\cos\phi\,\boldsymbol{k}$, $0\le\theta\le\pi$, $0\le\phi\le\pi$. Then $\boldsymbol{r}_\phi\times\boldsymbol{r}_\theta=\sin^2\phi\cos\theta\,\boldsymbol{i}+\sin^2\phi\sin\theta\,\boldsymbol{j}+\sin\phi\cos\phi\,\boldsymbol{k}$ and

$$\begin{aligned}\iint_S \text{curl}\boldsymbol{F}\cdot d\boldsymbol{S}&=\iint_{x^2+z^2\le 1}\text{curl}\boldsymbol{F}\cdot(\boldsymbol{r}_\phi\times\boldsymbol{r}_\theta)dA\\ &=\int_0^\pi\int_0^\pi(-\sin^2\phi\cos\theta-\sin^2\phi\sin\theta-\sin\phi\cos\phi)d\theta\,d\phi\\ &=\int_0^\pi(-2\sin^2\phi-\pi\sin\phi\cos\phi)d\phi=\left[\frac{1}{2}\sin 2\phi-\phi-\frac{\pi}{2}\sin^2\phi\right]_0^\pi=-\pi\end{aligned}$$

16. The components of $\boldsymbol{F}$ are polynomials, which have continuous partial derivatives throughout R^3 , and both the curve C and the surface S meet the requirements of Stokes' Theorem. If there is a vector field $\boldsymbol{G}$ where $\boldsymbol{F}=\text{curl}\boldsymbol{G}$, then Stokes' Theorem says $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_S$ curl, $\boldsymbol{G}\cdot d\boldsymbol{S}$ depends only on the values of $\boldsymbol{G}$ on C , and hence is independent of the choice of S . By Theorem 17.5.11 [ET 16.5.11], div curl$\boldsymbol{G}=0$, so div$\boldsymbol{F}=0\Leftrightarrow(3ax^2-3z^2)+(x^2+3by^2)+(3cz^2)=0\Leftrightarrow(3a+1)x^2+3by^2+(3c-3)z^2=0\Leftrightarrow$ $a=-\frac{1}{3}$, $b=0$, $c=1$.

17. $\text{curl}\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ x^x+z^2&y^y+x^2&z^z+y^2\end{vmatrix}=2y\boldsymbol{i}+2z\boldsymbol{j}+2x\boldsymbol{k}$ and $W=\int_C\boldsymbol{F}\cdot d\boldsymbol{r}=\iint_S\text{curl}\boldsymbol{F}\cdot d\boldsymbol{S}$.
To parametrize the surface, let $x=2\cos\theta\sin\phi$, $y=2\sin\theta\sin\phi$, $z=2\cos\phi$, so that $\boldsymbol{r}(\phi,\theta)=2\sin\phi\cos\theta\,\boldsymbol{i}+2\sin\phi\sin\theta\,\boldsymbol{j}+2\cos\phi\,\boldsymbol{k}$, $0\le\phi\le\frac{\pi}{2}$, $0\le\theta\le\frac{\pi}{2}$, and $\boldsymbol{r}_\phi\times\boldsymbol{r}_\theta=4\sin^2\phi\cos\theta\,\boldsymbol{i}+4\sin^2\phi\sin\theta\,\boldsymbol{j}+4\sin\phi\cos\phi\,\boldsymbol{k}$. Then
$\text{curl}\boldsymbol{F}(\boldsymbol{r}(\phi,\theta))=4\sin\phi\sin\theta\,\boldsymbol{i}+4\cos\phi\,\boldsymbol{j}+4\sin\phi\cos\theta\,\boldsymbol{k}$, and
$\text{curl}\boldsymbol{F}\cdot(\boldsymbol{r}_\phi\times\boldsymbol{r}_\theta)=16\sin^3\phi\sin\theta\cos\theta+16\cos\phi\sin^2\phi\sin\theta+16\sin^2\phi\cos\phi\cos\theta$. Therefore

$$\iint_S \operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_D \operatorname{curl}\boldsymbol{F}\cdot(\boldsymbol{r}_\phi\times\boldsymbol{r}_\theta)dA$$

$$=16\left[\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\right]\left[\int_0^{\pi/2}\sin^3\phi\,d\phi\right]+16\left[\int_0^{\pi/2}\sin\theta\,d\theta\right]\left[\int_0^{\pi/2}\sin^2\phi\cos\phi\,d\phi\right]$$

$$+16\left[\int_0^{\pi/2}\cos\theta\,d\theta\right]\left[\int_0^{\pi/2}\sin^2\phi\cos\phi\,d\phi\right]$$

$$=8\left[-\cos\phi+\frac{1}{3}\cos^3\phi\right]_0^{\pi/2}+16(1)\left[\frac{1}{3}\sin^3\phi\right]_0^{\pi/2}+16(1)\left[\frac{1}{3}\sin^3\phi\right]_0^{\pi/2}$$

$$=8\left[0+1+0-\frac{1}{3}\right]+16\left(\frac{1}{3}\right)+16\left(\frac{1}{3}\right)=\frac{16}{3}+\frac{16}{3}+\frac{16}{3}=16$$

18. $\int_C(y+\sin x)dx+(z^2+\cos y)dy+x^3dz=\int_C\boldsymbol{F}\cdot d\boldsymbol{r}$, where $\boldsymbol{F}(x,y,z)=(y+\sin x)\boldsymbol{i}+(z^2+\cos y)\boldsymbol{j}+x^3\boldsymbol{k}\Rightarrow$ $\operatorname{curl}\boldsymbol{F}=-2z\boldsymbol{i}-3x^2\boldsymbol{j}-\boldsymbol{k}$. Since $\sin 2t=2\sin t\cos t$, C lies on the surface $z=2xy$. Let S be the part of this surface that is bounded by C . Then the projection of S onto the xy –plane is the unit disk $D(x^2+y^2\leq 1)$. C is traversed clockwise (when viewed from above) so S is oriented downward. Using Equation 17.7.8 [ET 16.7.8] with $g(x,y)=2xy$, $P=-2(2xy)=-4xy$, $Q=-3x^2$, $R=-1$, we have

$$\int_C\boldsymbol{F}\cdot d\boldsymbol{r}=-\iint_S\operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}=-\iint_D\left[-(-4xy)(2y)-(-3x^2)(2x)-1\right]dA$$

$$=-\iint_D(8xy^2+6x^3-1)dA=-\int_0^{2\pi}\int_0^1(8r^3\cos\theta\sin^2\theta+6r^3\cos^3\theta-1)r\,dr\,d\theta$$

$$=-\int_0^{2\pi}\left(\frac{8}{5}\cos\theta\sin^2\theta+\frac{6}{5}\cos^3\theta-\frac{1}{2}\right)r\,dr\,d\theta$$

$$=-\left[\frac{8}{15}\sin^3\theta+\frac{6}{5}\left(\sin\theta-\frac{1}{3}\sin^3\theta\right)-\frac{1}{2}\theta\right]_0^{2\pi}=\pi$$

19. Assume S is centered at the origin with radius a and let H_1 and H_2 be the upper and lower hemispheres, respectively, of S . Then $\iint_S\operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{H_1}\operatorname{curl},\ \boldsymbol{F}\cdot d\boldsymbol{S}+\iint_{H_2}\operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}=\oint_{C_1}\boldsymbol{F}\cdot d\boldsymbol{r}+\oint_{C_2}\boldsymbol{F}\cdot d\boldsymbol{r}$ by Stokes' Theorem. But C_1 is the circle $x^2+y^2=a^2$ oriented in the counterclockwise direction while C_2 is the same circle oriented in the clockwise direction. Hence

$\oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} = -\oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r}$ so $\iint_S \text{curl} \boldsymbol{F} \cdot d\boldsymbol{S} = 0$ as desired.

20. **(a)** By Exercise 17.5.26 [ET 16.5.26], $\text{curl}(f\nabla g) = f\,\text{curl}(\nabla g) + \nabla f \times \nabla g = \nabla f \times \nabla g$ since $\text{curl}(\nabla g) = \mathbf{0}$. Hence by Stokes' Theorem $\int_C (f\nabla g) \cdot d\boldsymbol{r} = \iint_S (\nabla f \times \nabla g) \cdot d\boldsymbol{S}$.

(b) As in (a), $\text{curl}(f\nabla f) = \nabla f \times \nabla f = \mathbf{0}$, so by Stokes' Theorem, $\int_C (f\nabla f) \cdot d\boldsymbol{r} = \iint_S [\text{curl}(f\nabla f)] \cdot d\boldsymbol{S} = 0$.

(c) As in part (a),

$$\begin{aligned} \text{curl}(f\nabla g + g\nabla f) &= \text{curl}(f\nabla g) + \text{curl}(g\nabla f) \quad \text{(by Exercise 17.5.24)} \\ &= (\nabla f \times \nabla g) + (\nabla g \times \nabla f) = \mathbf{0} [\text{since}\ \boldsymbol{u} \times \boldsymbol{v} = -(\boldsymbol{v} \times \boldsymbol{u})] \end{aligned}$$

Hence by Stokes' Theorem, $\int_C (f\nabla g + g\nabla f) \cdot d\boldsymbol{r} = \iint_S \text{curl}(f\nabla g + g\nabla f) \cdot d\boldsymbol{S} = 0$.

1. The vectors that end near P_1 are longer than the vectors that start near P_1 , so the net flow is inward near P_1 and div$\boldsymbol{F}(P_1)$ is negative. The vectors that end near P_2 are shorter than the vectors that start near P_2 , so the net flow is outward near P_2 and div$\boldsymbol{F}(P_2)$ is positive.

2. **(a)** The vectors that end near P_1 are shorter than the vectors that start near P_1 , so the net flow is outward and P_1 is a source. The vectors that end near P_2 are longer than the vectors that start near P_2 , so the net flow is inward and P_2 is a sink.

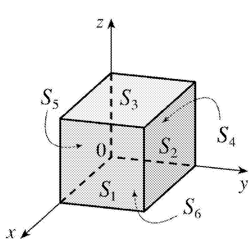
(b) $\boldsymbol{F}(x, y)=\left\langle x, y^2\right\rangle \Rightarrow \text{div}\boldsymbol{F}=\nabla\cdot\boldsymbol{F}=1+2y$. The y –value at P_1 is positive, so div$\boldsymbol{F}=1+2y$ is positive, thus P_1 is a source. At P_2 , $y<-1$, so div$\boldsymbol{F}=1+2y$ is negative, and P_2 is a sink.

3. div$\boldsymbol{F}=3+x+2x=3+3x$, so $\iiint_E \text{div}\boldsymbol{F}dV=\int_0^1\int_0^1\int_0^1(3x+3)dx\,dy\,dz=\frac{9}{2}$ (notice the triple integral is three times the volume of the cube plus three times $\bar{x}$).

To compute $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$ on S_1: $\boldsymbol{n}=\boldsymbol{i}$, $\boldsymbol{F}=3\boldsymbol{i}+y\boldsymbol{j}+2z\boldsymbol{k}$, and $\iint_{S_1}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_1}3dS=3$;

S_2: $\boldsymbol{F}=3x\boldsymbol{i}+x\boldsymbol{j}+2xz\boldsymbol{k}$, $\boldsymbol{n}=\boldsymbol{j}$ and $\iint_{S_2}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_2}x\,dS=\frac{1}{2}$;

S_3: $\boldsymbol{F}=3x\boldsymbol{i}+xy\boldsymbol{j}+2x\boldsymbol{k}$, $\boldsymbol{n}=\boldsymbol{k}$ and $\iint_{S_3}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_3}2x\,dS=1$;

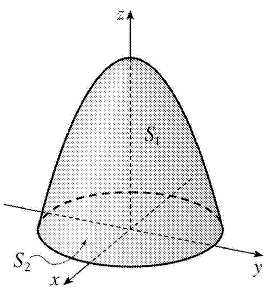


S_4: $\boldsymbol{F}=\boldsymbol{0}$, $\iint_{S_4}\boldsymbol{F}\cdot d\boldsymbol{S}=0$; S_5: $\boldsymbol{F}=3x\boldsymbol{i}+2x\boldsymbol{k}$, $\boldsymbol{n}=-\boldsymbol{j}$ and $\iint_{S_5}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_5}0dS=0$;

S_6: $\boldsymbol{F}=3x\boldsymbol{i}+xy\boldsymbol{j}$, $\boldsymbol{n}=-\boldsymbol{k}$ and $\iint_{S_6}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_6}0dS=0$. Thus $\iint_S\boldsymbol{F}\cdot d\boldsymbol{S}=\frac{9}{2}$.

4. div$\boldsymbol{F}=2x+x+1=3x+1$ so

$$\iiint_E \operatorname{div} \boldsymbol{F} dV = \iiint_E (3x+1)dV = \int_0^{2\pi}\int_0^{2}\int_0^{4-r^2} (3r\cos\theta+1)r\,dz\,dr\,d\theta$$
$$=\int_0^{2}\int_0^{2\pi} r(3r\cos\theta+1)(4-r^2)d\theta\,dr$$
$$=\int_0^{2\pi} r(4-r^2)\left[3r\sin\theta+\theta\right]_{\theta=0}^{\theta=2\pi} dr$$
$$=2\pi\int_0^{2}(4r-r^3)dr=2\pi\left[2r^2-\frac{1}{4}r^4\right]_0^2$$
$$=2\pi(8-4)=8\pi$$



On S_1 : The surface is $z=4-x^2-y^2, x^2+y^2\le 4$, with upward orientation, and $\boldsymbol{F}=x^2\boldsymbol{i}+xy\boldsymbol{j}+(4-x^2-y^2)\boldsymbol{k}$. Then

$$\iint_{S_1} \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_D [-(x^2)(-2x)-(xy)(-2y)+(4-x^2-y^2)]dA$$
$$=\iint_D \left[2x(x^2+y^2)+4-(x^2+y^2)\right]dA=\int_0^{2\pi}\int_0^{2}(2r\cos\theta\cdot r^2+4-r^2)r\,dr\,d\theta$$
$$=\int_0^{2\pi}\left[\frac{2}{5}r^5\cos\theta+2r^2-\frac{1}{4}r^4\right]_{r=0}^{r=2}d\theta=\int_0^{2\pi}\left(\frac{64}{5}\cos\theta+4\right)d\theta$$
$$=\left[\frac{64}{5}\sin\theta+4\theta\right]_0^{2\pi}=8\pi$$

On S_2: The surface is $z=0$ with downward orientation, so $\boldsymbol{F}=x^2\boldsymbol{i}+xy\boldsymbol{j}$, $\boldsymbol{n}=-\boldsymbol{k}$ and

$$\iint_{S_2} \boldsymbol{F}\cdot\boldsymbol{n}dS=\iint_{S_2} 0\,dS=0.$$

Thus

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_1} \boldsymbol{F}\cdot d\boldsymbol{S}+\iint_{S_2} \boldsymbol{F}\cdot d\boldsymbol{S}=8\pi .$$

5. div $\boldsymbol{F}=x+y+z$, so

$$\iiint_E \operatorname{div}\boldsymbol{F}dV = \int_0^{2\pi}\int_0^1\int_0^1 (r\cos\theta+r\sin\theta+z)r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^1 \left(r^2\cos\theta+r^2\sin\theta+\frac{1}{2}r\right)dr\,d\theta$$
$$=\int_0^{2\pi}\left(\frac{1}{3}\cos\theta+\frac{1}{3}\sin\theta+\frac{1}{4}\right)d\theta=\frac{1}{4}(2\pi)=\frac{\pi}{2}$$

Let S_1 be the top of the cylinder, S_2 the bottom, and S_3 the vertical edge. On S_1 , $z=1$, $\boldsymbol{n}=\boldsymbol{k}$, and

$\boldsymbol{F}=xy\boldsymbol{i}+y\boldsymbol{j}+x\boldsymbol{k}$, so $\iint_{S_1} \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_1} \boldsymbol{F}\cdot \boldsymbol{n}dS=\iint_{S_1} x\,dS=\int_0^{2\pi}\int_0^1 (r\cos\theta)r\,dr\,d\theta=[\sin\theta]_0^{2\pi}\left[\frac{1}{3}r^3\right]_0^1=0$. On S_2,

$z=0$, $\boldsymbol{n}=-\boldsymbol{k}$, and $\boldsymbol{F}=xy\boldsymbol{i}$ so $\iint_{S_2} \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_2} 0\,dS=0$. S_3 is given by $\boldsymbol{r}(\theta, z)=\cos\theta\,\boldsymbol{i}+\sin\theta\,\boldsymbol{j}+z\boldsymbol{k}$,

$0\le\theta\le 2\pi$, $0\le z\le 1$. Then $\boldsymbol{r}_\theta\times\boldsymbol{r}_z=\cos\theta\,\boldsymbol{i}+\sin\theta\,\boldsymbol{j}$ and

$$\iint_{S_3} \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_D \boldsymbol{F}\cdot(\boldsymbol{r}_\theta\times\boldsymbol{r}_z)dA=\int_0^{2\pi}\int_0^1(\cos^2\theta\sin\theta+z\sin^2\theta)dz\,d\theta$$
$$=\int_0^{2\pi}\left(\cos^2\theta\sin\theta+\frac{1}{2}\sin^2\theta\right)d\theta=\left[-\frac{1}{3}\cos^3\theta+\frac{1}{4}\left(\theta-\frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi}=\frac{\pi}{2}$$

Thus $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=0+0+\frac{\pi}{2}=\frac{\pi}{2}$.

6. div $\boldsymbol{F}=1+1+1=3$, so $\iiint_E \operatorname{div}\boldsymbol{F}dV=\iiint_E 3dV=3$ (volume of ball) $=3\left(\frac{4}{3}\pi\right)=4\pi$. To find $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$ we use spherical coordinates. S is the unit sphere, represented by $\boldsymbol{r}(\phi,\theta)=\sin\phi\cos\theta\,\boldsymbol{i}+\sin\phi\sin\theta\,\boldsymbol{j}+\cos\phi\,\boldsymbol{k}$, $0\le\phi\le\pi$, $0\le\theta\le 2\pi$. Then $\boldsymbol{r}_\phi\times\boldsymbol{r}_\theta=\sin^2\phi\cos\theta\,\boldsymbol{i}+\sin^2\phi\sin\theta\,\boldsymbol{j}+\sin\phi\cos\phi\,\boldsymbol{k}$ (see Example 17.6.10 [ET 16.6.10]) and $\boldsymbol{F}(\boldsymbol{r}(\phi,\theta))=\sin\phi\cos\theta\,\boldsymbol{i}+\sin\phi\sin\theta\,\boldsymbol{j}+\cos\phi\,\boldsymbol{k}$. Thus

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_D \boldsymbol{F}\cdot(\boldsymbol{r}_\phi\times\boldsymbol{r}_\theta)dA=\int_0^{2\pi}\int_0^{\pi}(\sin^3\phi\cos^2\theta+\sin^3\phi\sin^2\theta+\sin\phi\cos^2\phi)d\phi\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi = (2\pi)(2) = 4\pi$$

7. div $\boldsymbol{F} = \frac{\partial}{\partial x}(e^x \sin y) + \frac{\partial}{\partial y}(e^x \cos y) + \frac{\partial}{\partial z}(yz^2) = e^x \sin y - e^x \sin y + 2yz = 2yz$, so by the Divergence Theorem,

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E \operatorname{div} \boldsymbol{F} dV = \int_0^1 \int_0^1 \int_0^2 2yz\,dz\,dy\,dx = 2\int_0^1 dx \int_0^1 y\,dy \int_0^1 z\,dz$$

$$= 2\left[x\right]_0^1 \left[\frac{1}{2}y^2\right]_0^1 \left[\frac{1}{2}z^2\right]_0^2 = 2$$

8. div $\boldsymbol{F} = \frac{\partial}{\partial x}(x^2 z^3) + \frac{\partial}{\partial y}(2xyz^3) + \frac{\partial}{\partial z}(xz^4) = 2xz^3 + 2xz^3 + 4xz^3 = 8xz^3$, so by the Divergence Theorem,

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E \operatorname{div} \boldsymbol{F} dV = \int_{-1}^1 \int_{-2}^2 \int_{-3}^3 8xz^3\,dz\,dy\,dx = 8\int_{-1}^1 x\,dx \int_{-2}^2 dy \int_{-3}^3 z^3\,dz$$

$$= 8\left[\frac{1}{2}x^2\right]_{-1}^1 \left[y\right]_{-2}^2 \left[\frac{1}{4}z^4\right]_{-3}^3 = 0$$

9. div $\boldsymbol{F} = 3y^2 + 0 + 3z^2$, so using cylindrical coordinates with $y = r\cos\theta$, $z = r\sin\theta$, $x = x$ we have

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E (3y^2 + 3z^2)\,dV = \int_0^{2\pi} \int_0^1 \int_{-1}^2 (3r^2 \cos^2\theta + 3r^2 \sin^2\theta)\,r\,dx\,dr\,d\theta$$

$$= 3\int_0^{2\pi} d\theta \int_0^1 r^3\,dr \int_{-1}^2 dx = 3(2\pi)\left(\frac{1}{4}\right)(3) = \frac{9\pi}{2}$$

10. div $\boldsymbol{F} = 3x^2 y - 2x^2 y - x^2 y = 0$, so $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E 0\,dV = 0$.

11. div $\boldsymbol{F} = y\sin z + 0 - y\sin z = 0$, so by the Divergence Theorem, $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E 0\,dV = 0$.

12. div$\boldsymbol{F} = 2xy + 2xy + 2xy = 6xy$, so

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E 6xy\,dV$$

$$=\int_0^1\int_0^{2-2y}\int_0^{2-x-2y} 6xydz\,dx\,dy=\int_0^1\int_0^{2-2y} 6xy(2-x-2y)dx\,dy$$

$$=\int_0^1\int_0^{2-2y} (12xy-6x^2y-12xy^2)dx\,dy=\int_0^1\left[6x^2y-2x^3y-6x^2y^2\right]_{x=0}^{x=2-2y}dy$$

$$=\int_0^1 y(2-2y)^3dy=\left[-\frac{8}{5}y^5+6y^4-8y^3+4y^2\right]_0^1=\frac{2}{5}$$

13. div $\boldsymbol{F}=y^2+0+x^2=x^2+y^2$ so

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iiint_E (x^2+y^2)dV=\int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\cdot r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^2 r^3(4-r^2)dr\,d\theta$$

$$=\int_0^{2\pi} d\theta\int_0^2(4r^3-r^5)dr=2\pi\left[r^4-\frac{1}{6}r^6\right]_0^2=\frac{32}{3}\pi$$

14. div $\boldsymbol{F}=4x^3+4xy^2$ so

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iiint_E 4x(x^2+y^2)dV=\int_0^{2\pi}\int_0^1\int_0^{r\cos\theta+2}(4r^3\cos\theta)r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1(4r^5\cos^2\theta+8r^4\cos\theta)dr\,d\theta=\int_0^{2\pi}\left(\frac{2}{3}\cos^2\theta+\frac{8}{5}\cos\theta\right)d\theta=\frac{2}{3}\pi$$

15. div $\boldsymbol{F}=12x^2z+12y^2z+12z^3$ so

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iiint_E 12z(x^2+y^2+z^2)dV=\int_0^{2\pi}\int_0^{\pi}\int_0^R 12(\rho\cos\phi)(\rho^2)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=12\int_0^{2\pi}d\theta\int_0^{\pi}\sin\phi\cos\phi\,d\phi\int_0^R\rho^5d\rho=12(2\pi)\left[\frac{1}{2}\sin^2\phi\right]_0^{\pi}\left[\frac{1}{6}\rho^6\right]_0^R=0$$

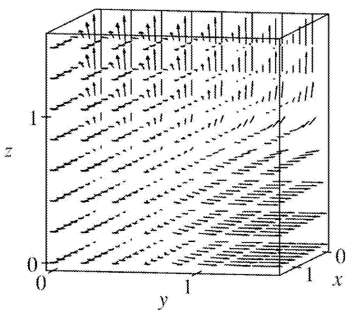
16.

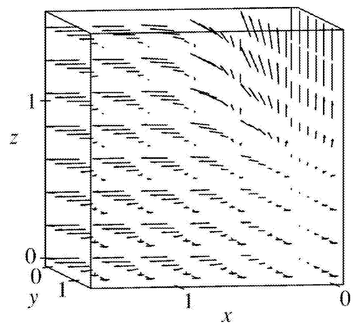
$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iiint_E 3(x^2+y^2+1)dV=\int_0^{2\pi}\int_0^{\pi/2}\int_1^2 3(\rho^2\sin^2\phi+1)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=2\pi\int_0^{\pi/2}\left[\frac{93}{5}\sin^3\phi+7\sin\phi\right]d\phi=2\pi\left[\frac{93}{5}\left(-\cos\phi+\frac{1}{3}\cos^3\phi\right)-7\cos\phi\right]_0^{\pi/2}=\frac{194}{5}\pi$$

17. $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E \sqrt{3-x^2}\,dV = \int_{-1}^{1}\int_{-1}^{1}\int_{0}^{2-x^4-y^4}\sqrt{3-x^2}\,dz\,dy\,dx = \frac{341}{60}\sqrt{2}+\frac{81}{20}\sin^{-1}\left(\frac{\sqrt{3}}{3}\right)$

18.





By the Divergence Theorem, the flux of $\boldsymbol{F}$ across the surface of the cube is

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2}\left[\cos x\cos^2 y + 3\sin^2 y\cos y\cos^4 z + 5\sin^4 z\cos z\cos^6 x\right]dz\,dy\,dx = \frac{19}{64}\pi^2.$$

19. For S_1 we have $\boldsymbol{n}=-\boldsymbol{k}$, so $\boldsymbol{F}\cdot\boldsymbol{n}=\boldsymbol{F}\cdot(-\boldsymbol{k})=-x^2z-y^2=-y^2$ (since $z=0$ on S_1). So if D is the unit disk, we get $\iint_{S_1}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_1}\boldsymbol{F}\cdot\boldsymbol{n}\,dS=\iint_D(-y^2)\,dA=-\int_0^{2\pi}\int_0^1 r^2(\sin^2\theta)\,r\,dr\,d\theta=-\frac{1}{4}\pi$. Now since S_2 is closed, we can use the Divergence Theorem. Since $\text{div}\boldsymbol{F}=\frac{\partial}{\partial x}(z^2x)+\frac{\partial}{\partial y}\left(\frac{1}{3}y^3+\tan z\right)+\frac{\partial}{\partial z}(x^2z+y^2)=z^2+y^2+x^2$, we use spherical coordinates to get $\iint_{S_2}\boldsymbol{F}\cdot d\boldsymbol{S}=\iiint_E \text{div}\boldsymbol{F}\,dV=\int_0^{2\pi}\int_0^{\pi/2}\int_0^1\rho^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\frac{2}{5}\pi$. Finally

$$\iint_S\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_2}\boldsymbol{F}\cdot d\boldsymbol{S}-\iint_{S_1}\boldsymbol{F}\cdot d\boldsymbol{S}=\frac{2}{5}\pi-\left(-\frac{1}{4}\pi\right)=\frac{13}{20}\pi.$$

20. As in the hint to Exercise 19, we create a closed surface $S_2=S\cup S_1$, where S is the part of the paraboloid $x^2+y^2+z=2$ that lies above the plane $z=1$, and S_1 is the disk $x^2+y^2=1$ on the plane $z=1$ oriented downward, and we then apply the Divergence Theorem. Since the disk S_1 is oriented downward, its unit normal vector is $\boldsymbol{n}=-\boldsymbol{k}$ and $\boldsymbol{F}\cdot(-\boldsymbol{k})=-z=-1$ on S_1. So

$\iint_{S_1}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_1}\boldsymbol{F}\cdot\boldsymbol{n}\,dS=\iint_{S_1}(-1)\,dS=-A(S_1)=-\pi$. Let E be the region bounded by S_2. Then

$$\iint_{S_2} \boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E \operatorname{div} \boldsymbol{F} dV = \iiint_E 1 dV = \int_0^1 \int_0^{2\pi} \int_1^{2-r^2} r dz d\theta\, dr = \int_0^1 \int_0^{2\pi} (r-r^3) d\theta\, dr$$

$$= (2\pi)\frac{1}{4} = \frac{\pi}{2}$$

Thus the flux of $\boldsymbol{F}$ across S is $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_{S_2} \boldsymbol{F}\cdot d\boldsymbol{S} - \iint_{S_1} \boldsymbol{F}\cdot d\boldsymbol{S} = \frac{\pi}{2} - (-\pi) = \frac{3\pi}{2}$.

21. Since $\frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} = \frac{x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}}{(x^2+y^2+z^2)^{3/2}}$ and $\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{(x^2+y^2+z^2)-3x^2}{(x^2+y^2+z^2)^{5/2}}$ with similar expressions for $\frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right)$ and $\frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$, we have

$\operatorname{div}\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|^3}\right) = \frac{3(x^2+y^2+z^2)-3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = 0$, except at (0, 0, 0) where it is undefined.

22. We first need to find $\boldsymbol{F}$ so that $\iint_S \boldsymbol{F}\cdot \boldsymbol{n} dS = \iint_S (2x+2y+z^2) dS$, so $\boldsymbol{F}\cdot\boldsymbol{n} = 2x+2y+z^2$.

But for S, $\boldsymbol{n} = \frac{x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}}{\sqrt{x^2+y^2+z^2}} = x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$. Thus $\boldsymbol{F} = 2\boldsymbol{i}+2\boldsymbol{j}+z\boldsymbol{k}$ and $\operatorname{div}\boldsymbol{F} = 1$. If

$B = \{(x, y, z) | x^2+y^2+z^2 \leq 1\}$, then $\iint_S (2x+2y+z^2) dS = \iiint_B dV = V(B) = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$.

23. $\iint_S \boldsymbol{a}\cdot\boldsymbol{n} dS = \iiint_E \operatorname{div} \boldsymbol{a} dV = 0$ since $\operatorname{div} \boldsymbol{a} = 0$.

24. $\frac{1}{3}\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \frac{1}{3}\iiint_E \operatorname{div}\boldsymbol{F} dV = \frac{1}{3}\iiint_E 3 dV = V(E)$

25. $\iint_S \operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E \operatorname{div}(\operatorname{curl}\boldsymbol{F}) dV = 0$ by Theorem 17.5.11 [ET 16.5.11].

26.

$$\iint_S D_{\boldsymbol{n}} f dS = \iint_S (\nabla f \cdot \boldsymbol{n}) dS = \iiint_E \operatorname{div}(\nabla f) dV = \iiint_E \nabla^2 f dV$$

27. $\iint_S (f \nabla g) \cdot \boldsymbol{n} dS = \iiint_E \operatorname{div}(f \nabla g) dV = \iiint_E (f \nabla^2 g + \nabla g \cdot \nabla f) dV$ by Exercise 17.5.25 [ET 16.5.25].

28. $\iint_S (f \nabla g - g \nabla f) \cdot \boldsymbol{n} dS = \iiint_E \left[(f \nabla^2 g + \nabla g \cdot \nabla f) - (g \nabla^2 f + \nabla g \cdot \nabla f) \right] dV$ [by Exercise 27].

But $\nabla g \cdot \nabla f = \nabla f \cdot \nabla g$, so that $\iint_S (f \nabla g - g \nabla f) \cdot \boldsymbol{n} dS = \iiint_E (f \nabla^2 g - g \nabla^2 f) dV$.

29. If $\boldsymbol{c} = c_1 \boldsymbol{i} + c_2 \boldsymbol{j} + c_3 \boldsymbol{k}$ is an arbitrary constant vector, we define $\boldsymbol{F} = f \boldsymbol{c} = f c_1 \boldsymbol{i} + f c_2 \boldsymbol{j} + f c_3 \boldsymbol{k}$. Then

$\operatorname{div} \boldsymbol{F} = \operatorname{div} f \boldsymbol{c} = \frac{\partial f}{\partial x} c_1 + \frac{\partial f}{\partial y} c_2 + \frac{\partial f}{\partial z} c_3 = \nabla f \cdot \boldsymbol{c}$ and the Divergence Theorem says $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_E \operatorname{div} \boldsymbol{F} dV$

$\Rightarrow \iint_S \boldsymbol{F} \cdot \boldsymbol{n} dS = \iiint_E \nabla f \cdot \boldsymbol{c} dV$. In particular, if $\boldsymbol{c} = \boldsymbol{i}$ then $\iint_S f \boldsymbol{i} \cdot \boldsymbol{n} dS = \iiint_E \nabla f \cdot \boldsymbol{i} dV \Rightarrow$

$\iint_S f n_1 dS = \iiint_E \frac{\partial f}{\partial x} dV$ (where $\boldsymbol{n} = n_1 \boldsymbol{i} + n_2 \boldsymbol{j} + n_3 \boldsymbol{k}$). Similarly, if $\boldsymbol{c} = \boldsymbol{j}$ we have $\iint_S f n_2 dS = \iiint_E \frac{\partial f}{\partial y} dV$,

and $\boldsymbol{c} = \boldsymbol{k}$ gives $\iint_S f n_3 dS = \iiint_E \frac{\partial f}{\partial z} dV$. Then

$$\begin{aligned}
\iint_S f \boldsymbol{n} dS &= \left(\iint_S f n_1 dS \right) \boldsymbol{i} + \left(\iint_S f n_2 dS \right) \boldsymbol{j} + \left(\iint_S f n_3 dS \right) \boldsymbol{k} \\
&= \left(\iiint_E \frac{\partial f}{\partial x} dV \right) \boldsymbol{i} + \left(\iiint_E \frac{\partial f}{\partial y} dV \right) \boldsymbol{j} + \left(\iiint_E \frac{\partial f}{\partial z} dV \right) \boldsymbol{k} \\
&= \iiint_E \left(\frac{\partial f}{\partial x} \boldsymbol{i} + \frac{\partial f}{\partial y} \boldsymbol{j} + \frac{\partial f}{\partial z} \boldsymbol{k} \right) dV = \iiint_E \nabla f dV
\end{aligned}$$

as desired.

30. By Exercise 29, $\iint_S p \boldsymbol{n} dS = \iiint_E \nabla p dV$, so

$$\boldsymbol{F} = -\iint_S p\boldsymbol{n}dS = -\iiint_E \nabla p dV = -\iiint_E \nabla(\rho g z) dV = -\iiint_E (\rho g \boldsymbol{k}) dV$$
$$= -\rho g \left(\iiint_E dV \right) \boldsymbol{k} = -\rho g V(E) \boldsymbol{k}$$

But the weight of the displaced liquid is volume × density × $g = \rho g V(E)$, thus $\boldsymbol{F} = -W\boldsymbol{k}$ as desired.

1. The auxiliary equation is $r^2-6r+8=0 \Rightarrow (r-4)(r-2)=0 \Rightarrow r=4$, $r=2$. Then by (8) the general solution is $y=c_1e^{4x}+c_2e^{2x}$.

2. The auxiliary equation is $r^2-4r+8=0 \Rightarrow r=2\pm 2i$. Then by (11) the general solution is $y=e^{2x}(c_1\cos 2x+c_2\sin 2x)$.

3. The auxiliary equation is $r^2+8r+41=0 \Rightarrow r=-4\pm 5i$. Then by (11) the general solution is $y=e^{-4x}(c_1\cos 5x+c_2\sin 5x)$.

4. The auxiliary equation is $2r^2-r-1=(2r+1)(r-1)=0 \Rightarrow r=1$, $r=-\frac{1}{2}$. Then the general solution is $y=c_1e^{x}+c_2e^{-x/2}$.

5. The auxiliary equation is $r^2-2r+1=(r-1)^2=0 \Rightarrow r=1$. Then by (10), the general solution is $y=c_1e^{x}+c_2xe^{x}$.

6. The auxiliary equation is $3r^2-5r=r(3r-5)=0 \Rightarrow r=0$, $r=\frac{5}{3}$, so $y=c_1+c_2e^{5x/3}$.

7. The auxiliary equation is $4r^2+1=0 \Rightarrow r=\pm \frac{1}{2}i$, so $y=c_1\cos\left(\frac{1}{2}x\right)+c_2\sin\left(\frac{1}{2}x\right)$.

8. The auxiliary equation is $16r^2+24r+9=(4r+3)^2=0 \Rightarrow r=-\frac{3}{4}$, so $y=c_1e^{-3x/4}+c_2xe^{-3x/4}$.

9. The auxiliary equation is $4r^2+r=r(4r+1)=0 \Rightarrow r=0$, $r=-\frac{1}{4}$, so $y=c_1+c_2e^{-x/4}$.

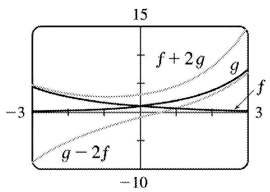
10. The auxiliary equation is $9r^2+4=0 \Rightarrow r=\pm \frac{2}{3}i$, so $y=c_1\cos\left(\frac{2}{3}x\right)+c_2\sin\left(\frac{2}{3}x\right)$.

11. The auxiliary equation is $r^2-2r-1=0 \Rightarrow r=1\pm\sqrt{2}$, so $y=c_1e^{(1+\sqrt{2})t}+c_2e^{(1-\sqrt{2})t}$.

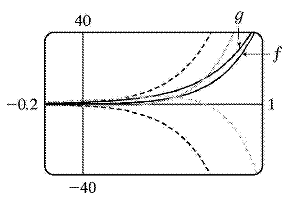
12. The auxiliary equation is $r^2-6r+4=0 \Rightarrow r=3\pm\sqrt{5}$, so $y=c_1e^{(3+\sqrt{5})t}+c_2e^{(3-\sqrt{5})t}$.

13. The auxiliary equation is $r^2+r+1=0 \Rightarrow r=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$, so $y=e^{-t/2}\left[c_1\cos\left(\frac{\sqrt{3}}{2}t\right)+c_2\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$.

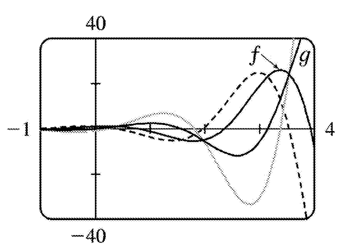
14. $6r^2-r-2=(2r+1)(3r-2)=0$ so $y=c_1e^{-x/2}+c_2e^{2x/3}$. The solutions $(c_1,c_2)=(0,1)$, $(1,0)$, $(1,2)$, $(-2,1)$ are shown. Each solution consists of a single continuous curve that approaches either 0 or $\pm\infty$ as $x\to\pm\infty$.



15. $r^2-8r+16=(r-4)^2=0$ so $y=c_1e^{4x}+c_2xe^{4x}$. The graphs are all asymptotic to the x –axis as $x\to-\infty$, and as $x\to\infty$ the solutions tend to $\pm\infty$.



16. $r^2-2r+5=0 \Rightarrow r=1\pm 2i$ and the solution is $y=e^x(c_1\cos 2x+c_2\sin 2x)$. Graphs for $(c_1,c_2)=(1,0)$, $(0,1)$, $(1,-1)$, $(-1,2)$ are shown. The solutions are all asymptotic to the x –axis as $x\to-\infty$ and they all oscillate. The amplitudes of the oscillations become arbitrarily large as $x\to\infty$ and arbitrarily small as $x\to-\infty$.



17. $2r^2+5r+3=(2r+3)(r+1)=0$, so $r=-\frac{3}{2}$, $r=-1$ and the general solution is $y=c_1e^{-3x/2}+c_2e^{-x}$. Then $y(0)=3\Rightarrow c_1+c_2=3$ and $y'(0)=-4\Rightarrow -\frac{3}{2}c_1-c_2=-4$, so $c_1=2$ and $c_2=1$. Thus the solution to the initial–

value problem is $y=2e^{-3x/2}+e^{-x}$.

18. $r^2+3=0 \Rightarrow r=\pm\sqrt{3}\,i$ and the general solution is
$y=e^{0x}\left(c_1\cos\left(\sqrt{3}\,x\right)+c_2\sin\left(\sqrt{3}\,x\right)\right)=c_1\cos\left(\sqrt{3}\,x\right)+c_2\sin\left(\sqrt{3}\,x\right)$. Then $y(0)=1 \Rightarrow c_1=1$ and $y'(0)=3$ $\Rightarrow c_2=\sqrt{3}$, so the solution to the initial–value problem is $y=\cos\left(\sqrt{3}\,x\right)+\sqrt{3}\sin\left(\sqrt{3}\,x\right)$.

19. $4r^2-4r+1=(2r-1)^2=0 \Rightarrow r=\frac{1}{2}$ and the general solution is $y=c_1e^{x/2}+c_2xe^{x/2}$. Then $y(0)=1 \Rightarrow c_1=1$ and $y'(0)=-1.5 \Rightarrow \frac{1}{2}c_1+c_2=-1.5$, so $c_2=-2$ and the solution to the initial–value problem is
$y=e^{x/2}-2xe^{x/2}$.

20. $2r^2+5r-3=(2r-1)(r+3)=0 \Rightarrow r=\frac{1}{2}$, $r=-3$ and the general solution is $y=c_1e^{x/2}+c_2e^{-3x}$. Then $1=y(0)=c_1+c_2$ and $4=y'(0)=\frac{1}{2}c_1-3c_2$ so $c_1=2$, $c_2=-1$ and the solution to the initial–value problem is
$y=2e^{x/2}-e^{-3x}$.

21. $r^2+16=0 \Rightarrow r=\pm 4i$ and the general solution is $y=e^{0x}(c_1\cos 4x+c_2\sin 4x)=c_1\cos 4x+c_2\sin 4x$. Then $y\left(\frac{\pi}{4}\right)=-3 \Rightarrow -c_1=-3 \Rightarrow c_1=3$ and $y'\left(\frac{\pi}{4}\right)=4 \Rightarrow -4c_2=4 \Rightarrow c_2=-1$, so the solution to the initial–value problem is $y=3\cos 4x-\sin 4x$.

22. $r^2-2r+5=0 \Rightarrow r=1\pm 2i$ and the general solution is $y=e^{x}(c_1\cos 2x+c_2\sin 2x)$. Then $0=y(\pi)=e^{\pi}(c_1+0)$ $\Rightarrow c_1=0$ and $2=y'(\pi)=(c_1+2c_2)e^{\pi} \Rightarrow c_2=1/e^{\pi}$ and the solution to the initial–value problem is
$y=\frac{e^{x}}{e^{\pi}}\sin 2x=e^{x-\pi}\sin 2x$.

23. $r^2+2r+2=0 \Rightarrow r=-1\pm i$ and the general solution is $y=e^{-x}(c_1\cos x+c_2\sin x)$. Then $2=y(0)=c_1$ and $1=y'(0)=c_2-c_1 \Rightarrow c_2=3$ and the solution to the initial–value problem is $y=e^{-x}(2\cos x+3\sin x)$.

24. $r^2+12r+36=(r+6)^2=0 \Rightarrow r=-6$ and the general solution is $y=c_1e^{-6x}+c_2xe^{-6x}$. Then

$0=y(1)=c_1e^{-6}+c_2e^{-6}\Rightarrow c_1+c_2=0$ and $1=y'(1)=-6c_1e^{-6}-5c_2e^{-6}\Rightarrow 6c_1+5c_2=-e^6$, so $c_1=-e^6$ and $c_2=e^6$. The solution to the initial–value problem is $y=-e^6e^{-6x}+e^6xe^{-6x}=(x-1)e^{6-6x}$.

25. $4r^2+1=0\Rightarrow r=\pm\frac{1}{2}i$ and the general solution is $y=c_1\cos\left(\frac{1}{2}x\right)+c_2\sin\left(\frac{1}{2}x\right)$. Then $3=y(0)=c_1$ and $-4=y(\pi)=c_2$, so the solution of the boundary–value problem is $y=3\cos\left(\frac{1}{2}x\right)-4\sin\left(\frac{1}{2}x\right)$.

26. $r^2+2r=r(2+r)=0\Rightarrow r=0$, $r=-2$ and the general solution is $y=c_1+c_2e^{-2x}$. Then $1=y(0)=c_1+c_2$ and $2=y(1)=c_1+c_2e^{-2}$ so $c_2=\frac{e^2}{1-e^2}$, $c_1=\frac{1-2e^2}{1-e^2}$. The solution of the boundary–value problem is $y=\frac{1-2e^2}{1-e^2}+\frac{e^2}{1-e^2}\cdot e^{-2x}$.

27. $r^2-3r+2=(r-2)(r-1)=0\Rightarrow r=1$, $r=2$ and the general solution is $y=c_1e^x+c_2e^{2x}$. Then $1=y(0)=c_1+c_2$ and $0=y(3)=c_1e^3+c_2e^6$ so $c_2=1/(1-e^3)$ and $c_1=e^3/(e^3-1)$. The solution of the boundary–value problem is $y=\frac{e^{x+3}}{e^3-1}+\frac{e^{2x}}{1-e^3}$.

28. $r^2+100=0\Rightarrow r=\pm 10i$ and the general solution is $y=c_1\cos 10x+c_2\sin 10x$. But $2=y(0)=c_1$ and $5=y(\pi)=c_1$, so there is no solution.

29. $r^2-6r+25=0\Rightarrow r=3\pm 4i$ and the general solution is $y=e^{3x}(c_1\cos 4x+c_2\sin 4x)$. But $1=y(0)=c_1$ and $2=y(\pi)=c_1e^{3\pi}\Rightarrow c_1=2/e^{3\pi}$, so there is no solution.

30. $r^2-6r+9=(r-3)^2=0\Rightarrow r=3$ and the general solution is $y=c_1e^{3x}+c_2xe^{3x}$. Then $1=y(0)=c_1$ and $0=y(1)=c_1e^3+c_2e^3\Rightarrow c_2=-1$. The solution of the boundary–value problem is $y=e^{3x}-xe^{3x}$.

31. $r^2+4r+13=0\Rightarrow r=-2\pm 3i$ and the general solution is $y=e^{-2x}(c_1\cos 3x+c_2\sin 3x)$. But $2=y(0)=c_1$ and $1=y\left(\frac{\pi}{2}\right)=e^{-\pi}(-c_2)$, so the solution to the boundary–value problem is $y=e^{-2x}(2\cos 3x-e^{\pi}\sin 3x)$.

32. $9r^2-18r+10=0 \Rightarrow r=1\pm\frac{1}{3}i$ and the general solution is $y=e^x\left(c_1\cos\frac{x}{3}+c_2\sin\frac{x}{3}\right)$. Then $0=y(0)=c_1$ and $1=y(\pi)=e^{\pi}\left(\frac{1}{2}c_1+\frac{\sqrt{3}}{2}c_2\right)\Rightarrow c_2=\frac{2}{\sqrt{3}\,e^{\pi}}$. The solution of the boundary–value problem is $y=\frac{2e^x}{\sqrt{3}\,e^{\pi}}\sin\left(\frac{x}{3}\right)=\frac{2}{\sqrt{3}}e^{x-\pi}\sin\left(\frac{x}{3}\right)$.

33. **(a)** *Case 1* ($\lambda=0$) : $y''+\lambda y=0\Rightarrow y''=0$ which has an auxiliary equation $r^2=0\Rightarrow r=0\Rightarrow y=c_1+c_2x$ where $y(0)=0$ and $y(L)=0$. Thus, $0=y(0)=c_1$ and $0=y(L)=c_2L\Rightarrow c_1=c_2=0$. Thus, $y=0$.
Case 2 ($\lambda<0$) : $y''+\lambda y=0$ has auxiliary equation $r^2=-\lambda\Rightarrow r=\pm\sqrt{-\lambda}$ (distinct and real since $\lambda<0$) $\Rightarrow y=c_1e^{\sqrt{-\lambda}\,x}+c_2e^{-\sqrt{-\lambda}\,x}$ where $y(0)=0$ and $y(L)=0$. Thus, $0=y(0)=c_1+c_2$ (*) and $0=y(L)=c_1e^{\sqrt{-\lambda}\,L}+c_2e^{-\sqrt{-\lambda}\,L}$ (**).
Multiplying (*) by $e^{\sqrt{-\lambda}\,L}$ and subtracting (**) gives $c_2\left(e^{\sqrt{-\lambda}\,L}-e^{-\sqrt{-\lambda}\,L}\right)=0\Rightarrow c_2=0$ and thus $c_1=0$ from (*). Thus, $y=0$ for the cases $\lambda=0$ and $\lambda<0$.
(b) $y''+\lambda y=0$ has an auxiliary equation $r^2+\lambda=0\Rightarrow r=\pm i\sqrt{\lambda}\Rightarrow y=c_1\cos\sqrt{\lambda}\,x+c_2\sin\sqrt{\lambda}\,x$ where $y(0)=0$ and $y(L)=0$. Thus, $0=y(0)=c_1$ and $0=y(L)=c_2\sin\sqrt{\lambda}\,L$ since $c_1=0$. Since we cannot have a trivial solution, $c_2\neq0$ and thus $\sin\sqrt{\lambda}\,L=0\Rightarrow\sqrt{\lambda}\,L=n\pi$ where n is an integer $\Rightarrow\lambda=n^2\pi^2/L^2$ and $y=c_2\sin(n\pi x/L)$ where n is an integer.

34. The auxiliary equation is $ar^2+br+c=0$. If $b^2-4ac>0$, then any solution is of the form $y(x)=c_1e^{r_1x}+c_2e^{r_2x}$ where $r_1=\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$. But a , b , and c are all positive so both r_1 and r_2 are negative and $y(x)=0$. If $b^2-4ac=0$, then any solution is of the form $y(x)=c_1e^{rx}+c_2xe^{rx}$ where $r=-b/(2a)<0$ since a , b are positive. Hence $y(x)=0$. Finally if $b^2-4ac<0$, then any solution is of the form $y(x)=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$ where $\alpha=-b/(2a)<0$ since a and b are positive. Thus $y(x)=0$.

1. The auxiliary equation is $r^2+3r+2=(r+2)(r+1)=0$, so the complementary solution is $y_c(x)=c_1e^{-2x}+c_2e^{-x}$. We try the particular solution $y_p(x)=Ax^2+Bx+C$, so $y_p'=2Ax+B$ and $y_p''=2A$. Substituting into the differential equation, we have $(2A)+3(2Ax+B)+2(Ax^2+Bx+C)=x^2$ or $2Ax^2+(6A+2B)x+(2A+3B+2C)=x^2$. Comparing coefficients gives $2A=1$, $6A+2B=0$, and $2A+3B+2C=0$, so $A=\frac{1}{2}$, $B=-\frac{3}{2}$, and $C=\frac{7}{4}$. Thus the general solution is $y(x)=y_c(x)+y_p(x)=c_1e^{-2x}+c_2e^{-x}+\frac{1}{2}x^2-\frac{3}{2}x+\frac{7}{4}$.

2. The auxiliary equation is $r^2+9=0$ with roots $r=\pm 3i$, so the complementary solution is $y_c(x)=c_1\cos(3x)+c_2\sin(3x)$. Try the particular solution $y_p(x)=Ae^{3x}$, so $y_p'=3Ae^{3x}$ and $y_p''=9Ae^{3x}$. Substitution into the differential equation gives $9Ae^{3x}+9(Ae^{3x})=e^{3x}$ or $18Ae^{3x}=e^{3x}$. Thus $A=\frac{1}{18}$ and the general solution is $y(x)=y_c(x)+y_p(x)=c_1\cos(3x)+c_2\sin(3x)+\frac{1}{18}e^{3x}$.

3. The auxiliary equation is $r^2-2r=r(r-2)=0$, so the complementary solution is $y_c(x)=c_1+c_2e^{2x}$. Try the particular solution $y_p(x)=A\cos 4x+B\sin 4x$, so $y_p'=-4A\sin 4x+4B\cos 4x$ and $y_p''=-16A\cos 4x-16B\sin 4x$. Substitution into the differential equation gives $(-16A\cos 4x-16B\sin 4x)-2(-4A\sin 4x+4B\cos 4x)=\sin 4x\Rightarrow(-16A-8B)\cos 4x+(8A-16B)\sin 4x=\sin 4x$. Then $-16A-8B=0$ and $8A-16B=1\Rightarrow A=\frac{1}{40}$ and $B=-\frac{1}{20}$. Thus the general solution is $y(x)=y_c(x)+y_p(x)=c_1+c_2e^{2x}+\frac{1}{40}\cos 4x-\frac{1}{20}\sin 4x$.

4. The auxiliary equation is $r^2+6r+9=(r+3)^2=0$, so the complementary solution is $y_c(x)=c_1e^{-3x}+c_2xe^{-3x}$. Try the particular solution $y_p(x)=Ax+B$, so $y_p'=A$ and $y_p''=0$. Substitution into the differential equation gives $0+6A+9(Ax+B)=1+x$ or $(9A)x+(6A+9B)=1+x$. Comparing coefficients, we have $9A=1$ and $6A+9B=1$, so $A=\frac{1}{9}$ and $B=\frac{1}{27}$. Thus the general solution is $y(x)=c_1e^{-3x}+c_2xe^{-3x}+\frac{1}{9}x+\frac{1}{27}$.

5. The auxiliary equation is $r^2-4r+5=0$ with roots $r=2\pm i$, so the complementary solution is

$y_c(x)=e^{2x}(c_1\cos x+c_2\sin x)$. Try $y_p(x)=Ae^{-x}$, so $y_p'=-Ae^{-x}$ and $y_p''=Ae^{-x}$. Substitution gives $Ae^{-x}-4(-Ae^{-x})+5(Ae^{-x})=e^{-x}\Rightarrow 10Ae^{-x}=e^{-x}\Rightarrow A=\frac{1}{10}$. Thus the general solution is $y(x)=e^{2x}(c_1\cos x+c_2\sin x)+\frac{1}{10}e^{-x}$.

6. $y_c(x)=e^{-x}(c_1x+c_2)$. Try $y_p(x)=x^2(Ax+B)e^{-x}$ so that no term in y_p is a solution of the complementary equation. Then $y_p'=[-Ax^3+(3A-B)x^2+2Bx]e^{-x}$, $y_p''=[Ax^3+(B-6A)x^2+(6A-4B)x+2B]e^{-x}$ and substitution gives $[Ax^3+(B-6A)x^2+(6A-4B)x+2B]+2[-Ax^3+(3A-B)x^2+2Bx]+(Ax^3+Bx^2)=x\Rightarrow$ $6Ax+2B=x$. So $y_p(x)=x^2\left(\frac{1}{6}x\right)e^{-x}$ and the general solution is $y(x)=e^{-x}\left(c_1x+c_2\right)+\frac{1}{6}x^3e^{-x}$.

7. The auxiliary equation is $r^2+1=0$ with roots $r=\pm i$, so the complementary solution is $y_c(x)=c_1\cos x+c_2\sin x$. For $y''+y=e^x$ try $y_{p_1}(x)=Ae^x$. Then $y_{p_1}'=y_{p_1}''=Ae^x$ and substitution gives $Ae^x+Ae^x=e^x\Rightarrow A=\frac{1}{2}$, so $y_{p_1}(x)=\frac{1}{2}e^x$. For $y''+y=x^3$ try $y_{p_2}(x)=Ax^3+Bx^2+Cx+D$. Then $y_{p_2}'=3Ax^2+2Bx+C$ and $y_{p_2}''=6Ax+2B$. Substituting, we have $6Ax+2B+Ax^3+Bx^2+Cx+D=x^3$, so $A=1$, $B=0$, $6A+C=0\Rightarrow C=-6$, and $2B+D=0\Rightarrow D=0$. Thus $y_{p_2}(x)=x^3-6x$ and the general solution is $y(x)=y_c(x)+y_{p_1}(x)+y_{p_2}(x)=c_1\cos x+c_2\sin x+\frac{1}{2}e^x+x^3-6x$. But $2=y(0)=c_1+\frac{1}{2}\Rightarrow c_1=\frac{3}{2}$ and $0=y'(0)=c_2+\frac{1}{2}-6\Rightarrow c_2=\frac{11}{2}$. Thus the solution to the initial-value problem is $y(x)=\frac{3}{2}\cos x+\frac{11}{2}\sin x+\frac{1}{2}e^x+x^3-6x$.

8. The auxiliary equation is $r^2-4=0$ with roots $r=\pm 2$, so the complementary solution is $y_c(x)=c_1e^{2x}+c_2e^{-2x}$. Try $y_p(x)=e^x(A\cos x+B\sin x)$, so $y_p'=e^x(A\cos x+B\sin x+B\cos x-A\sin x)$ and $y_p''=e^x(2B\cos x-2A\sin x)$. Substitution gives $e^x(2B\cos x-2A\sin x)-4e^x(A\cos x+B\sin x)=e^x\cos x\Rightarrow(2B-4A)e^x\cos x+(-2A-4B)e^x\sin x=e^x\cos x\Rightarrow$

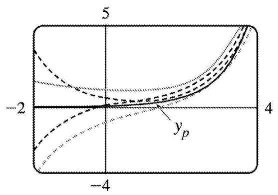
$A=-\frac{1}{5}$, $B=\frac{1}{10}$. Thus the general solution is $y(x)=c_1e^{2x}+c_2e^{-2x}+e^{x}\left(-\frac{1}{5}\cos x+\frac{1}{10}\sin x\right)$. But $1=y(0)=c_1+c_2-\frac{1}{5}$ and $2=y'(0)=2c_1-2c_2-\frac{1}{10}$. Then $c_1=\frac{9}{8}$, $c_2=\frac{3}{40}$, and the solution to the initial–value problem is $y(x)=\frac{9}{8}e^{2x}+\frac{3}{40}e^{-2x}+e^{x}\left(-\frac{1}{5}\cos x+\frac{1}{10}\sin x\right)$.

9. The auxiliary equation is $r^2-r=0$ with roots $r=0$, $r=1$ so the complementary solution is $y_c(x)=c_1+c_2e^{x}$. Try $y_p(x)=x(Ax+B)e^{x}$ so that no term in y_p is a solution of the complementary equation. Then $y_p'=(Ax^2+(2A+B)x+B)e^{x}$ and $y_p''=(Ax^2+(4A+B)x+(2A+2B))e^{x}$. Substitution into the differential equation gives $(Ax^2+(4A+B)x+(2A+2B))e^{x}-(Ax^2+(2A+B)x+B)e^{x}=xe^{x}\Rightarrow$ $(2Ax+(2A+B))e^{x}=xe^{x}\Rightarrow A=\frac{1}{2}$, $B=-1$. Thus $y_p(x)=\left(\frac{1}{2}x^2-x\right)e^{x}$ and the general solution is $y(x)=c_1+c_2e^{x}+\left(\frac{1}{2}x^2-x\right)e^{x}$. But $2=y(0)=c_1+c_2$ and $1=y'(0)=c_2-1$, so $c_2=2$ and $c_1=0$. The solution to the initial–value problem is $y(x)=2e^{x}+\left(\frac{1}{2}x^2-x\right)e^{x}=e^{x}\left(\frac{1}{2}x^2-x+2\right)$.

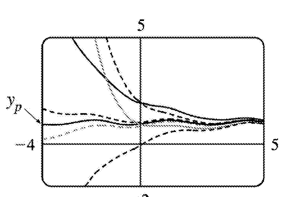
10. $y_c(x)=c_1e^{x}+c_2e^{-2x}$. For $y''+y'-2y=x$ try $y_{p_1}(x)=Ax+B$. Then $y_{p_1}'=A$, $y_{p_1}''=0$, and substitution gives $0+A-2(Ax+B)=x\Rightarrow A=-\frac{1}{2}$, $B=-\frac{1}{4}$, so $y_{p_1}(x)=-\frac{1}{2}x-\frac{1}{4}$. For $y''+y'-2y=\sin 2x$ try $y_{p_2}(x)=A\cos 2x+B\sin 2x$. Then $y_{p_2}'=-2A\sin 2x+2B\cos 2x$, $y_{p_2}''=-4A\cos 2x-4B\sin 2x$, and substitution gives $(-4A\cos 2x-4B\sin 2x)+(-2A\sin 2x+2B\cos 2x)-2(A\cos 2x+B\sin 2x)=\sin 2x\Rightarrow A=-\frac{1}{20}$, $B=-\frac{3}{20}$. Thus $y_{p_2}(x)=-\frac{1}{20}\cos 2x+-\frac{3}{20}\sin 2x$ and the general solution is $y(x)=c_1e^{x}+c_2e^{-2x}-\frac{1}{2}x-\frac{1}{4}-\frac{1}{20}\cos 2x-\frac{3}{20}\sin 2x$. But $1=y(0)=c_1+c_2-\frac{1}{4}-\frac{1}{20}$ and $0=y'(0)=c_1-2c_2-\frac{1}{2}-\frac{3}{10}\Rightarrow c_1=\frac{17}{15}$ and $c_2=\frac{1}{6}$. Thus the solution to the initial–value problem is $y(x)=\frac{17}{15}e^{x}+\frac{1}{6}e^{-2x}-\frac{1}{2}x-\frac{1}{4}-\frac{1}{20}\cos 2x-\frac{3}{20}\sin 2x$.

11. $y_c(x)=c_1e^{-x/4}+c_2e^{-x}$. Try $y_p(x)=Ae^{x}$. Then $10Ae^{x}=e^{x}$, so $A=\frac{1}{10}$ and the general solution is

$y(x)=c_1e^{-x/4}+c_2e^{-x}+\frac{1}{10}e^{x}$. The solutions are all composed of exponential curves and with the exception of the particular solution (which approaches 0 as $x\to-\infty$), they all approach either ∞ or $-\infty$ as $x\to-\infty$. As $x\to\infty$, all solutions are asymptotic to $y_p=\frac{1}{10}e^{x}$.



12. The auxiliary equation is $(2r+1)(r+1)=0$, so $r=-1$, $-\frac{1}{2}$ and $y_c(x)=c_1e^{-x}+c_2e^{-x/2}$. For $2y''+3y'+y=1$, try $y_{p_1}(x)=A$; substituting gives $y_{p_1}(x)=1$. For $2y''+3y'+y=\cos 2x$ try $y_{p_2}=A\cos 2x+B\sin 2x\Rightarrow y_{p_2}'=-2A\sin 2x+2B\cos 2x$, $y_{p_2}''=-4A\cos 2x-4B\sin 2x$.
Substituting into the differential equation gives $\cos 2x=(6B-7A)\cos 2x+(-7B-6A)\sin 2x$.
Then solving the equations $6B-7A=1$ and $-7B-6A=0$ gives $A=-\frac{7}{85}$, $B=\frac{6}{85}$. Thus,
$y_{p_2}(x)=-\frac{7}{85}\cos 2x+\frac{6}{85}\sin 2x$ and the general solution is $y(x)=c_1e^{-x}+c_2e^{-x/2}+1-\frac{7}{85}\cos 2x+\frac{6}{85}\sin 2x$.
The graph shows $y_p=y_{p_1}+y_{p_2}$ and several other solutions. Notice that all solutions are asymptotic to y_p as $x\to\infty$.



13. Here $y_c(x)=c_1\cos 3x+c_2\sin 3x$. For $y''+9y=e^{2x}$ try $y_{p_1}(x)=Ae^{2x}$ and for $y''+9y=x^2\sin x$ try $y_{p_2}(x)=(Bx^2+Cx+D)\cos x+(Ex^2+Fx+G)\sin x$. Thus a trial solution is
$y_p(x)=y_{p_1}(x)+y_{p_2}(x)=Ae^{2x}+(Bx^2+Cx+D)\cos x+(Ex^2+Fx+G)\sin x$.

14. Since $y_c(x)=c_1+c_2e^{-9x}$, try $y_p(x)=(Ax+B)e^{-x}\cos\pi x+(Cx+D)e^{-x}\sin\pi x$.

15. Here $y_c(x)=c_1+c_2e^{-9x}$. For $y''+9y'=1$ try $y_{p_1}(x)=Ax$ (since $y=A$ is a solution to the complementary equation) and for $y''+9y'=xe^{9x}$ try $y_{p_2}(x)=(Bx+C)e^{9x}$.

16. Since $y_c(x)=c_1e^{x}+c_2e^{-4x}$ try $y_p(x)=x(Ax^3+Bx^2+Cx+D)e^{x}$ so that no term of $y_p(x)$ satisfies the complementary equation.

17. Since $y_c(x)=e^{-x}(c_1\cos 3x+c_2\sin 3x)$ we try $y_p(x)=x(Ax^2+Bx+C)e^{-x}\cos 3x+x(Dx^2+Ex+F)e^{-x}\sin 3x$ (so that no term of y_p is a solution of the complementary equation).

18. Here $y_c(x)=c_1\cos 2x+c_2\sin 2x$. For $y''+4y=e^{3x}$ try $y_{p_1}(x)=Ae^{3x}$ and for $y''+4y=x\sin 2x$ try $y_{p_2}(x)=x(Bx+C)\cos 2x+x(Dx+E)\sin 2x$ (so that no term of y_{p_2} is a solution of the complementary equation).

19. **(a)** The complementary solution is $y_c(x)=c_1\cos 2x+c_2\sin 2x$. A particular solution is of the form $y_p(x)=Ax+B$. Thus, $4Ax+4B=x\Rightarrow A=\frac{1}{4}$ and $B=0\Rightarrow y_p(x)=\frac{1}{4}x$. Thus, the general solution is $y=y_c+y_p=c_1\cos 2x+c_2\sin 2x+\frac{1}{4}x$.

(b) In (a), $y_c(x)=c_1\cos 2x+c_2\sin 2x$, so set $y_1=\cos 2x$, $y_2=\sin 2x$. Then $y_1y_2'-y_2y_1'=2\cos^2 2x+2\sin^2 2x=2$ so $u_1'=-\frac{1}{2}x\sin 2x\Rightarrow$ $u_1(x)=-\frac{1}{2}\int x\sin 2x\,dx=-\frac{1}{4}\left(-x\cos 2x+\frac{1}{2}\sin 2x\right)$ and $u_2'=\frac{1}{2}x\cos 2x\Rightarrow$ $u_2(x)=\frac{1}{2}\int x\cos 2x\,dx=\frac{1}{4}\left(x\sin 2x+\frac{1}{2}\cos 2x\right)$. Hence $y_p(x)=-\frac{1}{4}\left(-x\cos 2x+\frac{1}{2}\sin 2x\right)\cos 2x+\frac{1}{4}\left(x\sin 2x+\frac{1}{2}\cos 2x\right)\sin 2x=\frac{1}{4}x$. Thus $y(x)=y_c(x)+y_p(x)=c_1\cos 2x+c_2\sin 2x+\frac{1}{4}x$.

20. **(a)** Here $r^2-3r+2=0\Rightarrow r=1$ or 2 and

$y_c(x)=c_1e^{2x}+c_2e^{x}$. We try a particular solution of the form $y_p(x)=A\cos x+B\sin x\Rightarrow y_p'=-A\sin x+B\cos x$ and $y_p''=-A\cos x-B\sin x$. Then the equation $y''-3y'+2y=\sin x$ becomes

$(A-3B)\cos x+(B+3A)\sin x=\sin x\Rightarrow A-3B=0$ and $B+3A=1\Rightarrow A=\frac{3}{10}$ and $B=\frac{1}{10}$. Thus,

$y_p(x)=\frac{3}{10}\cos x+\frac{1}{10}\sin x$. Therefore, the general solution is

$y(x)=y_c(x)+y_p(x)=c_1e^{2x}+c_2e^{x}+\frac{3}{10}\cos x+\frac{1}{10}\sin x$.

(b) From (a) we know that $y_c(x)=c_1e^{2x}+c_2e^{x}$. Setting $y_1=e^{2x}$, $y_2=e^{x}$, we have

$y_1y_2'-y_2y_1'=e^{3x}-2e^{3x}=-e^{3x}$. Thus $u_1'=-\frac{\sin xe^{x}}{-e^{3x}}=\sin xe^{-2x}$ and $u_2'=\frac{\sin xe^{2x}}{-e^{3x}}=-\sin xe^{-x}$. Then

$u_1(x)=\int e^{-2x}\sin x\,dx=\frac{1}{5}e^{-2x}(-2\sin x-\cos x)$ and $u_2(x)=-\int e^{-x}\sin x\,dx=-\frac{1}{2}e^{-x}(-\sin x-\cos x)$. Thus

$y_p(x)=\frac{1}{5}(-2\sin x-\cos x)+\frac{1}{2}(\sin x+\cos x)=\frac{1}{10}\sin x+\frac{3}{10}\cos x$ and the general solution is

$y(x)=y_c(x)+y_p(x)=c_1e^{2x}+c_2e^{x}+\frac{1}{10}\sin x+\frac{3}{10}\cos x$.

21. **(a)** $r^2-r=r(r-1)=0\Rightarrow r=0\ ,\ 1$, so the complementary solution is $y_c(x)=c_1e^{x}+c_2xe^{x}$. A particular solution is of the form $y_p(x)=Ae^{2x}$. Thus $4Ae^{2x}-4Ae^{2x}+Ae^{2x}=e^{2x}\Rightarrow Ae^{2x}=e^{2x}\Rightarrow A=1\Rightarrow y_p(x)=e^{2x}$. So a general solution is $y(x)=y_c(x)+y_p(x)=c_1e^{x}+c_2xe^{x}+e^{2x}$.

(b) From (a), $y_c(x)=c_1e^{x}+c_2xe^{x}$, so set $y_1=e^{x}$, $y_2=xe^{x}$. Then, $y_1y_2'-y_2y_1'=e^{2x}(1+x)-xe^{2x}=e^{2x}$ and so $u_1'=-xe^{x}\Rightarrow u_1(x)=-\int xe^{x}dx=-(x-1)e^{x}$ and $u_2'=e^{x}\Rightarrow u_2(x)=\int e^{x}dx=e^{x}$. Hence $y_p(x)=(1-x)e^{2x}+xe^{2x}=e^{2x}$ and the general solution is $y(x)=y_c(x)+y_p(x)=c_1e^{x}+c_2xe^{x}+e^{2x}$.

22. **(a)** Here $r^2-2r+1=(r+1)^2=0\Rightarrow r=-1$ and $y_c(x)=c_1+c_2e^{x}$ and so we try a particular solution of the form $y_p(x)=Axe^{x}$. Thus, after calculating the necessary derivatives, we get $y''-y'=e^{x}\Rightarrow$ $Ae^{x}(2+x)-Ae^{x}(1+x)=e^{x}\Rightarrow A=1$. Thus $y_p(x)=xe^{x}$ and the general solution is $y(x)=c_1+c_2e^{x}+xe^{x}$.

(b) From (a) we know that $y_c(x)=c_1+c_2e^{x}$, so setting $y_1=1$, $y_2=e^{x}$, then $y_1y_2'-y_2y_1'=e^{x}-0=e^{x}$. Thus $u_1'=-e^{2x}/e^{x}=-e^{x}$ and $u_2'=e^{x}/e^{x}=1$. Then $u_1(x)=-\int e^{x}dx=-e^{x}$ and $u_2(x)=x$. Thus $y_p(x)=-e^{x}+xe^{x}$ and the general solution is $y(x)=c_1+c_2e^x-e^x+xe^x=c_1+c_3e^x+xe^x$.

23. As in Example 6, $y_c(x)=c_1\sin x+c_2\cos x$, so set $y_1=\sin x$, $y_2=\cos x$. Then $y_1y_2'-y_2y_1'=-\sin^2 x-\cos^2 x=-1$, so $u_1'=-\dfrac{\sec x\cos x}{-1}=1\Rightarrow u_1(x)=x$ and $u_2'=\dfrac{\sec x\sin x}{-1}=-\tan x\Rightarrow$ $u_2(x)=-\int\tan x\,dx=\ln|\cos x|=\ln(\cos x)$ on $0<x<\dfrac{\pi}{2}$. Hence $y_p(x)=x\sin x+\cos x\ln(\cos x)$ and the general solution is $y(x)=(c_1+x)\sin x+\left[c_2+\ln(\cos x)\right]\cos x$.

24. Setting $y_1=\sin x$, $y_2=\cos x$, then $y_1y_2'-y_2y_1'=-\sin^2 x-\cos^2 x=-1$. Thus $u_1'=-\dfrac{\cot x\cos x}{-1}=\dfrac{\cos^2 x}{\sin x}$ and $u_2'=\dfrac{\cot x\sin x}{-1}=-\cos x$. Then $u_1(x)=\int\dfrac{\cos^2 x}{\sin x}\,dx=\int(\csc x-\sin x)dx=\ln(\csc x-\cot x)+\cos x$ and $u_2(x)=-\sin x$. Thus $y_p(x)=[\cos x+\ln(\csc x-\cot x)]\sin x+(-\sin x)(\cos x)$ and the general solution is $y(x)=c_1\sin x+c_2\cos x+\sin x\ln(\csc x-\cot x)$.

25. $y_1=e^x$, $y_2=e^{2x}$ and $y_1y_2'-y_2y_1'=e^{3x}$. So $u_1'=\dfrac{-e^{2x}}{(1+e^{-x})e^{3x}}=-\dfrac{e^{-x}}{1+e^{-x}}$ and $u_1(x)=\int-\dfrac{e^{-x}}{1+e^{-x}}\,dx=\ln(1+e^{-x})$. $u_2'=\dfrac{e^x}{(1+e^{-x})e^{3x}}=\dfrac{e^x}{e^{3x}+e^{2x}}$ so $u_2(x)=\int\dfrac{e^x}{e^{3x}+e^{2x}}\,dx=\ln\left(\dfrac{e^x+1}{e^x}\right)-e^{-x}=\ln(1+e^{-x})-e^{-x}$. Hence $y_p(x)=e^x\ln(1+e^{-x})+e^{2x}[\ln(1+e^{-x})-e^{-x}]$ and the general solution is $y(x)=[c_1+\ln(1+e^{-x})]e^x+[c_2-e^{-x}+\ln(1+e^{-x})]e^{2x}$.

26. $y_1=e^{-x}$, $y_2=e^{-2x}$ and $y_1y_2'-y_2y_1'=-e^{-3x}$. So $u_1'=-\dfrac{(\sin e^x)e^{-2x}}{-e^{-3x}}=e^x\sin e^x$ and $u_2'=\dfrac{(\sin e^x)e^{-x}}{-e^{-3x}}=-e^{2x}\sin e^x$. Hence $u_1(x)=\int e^x\sin e^x\,dx=-\cos e^x$ and $u_2(x)=\int-e^{2x}\sin e^x\,dx=e^x\cos e^x-\sin e^x$. Then $y_p(x)=-e^{-x}\cos e^x-e^{-2x}[\sin e^x-e^x\cos e^x]$ and the general solution is $y(x)=(c_1-\cos e^x)e^{-x}+[c_2-\sin e^x+e^x\cos e^x]e^{-2x}$.

27. $y_1=e^{-x}$, $y_2=e^x$ and $y_1y_2'-y_2y_1'=2$. So

$u_1'=-\frac{e^x}{2x}$, $u_2'=\frac{e^{-x}}{2x}$ and

$y_p(x)=-e^{-x}\int \frac{e^x}{2x}\,dx+e^x\int \frac{e^{-x}}{2x}\,dx$. Hence the general solution is

$y(x)=\left(c_1-\int \frac{e^x}{2x}\,dx\right)e^{-x}+\left(c_2+\int \frac{e^{-x}}{2x}\,dx\right)e^x$.

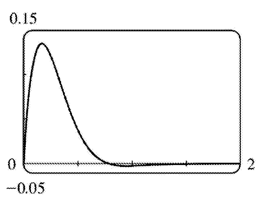
28. $y_1=e^{-2x}$, $y_2=xe^{-2x}$ and $y_1y_2'-y_2y_1'=e^{-4x}$. Then $u_1'=\frac{-e^{-2x}xe^{-2x}}{x^3e^{-4x}}=-\frac{1}{x^2}$ so $u_1(x)=x^{-1}$ and $u_2'=\frac{e^{-2x}e^{-2x}}{x^3e^{-4x}}=\frac{1}{x^3}$ so $u_2(x)=-\frac{1}{2x^2}$. Thus $y_p(x)=\frac{e^{-2x}}{x}-\frac{xe^{-2x}}{2x^2}=\frac{e^{-2x}}{2x}$ and the general solution is $y(x)=e^{-2x}[c_1+c_2x+1/(2x)]$.

1. By Hooke's Law $k(0.6)=20$ so $k=\frac{100}{3}$ is the spring constant and the differential equation is $3x''+\frac{100}{3}x=0$. The general solution is $x(t)=c_1\cos\left(\frac{10}{3}t\right)+c_2\sin\left(\frac{10}{3}t\right)$. But $0=x(0)=c_1$ and $1.2=x'(0)=\frac{10}{3}c_2$, so the position of the mass after t seconds is $x(t)=0.36\sin\left(\frac{10}{3}t\right)$.

2. $k(0.3)=24.3$ or $k=81$ is the spring constant and the resulting initial–value problem is $4x''+81x=0$, $x(0)=-0.5$ (since compressed), $x'(0)=0$. The general solution is $x(t)=c_1\cos\left(\frac{9}{2}t\right)+c_2\sin\left(\frac{9}{2}t\right)$. But $-0.2=x(0)=c_1$ and $0=x'(0)=\frac{9}{2}c_2$. Thus the position is given by $x(t)=-0.2\cos(4.5t)$.

3. $k(0.5)=6$ or $k=12$ is the spring constant, so the initial–value problem is $2x''+14x'+12x=0$, $x(0)=1$, $x'(0)=0$. The general solution is $x(t)=c_1e^{-6t}+c_2e^{-t}$. But $1=x(0)=c_1+c_2$ and $0=x'(0)=-6c_1-c_2$. Thus the position is given by $x(t)=-\frac{1}{5}e^{-6t}+\frac{6}{5}e^{-t}$.

4.
(a) The differential equation is $3x''+30x'+123x=0$ with general solution $x(t)=e^{-5t}(c_1\cos 4t+c_2\sin 4t)$. Then $0=x(0)=c_1$ and $2=x'(0)=4c_2$, so the position is given by $x(t)=\frac{1}{2}e^{-5t}\sin 4t$.

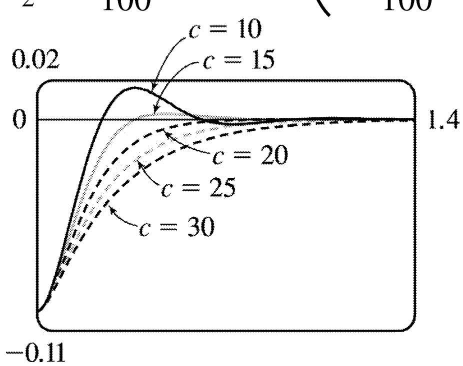
(b) 

5. For critical damping we need $c^2-4mk=0$ or $m=c^2/(4k)=14^2/(4\cdot 12)=\frac{49}{12}$ kg.

6. For critical damping we need $c^2=4mk$ or $c=2\sqrt{mk}=2\sqrt{3\cdot 123}=6\sqrt{41}$.

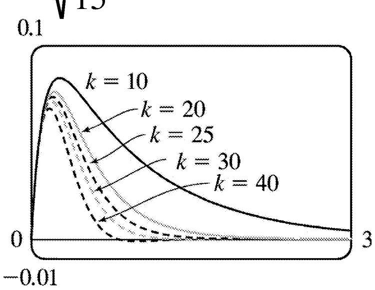
7. We are given $m=1$, $k=100$, $x(0)=-0.1$ and $x'(0)=0$. From (3), the differential equation is $\frac{d^2x}{dt^2}+c\frac{dx}{dt}+100x=0$ with auxiliary equation $r^2+cr+100=0$. If $c=10$, we have two complex roots

$r=-5\pm 5\sqrt{3}i$, so the motion is underdamped and the solution is $x=e^{-5t}\left[c_1\cos\left(5\sqrt{3}\,t\right)+c_2\sin\left(5\sqrt{3}\,t\right)\right]$. Then $-0.1=x(0)=c_1$ and $0=x'(0)=5\sqrt{3}\,c_2-5c_1 \Rightarrow c_2=-\dfrac{1}{10\sqrt{3}}$, so $x=e^{-5t}\left[-0.1\cos\left(5\sqrt{3}\,t\right)-\dfrac{1}{10\sqrt{3}}\sin\left(5\sqrt{3}\,t\right)\right]$. If $c=15$, we again have underdamping since the auxiliary equation has roots $r=-\dfrac{15}{2}\pm\dfrac{5\sqrt{7}}{2}i$. The general solution is $x=e^{-15t/2}\left[c_1\cos\left(\dfrac{5\sqrt{7}}{2}t\right)+c_2\sin\left(\dfrac{5\sqrt{7}}{2}t\right)\right]$, so $-0.1=x(0)=c_1$ and $0=x'(0)=\dfrac{5\sqrt{7}}{2}c_2-\dfrac{15}{2}c_1$ $\Rightarrow c_2=-\dfrac{3}{10\sqrt{7}}$. Thus $x=e^{-15t/2}\left[-0.1\cos\left(\dfrac{5\sqrt{7}}{2}t\right)-\dfrac{3}{10\sqrt{7}}\sin\left(\dfrac{5\sqrt{7}}{2}t\right)\right]$. For $c=20$, we have equal roots $r_1=r_2=-10$, so the oscillation is critically damped and the solution is $x=(c_1+c_2t)e^{-10t}$. Then $-0.1=x(0)=c_1$ and
$0=x'(0)=-10c_1+c_2\Rightarrow c_2=-1$, so $x=(-0.1-t)e^{-10t}$. If $c=25$ the auxiliary equation has roots $r_1=-5$, $r_2=-20$, so we have overdamping and the solution is $x=c_1e^{-5t}+c_2e^{-20t}$. Then $-0.1=x(0)=c_1+c_2$ and $0=x'(0)=-5c_1-20c_2\Rightarrow c_1=-\dfrac{2}{15}$ and $c_2=\dfrac{1}{30}$, so $x=-\dfrac{2}{15}e^{-5t}+\dfrac{1}{30}e^{-20t}$. If $c=30$ we have roots $r=-15\pm 5\sqrt{5}$, so the motion is overdamped and the solution is $x=c_1e^{\left(-15+5\sqrt{5}\right)t}+c_2e^{\left(-15-5\sqrt{5}\right)t}$.
Then $-0.1=x(0)=c_1+c_2$ and $0=x'(0)=\left(-15+5\sqrt{5}\right)c_1+\left(-15-5\sqrt{5}\right)c_2\Rightarrow c_1=\dfrac{-5-3\sqrt{5}}{100}$ and $c_2=\dfrac{-5+3\sqrt{5}}{100}$, so $x=\left(\dfrac{-5-3\sqrt{5}}{100}\right)e^{\left(-15+5\sqrt{5}\right)t}+\left(\dfrac{-5+3\sqrt{5}}{100}\right)e^{\left(-15-5\sqrt{5}\right)t}$.



8. We are given $m=1$, $c=10$, $x(0)=0$ and $x'(0)=1$. The differential equation is $\dfrac{d^2x}{dt^2}+10\dfrac{dx}{dt}+kx=0$ with auxiliary equation $r^2+10r+k=0$. $k=10$: the auxiliary equation has roots $r=-5\pm\sqrt{15}$ so we have

overdamping and the solution is $x=c_1e^{(-5+\sqrt{15})t}+c_2e^{(-5-\sqrt{15})t}$. Entering the initial conditions gives $c_1=\frac{1}{2\sqrt{15}}$ and $c_2=-\frac{1}{2\sqrt{15}}$, so $x=\frac{1}{2\sqrt{15}}e^{(-5+\sqrt{15})t}-\frac{1}{2\sqrt{15}}e^{(-5-\sqrt{15})t}$. $k=20$: $r=-5\pm\sqrt{5}$ and the solution is $x=c_1e^{(-5+\sqrt{5})t}+c_2e^{(-5-\sqrt{5})t}$ so again the motion is overdamped. The initial conditions give $c_1=\frac{1}{2\sqrt{5}}$ and $c_2=-\frac{1}{2\sqrt{5}}$, so $x=\frac{1}{2\sqrt{5}}e^{(-5+\sqrt{5})t}-\frac{1}{2\sqrt{5}}e^{(-5-\sqrt{5})t}$. $k=25$: we have equal roots $r_1=r_2=-5$, so the motion is critically damped and the solution is $x=(c_1+c_2t)e^{-5t}$. The initial conditions give $c_1=0$ and $c_2=1$, so $x=te^{-5t}$. $k=30$: $r=-5\pm\sqrt{5}\,i$ so the motion is underdamped and the solution is $x=e^{-5t}\left[c_1\cos\left(\sqrt{5}\,t\right)+c_2\sin\left(\sqrt{5}\,t\right)\right]$. The initial conditions give $c_1=0$ and $c_2=\frac{1}{\sqrt{5}}$, so $x=\frac{1}{\sqrt{5}}e^{-5t}\sin\left(\sqrt{5}\,t\right)$. $k=40$: $r=-5\pm\sqrt{15}\,i$ so we again have underdamping. The solution is $x=e^{-5t}\left[c_1\cos\left(\sqrt{15}\,t\right)+c_2\sin\left(\sqrt{15}\,t\right)\right]$, and the initial conditions give $c_1=0$ and $c_2=\frac{1}{\sqrt{15}}$. Thus $x=\frac{1}{\sqrt{15}}e^{-5t}\sin\left(\sqrt{15}\,t\right)$.



9. The differential equation is $mx''+kx=F_0\cos\omega_0t$ and $\omega_0\neq\omega=\sqrt{k/m}$. Here the auxiliary equation is $mr^2+k=0$ with roots $\pm\sqrt{k/m}\,i=\pm\omega i$ so $x_c(t)=c_1\cos\omega t+c_2\sin\omega t$. Since $\omega_0\neq\omega$, try $x_p(t)=A\cos\omega_0t+B\sin\omega_0t$. Then we need $(m)\left(-\omega_0^2\right)(A\cos\omega_0t+B\sin\omega_0t)+k(A\cos\omega_0t+B\sin\omega_0t)=F_0\cos\omega_0t$ or $A\left(k-m\omega_0^2\right)=F_0$ and $B\left(k-m\omega_0^2\right)=0$. Hence $B=0$ and $A=\frac{F_0}{k-m\omega_0^2}=\frac{F_0}{m\left(\omega^2-\omega_0^2\right)}$ since $\omega^2=\frac{k}{m}$. Thus the motion of the mass is given by

$$x(t)=c_1\cos\omega t+c_2\sin\omega t+\frac{F_0}{m\left(\omega^2-\omega_0^2\right)}\cos\omega_0 t\ .$$

10. As in Exercise 9, $x_c(t)=c_1\cos\omega t+c_2\sin\omega t$. But the natural frequency of the system equals the frequency of the external force, so try $x_p(t)=t(A\cos\omega t+B\sin\omega t)$. Then we need $m(2\omega B-\omega^2 At)\cos\omega t-m(2\omega A+\omega^2 Bt)\sin\omega t+kAt\cos\omega t+kBt\sin\omega t=F_0\cos\omega t$ or $2m\omega B=F_0$ and $-2m\omega A=0$ (noting $-m\omega^2 A+kA=0$ and $-m\omega^2 B+kB=0$ since $\omega^2=k/m$). Hence the general solution is $x(t)=c_1\cos\omega t+c_2\sin\omega t+\left[F_0 t/(2m\omega)\right]\sin\omega t$.

11. From Equation 6, $x(t)=f(t)+g(t)$ where $f(t)=c_1\cos\omega t+c_2\sin\omega t$ and $g(t)=\dfrac{F_0}{m\left(\omega^2-\omega_0^2\right)}\cos\omega_0 t$. Then f is periodic, with period $\dfrac{2\pi}{\omega}$, and if $\omega\neq\omega_0$, g is periodic with period $\dfrac{2\pi}{\omega_0}$. If $\dfrac{\omega}{\omega_0}$ is a rational number, then we can say $\dfrac{\omega}{\omega_0}=\dfrac{a}{b}\Rightarrow a=\dfrac{b\omega}{\omega_0}$ where a and b are non–zero integers. Then

$$x\left(t+a\cdot\frac{2\pi}{\omega}\right)=f\left(t+a\cdot\frac{2\pi}{\omega}\right)+g\left(t+a\cdot\frac{2\pi}{\omega}\right)=f(t)+g\left(t+\frac{b\omega}{\omega_0}\cdot\frac{2\pi}{\omega}\right)$$
$$=f(t)+g\left(t+b\cdot\frac{2\pi}{\omega_0}\right)=f(t)+g(t)=x(t)$$

so $x(t)$ is periodic.

12. **(a)** The graph of $x=c_1e^{rt}+c_2te^{rt}$ has a t –intercept when $c_1e^{rt}+c_2te^{rt}=0\Leftrightarrow e^{rt}(c_1+c_2t)=0\Leftrightarrow c_1=-c_2t$. Since $t>0$, x has a t –intercept if and only if c_1 and c_2 have opposite signs.

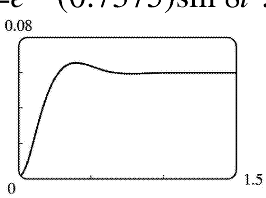
(b) For $t>0$, the graph of x crosses the t –axis when $c_1e^{r_1t}+c_2e^{r_2t}=0\Leftrightarrow c_2e^{r_2t}=-c_1e^{r_1t}\Leftrightarrow$ $c_2=-c_1\dfrac{e^{r_1t}}{e^{r_2t}}=-c_1e^{(r_1-r_2)t}$. But $r_1>r_2\Rightarrow r_1-r_2>0$ and since $t>0$, $e^{(r_1-r_2)t}>1$. Thus

$\left|c_2\right|=\left|c_1\right|e^{(r_1-r_2)t}>\left|c_1\right|$, and the graph of x can cross the t –axis only if $\left|c_2\right|>\left|c_1\right|$.

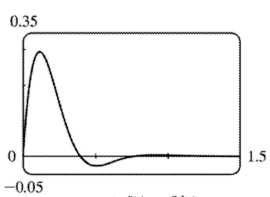
13. Here the initial–value problem for the charge is $Q''+20Q'+500Q=12$, $Q(0)=Q'(0)=0$. Then $Q_c(t)=e^{-10t}(c_1\cos 20t+c_2\sin 20t)$ and try $Q_p(t)=A\Rightarrow 500A=12$ or $A=\frac{3}{125}$. The general solution is $Q(t)=e^{-10t}(c_1\cos 20t+c_2\sin 20t)+\frac{3}{125}$. But $0=Q(0)=c_1+\frac{3}{125}$ and $Q'(t)=I(t)=e^{-10t}\left[(-10c_1+20c_2)\cos 20t+(-10c_2-20c_1)\sin 20t\right]$ but $0=Q'(0)=-10c_1+20c_2$. Thus the charge is $Q(t)=-\frac{1}{250}e^{-10t}(6\cos 20t+3\sin 20t)+\frac{3}{125}$ and the current is $I(t)=e^{-10t}\left(\frac{3}{5}\right)\sin 20t$.

14. **(a)** Here the initial–value problem for the charge is $2Q''+24Q'+200Q=12$ with $Q(0)=0.001$ and $Q'(0)=0$. Then $Q_c(t)=e^{-6t}(c_1\cos 8t+c_2\sin 8t)$ and try $Q_p(t)=A\Rightarrow A=\frac{3}{50}$ and the general solution is $Q(t)=e^{-6t}(c_1\cos 8t+c_2\sin 8t)+\frac{3}{50}$. But $0.001=Q(0)=c+\frac{3}{50}$ so $c_1=-0.059$. Also $Q'(t)=I(t)=e^{-6t}\left[(-6c_1+8c_2)\cos 8t+(-6c_2-8c_1)\sin 8t\right]$ and $0=Q'(0)=-6c_1+8c_2$ so $c_2=-0.04425$. Hence the charge is $Q(t)=-e^{-6t}(0.059\cos 8t+0.04425\sin 8t)+\frac{3}{50}$ and the current is $I(t)=e^{-6t}(0.7375)\sin 8t$.

(b)



charge, $Q(t)$



current, $I(t)=Q'(t)$

15. As in Exercise 13, $Q_c(t)=e^{-10t}(c_1\cos 20t+c_2\sin 20t)$ but $E(t)=12\sin 10t$ so try $Q_p(t)=A\cos 10t+B\sin 10t$. Substituting into the differential equation gives $(-100A+200B+500A)\cos 10t+(-100B-200A+500B)\sin 10t=12\sin 10t\Rightarrow 400A+200B=0$ and

$400B-200A=12$. Thus $A=-\frac{3}{250}$, $B=\frac{3}{125}$ and the general solution is

$Q(t)=e^{-10t}(c_1\cos 20t+c_2\sin 20t)-\frac{3}{250}\cos 10t+\frac{3}{125}\sin 10t$. But $0=Q(0)=c_1-\frac{3}{250}$ so $c_1=\frac{3}{250}$. Also

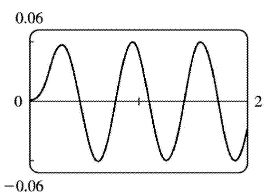
$Q'(t)=\frac{3}{25}\sin 10t+\frac{6}{25}\cos 10t+e^{-10t}\left[(-10c_1+20c_2)\cos 20t+(-10c_2-20c_1)\sin 20t\right]$ and

$0=Q'(0)=\frac{6}{25}-10c_1+20c_2$ so $c_2=-\frac{3}{500}$. Hence the charge is given by

$Q(t)=e^{-10t}\left[\frac{3}{250}\cos 20t-\frac{3}{500}\sin 20t\right]-\frac{3}{250}\cos 10t+\frac{3}{125}\sin 10t$.

16. **(a)** As in Exercise 14, $Q_c(t)=e^{-6t}(c_1\cos 8t+c_2\sin 8t)$ but try $Q_p(t)=A\cos 10t+B\sin 10t$.
Substituting into the differential equation gives
$(-200A+240B+200A)\cos 10t+(-200B-240A+200B)\sin 10t=12\sin 10t$, so $B=0$ and
$A=-\frac{1}{20}$. Hence, the general solution is $Q(t)=e^{-6t}(c_1\cos 8t+c_2\sin 8t)-\frac{1}{20}\cos 10t$. But
$0.001=Q(0)=c_1-\frac{1}{20}$, $Q'(t)=e^{-6t}\left[(-6c_1+8c_2)\cos 8t+(-6c_2-8c_1)\sin 8t\right]-\frac{1}{2}\sin 10t$ and
$0=Q'(0)=-6c_1+8c_2$, so $c_1=0.051$ and $c_2=0.03825$. Thus the charge is given by

$Q(t)=e^{-6t}(0.051\cos 8t+0.03825\sin 8t)-\frac{1}{20}\cos 10t$.

(b) 

17. $x(t)=A\cos(\omega t+\delta)\Leftrightarrow x(t)=A[\cos\omega t\cos\delta-\sin\omega t\sin\delta]\Leftrightarrow x(t)=A\left(\frac{c_1}{A}\cos\omega t+\frac{c_2}{A}\sin\omega t\right)$ where
$\cos\delta=c_1/A$ and $\sin\delta=-c_2/A\Leftrightarrow x(t)=c_1\cos\omega t+c_2\sin\omega t$. (Note that $\cos^2\delta+\sin^2\delta=1\Rightarrow c_1^2+c_2^2=A^2$.)

18. **(a)** We approximate $\sin\theta$ by θ and, with $L=1$ and $g=9.8$, the differential equation becomes
$\frac{d^2\theta}{dt^2}+9.8\theta=0$. The auxiliary equation is $r^2+9.8=0\Rightarrow r=\pm\sqrt{9.8}\,i$, so the general solution is

$\theta(t)=c_1\cos\left(\sqrt{9.8}\,t\right)+c_2\sin\left(\sqrt{9.8}\,t\right)$. Then $0.2=\theta(0)=c_1$ and $1=\theta'(0)=\sqrt{9.8}\,c_2\Rightarrow c_2=\frac{1}{\sqrt{9.8}}$, so the equation is $\theta(t)=0.2\cos\left(\sqrt{9.8}\,t\right)+\dfrac{1}{\sqrt{9.8}}\sin\left(\sqrt{9.8}\,t\right)$.

(b) $\theta'(t)=-0.2\sqrt{9.8}\sin\left(\sqrt{9.8}\,t\right)+\cos\left(\sqrt{9.8}\,t\right)=0$ or $\tan\left(\sqrt{9.8}\,t\right)=\dfrac{5}{\sqrt{9.8}}$, so the critical numbers are $t=\dfrac{1}{\sqrt{9.8}}\tan^{-1}\left(\dfrac{5}{\sqrt{9.8}}\right)+\dfrac{n}{\sqrt{9.8}}\pi$ (n any integer). The maximum angle from the vertical is $\theta\left(\dfrac{1}{\sqrt{9.8}}\tan^{-1}\left(\dfrac{5}{\sqrt{9.8}}\right)\right)\approx 0.377$ radians (or about 21.7°).

(c) From part (b), the critical numbers of $\theta(t)$ are spaced $\dfrac{\pi}{\sqrt{9.8}}$ apart, and the time between successive maximum values is $2\left(\dfrac{\pi}{\sqrt{9.8}}\right)$. Thus the period of the pendulum is $\dfrac{2\pi}{\sqrt{9.8}}\approx 2.007$ seconds.

(d) $\theta(t)=0\Rightarrow 0.2\cos\left(\sqrt{9.8}\,t\right)+\dfrac{1}{\sqrt{9.8}}\sin\left(\sqrt{9.8}\,t\right)=0\Rightarrow\tan\left(\sqrt{9.8}\,t\right)=-0.2\sqrt{9.8}\Rightarrow$
$t=\dfrac{1}{\sqrt{9.8}}\left[\tan^{-1}\left(-0.2\sqrt{9.8}\right)+\pi\right]\approx 0.825$ seconds.

(e) $\theta'(0.825)\approx -1.180$ rad/s.

1. Let $y(x)=\sum_{n=0}^{\infty} c_n x^n$. Then $y'(x)=\sum_{n=1}^{\infty} nc_n x^{n-1}$ and the given equation, $y'-y=0$, becomes $\sum_{n=1}^{\infty} nc_n x^{n-1}-\sum_{n=0}^{\infty} c_n x^n=0$. Replacing n by $n+1$ in the first sum gives $\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n-\sum_{n=0}^{\infty} c_n x^n=0$, so $\sum_{n=0}^{\infty}\left[(n+1)c_{n+1}-c_n\right]x^n=0$. Equating coefficients gives $(n+1)c_{n+1}-c_n=0$, so the recursion relation is $c_{n+1}=\frac{c_n}{n+1}$, $n=0,1,2,\ldots$. Then $c_1=c_0$, $c_2=\frac{1}{2}c_1=\frac{c_0}{2}$, $c_3=\frac{1}{3}c_2=\frac{1}{3}\cdot\frac{1}{2}c_0=\frac{c_0}{3!}$, $c_4=\frac{1}{4}c_3=\frac{c_0}{4!}$, and in general, $c_n=\frac{c_0}{n!}$. Thus, the solution is

$$y(x)=\sum_{n=0}^{\infty} c_n x^n=\sum_{n=0}^{\infty}\frac{c_0}{n!}x^n=c_0\sum_{n=0}^{\infty}\frac{x^n}{n!}=c_0e^x$$

2. Let $y(x)=\sum_{n=0}^{\infty} c_n x^n$. Then $y'=xy\Rightarrow y'-xy=0\Rightarrow \sum_{n=1}^{\infty} nc_n x^{n-1}-x\sum_{n=0}^{\infty} c_n x^n=0$ or $\sum_{n=1}^{\infty} nc_n x^{n-1}-\sum_{n=0}^{\infty} c_n x^{n+1}=0$. Replacing n with $n+1$ in the first sum and n with $n-1$ in the second gives $\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n-\sum_{n=1}^{\infty} c_{n-1}x^n=0$ or $c_1+\sum_{n=1}^{\infty}(n+1)c_{n+1}x^n-\sum_{n=1}^{\infty} c_{n-1}x^n=0$. Thus, $c_1+\sum_{n=1}^{\infty}\left[(n+1)c_{n+1}-c_{n-1}\right]x^n=0$. Equating coefficients gives $c_1=0$ and $(n+1)c_{n+1}-c_{n-1}=0$. Thus, the recursion relation is $c_{n+1}=\frac{c_{n-1}}{n+1}$, $n=1,2$, … . But $c_1=0$, so $c_3=0$ and $c_5=0$ and in general $c_{2n+1}=0$. Also, $c_2=\frac{c_0}{2}$, $c_4=\frac{c_2}{4}=\frac{c_0}{4\cdot 2}=\frac{c_0}{2^2\cdot 2!}$, $c_6=\frac{c_4}{6}=\frac{c_0}{6\cdot 4\cdot 2}=\frac{c_0}{2^3\cdot 3!}$ and in general $c_{2n}=\frac{c_0}{2^n\cdot n!}$. Thus, the solution is

$$y(x)=\sum_{n=0}^{\infty} c_n x^n=\sum_{n=0}^{\infty} c_{2n}x^{2n}=\sum_{n=0}^{\infty}\frac{c_0}{2^n\cdot n!}x^{2n}=c_0\sum_{n=0}^{\infty}\frac{\left(x^2/2\right)^n}{n!}=c_0e^{x^2/2}$$

3. Assuming $y(x)=\sum_{n=0}^{\infty} c_n x^n$, we have $y'(x)=\sum_{n=1}^{\infty} nc_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n$ and $-x^2y=-\sum_{n=0}^{\infty} c_n x^{n+2}=-\sum_{n=2}^{\infty} c_{n-2}x^n$. Hence, the equation $y'=x^2y$ becomes $\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n-\sum_{n=2}^{\infty} c_{n-2}x^n=0$ or $c_1+2c_2x+\sum_{n=2}^{\infty}\left[(n+1)c_{n+1}-c_{n-2}\right]x^n=0$. Equating coefficients gives $c_1=c_2=0$ and

$c_{n+1}=\dfrac{c_{n-2}}{n+1}$ for $n=2,3$, … . But $c_1=0$, so $c_4=0$ and $c_7=0$ and in general $c_{3n+1}=0$. Similarly $c_2=0$ so $c_{3n+2}=0$. Finally $c_3=\dfrac{c_0}{3}$, $c_6=\dfrac{c_3}{6}=\dfrac{c_0}{6\cdot 3}=\dfrac{c_0}{3^2\cdot 2!}$, $c_9=\dfrac{c_6}{9}=\dfrac{c_0}{9\cdot 6\cdot 3}=\dfrac{c_0}{3^3\cdot 3!}$, … , and $c_{3n}=\dfrac{c_0}{3^n\cdot n!}$.
Thus, the solution is

$$y(x)=\Sigma_{n=0}^{\infty}c_nx^n=\Sigma_{n=0}^{\infty}c_{3n}x^{3n}=\Sigma_{n=0}^{\infty}\frac{c_0}{3^n\cdot n!}x^{3n}=c_0\Sigma_{n=0}^{\infty}\frac{x^{3n}}{3^n\text{n!}}=c_0\Sigma_{n=0}^{\infty}\frac{\left(x^3/3\right)^n}{n!}=c_0e^{x^3/3}$$

4. Let $y(x)=\Sigma_{n=0}^{\infty}c_nx^n\Rightarrow y'(x)=\Sigma_{n=1}^{\infty}nc_nx^{n-1}=\Sigma_{n=0}^{\infty}(n+1)c_{n+1}x^n$. Then the differential equation becomes $(x-3)\Sigma_{n=0}^{\infty}(n+1)c_{n+1}x^n+2\Sigma_{n=0}^{\infty}c_nx^n=0\Rightarrow$
$\Sigma_{n=0}^{\infty}(n+1)c_{n+1}x^{n+1}-3\Sigma_{n=0}^{\infty}(n+1)c_{n+1}x^n+2\Sigma_{n=0}^{\infty}c_nx^n=0\Rightarrow\Sigma_{n=1}^{\infty}nc_nx^n-\Sigma_{n=0}^{\infty}3(n+1)c_{n+1}x^n+\Sigma_{n=0}^{\infty}2c_nx^n=0$
$\Rightarrow\Sigma_{n=0}^{\infty}\left[(n+2)c_n-3(n+1)c_{n+1}\right]x^n=0\left(\text{since}\Sigma_{n=1}^{\infty}nc_nx^n=\Sigma_{n=0}^{\infty}nc_nx^n\right)$. Equating coefficients gives
$(n+2)c_n-3(n+1)c_{n+1}=0$, thus the recursion relation is $c_{n+1}=\dfrac{(n+2)c_n}{3(n+1)}$, $n=0,1,2,\ldots$. Then $c_1=\dfrac{2c_0}{3}$, $c_2=\dfrac{3c_1}{3(2)}=\dfrac{3c_0}{3^2}$, $c_3=\dfrac{4c_2}{3(3)}=\dfrac{4c_0}{3^3}$, $c_4=\dfrac{5c_3}{3(4)}=\dfrac{5c_0}{3^4}$, and in general, $c_n=\dfrac{(n+1)c_0}{3^n}$. Thus the solution is
$y(x)=\Sigma_{n=0}^{\infty}c_nx^n=c_0\Sigma_{n=0}^{\infty}\dfrac{n+1}{3^n}x^n$. $\left[\text{Note that } c_0\Sigma_{n=0}^{\infty}\dfrac{n+1}{3^n}x^n=\dfrac{9c_0}{(3-x)^2}\text{ for}|x|<3.\right]$

5. Let $y(x)=\Sigma_{n=0}^{\infty}c_nx^n\Rightarrow y'(x)=\Sigma_{n=1}^{\infty}nc_nx^{n-1}$ and $y''(x)=\Sigma_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$. The differential equation becomes $\Sigma_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n+x\Sigma_{n=1}^{\infty}nc_nx^{n-1}+\Sigma_{n=0}^{\infty}c_nx^n=0$ or
$\Sigma_{n=0}^{\infty}\left[(n+2)(n+1)c_{n+2}+nc_n+c_n\right]x^n\left(\text{since}\Sigma_{n=1}^{\infty}nc_nx^n=\Sigma_{n=0}^{\infty}nc_nx^n\right)$. Equating coefficients gives
$(n+2)(n+1)c_{n+2}+(n+1)c_n=0$, thus the recursion relation is $c_{n+2}=\dfrac{-(n+1)c_n}{(n+2)(n+1)}=-\dfrac{c_n}{n+2}$, $n=0,1,2,\ldots$.
Then the even coefficients are given by $c_2=-\dfrac{c_0}{2}$, $c_4=-\dfrac{c_2}{4}=\dfrac{c_0}{2\cdot 4}$, $c_6=-\dfrac{c_4}{6}=-\dfrac{c_0}{2\cdot 4\cdot 6}$, and in general,

$c_{2n}=(-1)^n \dfrac{c_0}{2\cdot 4\cdot \cdots \cdot 2n}=\dfrac{(-1)^n c_0}{2^n n!}$. The odd coefficients are $c_3=-\dfrac{c_1}{3}$, $c_5=-\dfrac{c_3}{5}=\dfrac{c_1}{3\cdot 5}$,

$c_7=-\dfrac{c_5}{7}=-\dfrac{c_1}{3\cdot 5\cdot 7}$, and in general, $c_{2n+1}=(-1)^n \dfrac{c_1}{3\cdot 5\cdot 7\cdot \cdots \cdot (2n+1)}=\dfrac{(-2)^n n! c_1}{(2n+1)!}$. The solution is $y(x)=c_0\sum_{n=0}^{\infty} \dfrac{(-1)^n}{2^n n!} x^{2n}+c_1\sum_{n=0}^{\infty} \dfrac{(-2)^n n!}{(2n+1)!} x^{2n+1}$.

6. Let $y(x)=\sum_{n=0}^{\infty} c_n x^n$. Then $y''(x)=\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$. Hence, the equation $y''=y$ becomes $\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n-\sum_{n=0}^{\infty}c_n x^n=0$ or $\sum_{n=0}^{\infty}\left[(n+2)(n+1)c_{n+2}-c_n\right]x^n=0$. So the recursion relation is $c_{n+2}=\dfrac{c_n}{(n+2)(n+1)}$, $n=0,1,\ldots$. Given c_0 and c_1 , $c_2=\dfrac{c_0}{2\cdot 1}$, $c_4=\dfrac{c_2}{4\cdot 3}=\dfrac{c_0}{4!}$, $c_6=\dfrac{c_4}{6\cdot 5}=\dfrac{c_0}{6!}$, … , $c_{2n}=\dfrac{c_0}{(2n)!}$ and $c_3=\dfrac{c_1}{3\cdot 2}$, $c_5=\dfrac{c_3}{5\cdot 4}=\dfrac{c_1}{5\cdot 4\cdot 3\cdot 2}=\dfrac{c_1}{5!}$, $c_7=\dfrac{c_5}{7\cdot 6}=\dfrac{c_1}{7!}$, … , $c_{2n+1}=\dfrac{c_1}{(2n+1)!}$. Thus, the solution is

$$y(x)=\sum_{n=0}^{\infty}c_n x^n=\sum_{n=0}^{\infty}c_{2n}x^{2n}+\sum_{n=0}^{\infty}c_{2n+1}x^{2n+1}=c_0\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}+c_1\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

The solution can be written as
$y(x)=c_0\cosh x+c_1\sinh x\left[\text{ or } y(x)=c_0\dfrac{e^x+e^{-x}}{2}+c_1\dfrac{e^x-e^{-x}}{2}=\dfrac{c_0+c_1}{2}e^x+\dfrac{c_0-c_1}{2}e^{-x}\right]$.

7. Let $y(x)=\sum_{n=0}^{\infty}c_n x^n$. Then $y''=\sum_{n=0}^{\infty}n(n-1)c_n x^{n-2}$, $xy'=\sum_{n=0}^{\infty}nc_n x^n$ and $\left(x^2+1\right)y''=\sum_{n=0}^{\infty}n(n-1)c_n x^n+\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$. The differential equation becomes $\sum_{n=0}^{\infty}\left[(n+2)(n+1)c_{n+2}+[n(n-1)+n-1]c_n\right]x^n=0$. The recursion relation is $c_{n+2}=-\dfrac{(n-1)c_n}{n+2}$, $n=0,1,2,\ldots$. Given c_0 and c_1 , $c_2=\dfrac{c_0}{2}$, $c_4=-\dfrac{c_2}{4}=-\dfrac{c_0}{2^2\cdot 2!}$, $c_6=-\dfrac{3c_4}{6}=(-1)^2\dfrac{3c_0}{2^3\cdot 3!}$, … ,

$c_{2n}=(-1)^{n-1}\dfrac{1\cdot 3\cdot\cdots\cdot(2n-3)\,c_0}{2^n\text{n}!}=(-1)^{n-1}\dfrac{(2n-3)!\,c_0}{2^n2^{n-2}\text{n}!(n-2)!}=(-1)^{n-1}\dfrac{(2n-3)!\,c_0}{2^{2n-2}\text{n}!(n-2)!}$ for $n=2,3,\ldots$.

$c_3=\dfrac{0\cdot c_1}{3}=0\Rightarrow c_{2n+1}=0$ for $n=1,2,\ldots$. Thus the solution is

$y(x)=c_0+c_1x+c_0\dfrac{x^2}{2}+c_0\sum_{n=2}^{\infty}\dfrac{(-1)^{n-1}(2n-3)!}{2^{2n-2}\text{n}!(n-2)!}x^{2n}$.

8. Assuming $y(x)=\sum_{n=0}^{\infty}c_nx^n$, $y''(x)=\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$ and $-xy(x)=-\sum_{n=0}^{\infty}c_nx^{n+1}=-\sum_{n=1}^{\infty}c_{n-1}x^n$. The equation $y''=xy$ becomes $\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n-\sum_{n=1}^{\infty}c_{n-1}x^n=0$ or $2c_2+\sum_{n=1}^{\infty}\left[(n+2)(n+1)c_{n+2}-c_{n-1}\right]x^n=0$. Equating coefficients gives $c_2=0$ and $c_{n+2}=\dfrac{c_{n-1}}{(n+2)(n+1)}$ for $n=1,2$, … . Since $c_2=0$, $c_{3n+2}=0$ for $n=0,1,2,\ldots$. Given c_0 , $c_3=\dfrac{c_0}{3\cdot 2}$, $c_6=\dfrac{c_3}{6\cdot 5}=\dfrac{c_0}{6\cdot 5\cdot 3\cdot 2}$, … , $c_{3n}=\dfrac{c_0}{3n(3n-1)(3n-3)(3n-4)\cdot\cdots\cdot 6\cdot 5\cdot 3\cdot 2}$. Given c_1 , $c_4=\dfrac{c_1}{4\cdot 3}$, $c_7=\dfrac{c_4}{7\cdot 6}=\dfrac{c_1}{7\cdot 6\cdot 4\cdot 3}$, … , $c_{3n+1}=\dfrac{c_1}{(3n+1)3n(3n-2)(3n-3)\ldots 7\cdot 6\cdot 4\cdot 3}$. The solution can be written as

$$y(x)=c_0\sum_{n=0}^{\infty}\frac{(3n-2)(3n-5)\cdot\cdots\cdot 7\cdot 4\cdot 1}{(3n)!}x^{3n}+c_1\sum_{n=0}^{\infty}\frac{(3n-1)(3n-4)\cdot\cdots\cdot 8\cdot 5\cdot 2}{(3n+1)!}x^{3n+1}$$

9. Let $y(x)=\sum_{n=0}^{\infty}c_nx^n$. Then $-xy'(x)=-x\sum_{n=1}^{\infty}nc_nx^{n-1}=-\sum_{n=1}^{\infty}nc_nx^n=-\sum_{n=0}^{\infty}nc_nx^n$, $y''(x)=\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$, and the equation $y''-xy'-y=0$ becomes $\sum_{n=0}^{\infty}\left[(n+2)(n+1)c_{n+2}-nc_n-c_n\right]x^n=0$. Thus, the recursion relation is

$c_{n+2}=\dfrac{nc_n+c_n}{(n+2)(n+1)}=\dfrac{c_n(n+1)}{(n+2)(n+1)}=\dfrac{c_n}{n+2}$ for $n=0,1,2$, … . One of the given conditions is

$y(0)=1$. But $y(0)=\sum_{n=0}^{\infty}c_n(0)^n=c_0+0+0+\cdots=c_0$, so $c_0=1$. Hence, $c_2=\dfrac{c_0}{2}=\dfrac{1}{2}$, $c_4=\dfrac{c_2}{4}=\dfrac{1}{2\cdot 4}$,

$c_6=\dfrac{c_4}{6}=\dfrac{1}{2\cdot 4\cdot 6}$, … , $c_{2n}=\dfrac{1}{2^n n!}$. The other given condition is $y'(0)=0$. But

$y'(0)=\sum_{n=1}^{\infty} nc_n(0)^{n-1}=c_1+0+0+\cdots=c_1$, so $c_1=0$. By the recursion relation, $c_3=\frac{c_1}{3}=0$, $c_5=0$, … , $c_{2n+1}=0$ for $n=0$, 1 , 2 , … . Thus, the solution to the initial–value problem is

$$y(x)=\sum_{n=0}^{\infty} c_n x^n=\sum_{n=0}^{\infty} c_{2n} x^{2n}=\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}=\sum_{n=0}^{\infty} \frac{\left(x^2/2\right)^n}{n!}=e^{x^2/2}$$

10. Assuming that $y(x)=\sum_{n=0}^{\infty} c_n x^n$, we have $x^2y=\sum_{n=0}^{\infty} c_n x^{n+2}$ and

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}=\sum_{n=-2}^{\infty}(n+4)(n+3)c_{n+4}x^{n+2}=2c_2+6c_3x+\sum_{n=0}^{\infty}(n+4)(n+3)c_{n+4}x^{n+2}$$

Thus, the equation $y''+x^2y=0$ becomes $2c_2+6c_3x+\sum_{n=0}^{\infty}\left[(n+4)(n+3)c_{n+4}+c_n\right]x^{n+2}=0$. So $c_2=c_3=0$ and the recursion relation is $c_{n+4}=-\frac{c_n}{(n+4)(n+3)}$, $n=0,1,2$, … .
But $c_1=y'(0)=0=c_2=c_3$ and by the recursion relation, $c_{4n+1}=c_{4n+2}=c_{4n+3}=0$ for $n=0,1,2$, … .
Also, $c_0=y(0)=1$, so
$c_4=-\frac{c_0}{4\cdot 3}=-\frac{1}{4\cdot 3}$, $c_8=-\frac{c_4}{8\cdot 7}=\frac{(-1)^2}{8\cdot 7\cdot 4\cdot 3}$, … , $c_{4n}=\frac{(-1)^n}{4n(4n-1)(4n-4)(4n-5)\cdot\cdots\cdot 4\cdot 3}$.
Thus, the solution to the initial–value problem is

$$y(x)=\sum_{n=0}^{\infty} c_n x^n=c_0+\sum_{n=0}^{\infty} c_{4n}x^{4n}=1+\sum_{n=1}^{\infty}(-1)^n\frac{x^{4n}}{4n(4n-1)(4n-4)(4n-5)\cdot\cdots\cdot 4\cdot 3}$$

11. Assuming that $y(x)=\sum_{n=0}^{\infty} c_n x^n$, we have $xy=x\sum_{n=0}^{\infty} c_n x^n=\sum_{n=0}^{\infty} c_n x^{n+1}$,
$x^2y'=x^2\sum_{n=1}^{\infty} nc_n x^{n-1}=\sum_{n=0}^{\infty} nc_n x^{n+1}$,

$y''(x)=\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}=\sum_{n=-1}^{\infty}(n+3)(n+2)c_{n+3}x^{n+1}$ [replace n with $n+3$] =
$2c_2+\sum_{n=0}^{\infty}(n+3)(n+2)c_{n+3}x^{n+1}$,
and the equation $y''+x^2y'+xy=0$ becomes $2c_2+\sum_{n=0}^{\infty}\left[(n+3)(n+2)c_{n+3}+nc_n+c_n\right]x^{n+1}=0$.
So

$c_2=0$ and the recursion relation is $c_{n+3}=\dfrac{-nc_n-c_n}{(n+3)(n+2)}=-\dfrac{(n+1)c_n}{(n+3)(n+2)}$, $n=0,1,2$, … .
But $c_0=y(0)=0=c_2$ and by the recursion relation, $c_{3n}=c_{3n+2}=0$ for $n=0$, 1 , 2 , … .
Also, $c_1=y'(0)=1$, so

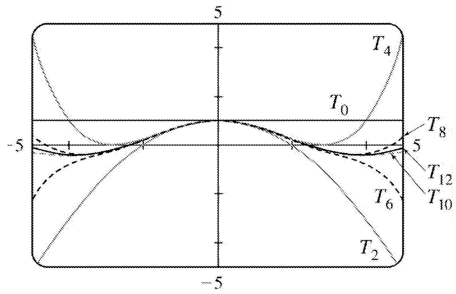
$c_4=-\dfrac{2c_1}{4\cdot 3}=-\dfrac{2}{4\cdot 3}$, $c_7=-\dfrac{5c_4}{7\cdot 6}=(-1)^2\dfrac{2\cdot 5}{7\cdot 6\cdot 4\cdot 3}=(-1)^2\dfrac{2^2 5^2}{7!}$, … , $c_{3n+1}=(-1)^n\dfrac{2^2 5^2\cdot\cdots\cdot(3n-1)^2}{(3n+1)!}$.
Thus, the solution is

$$y(x)=\Sigma_{n=0}^{\infty}c_n x^n=x+\Sigma_{n=1}^{\infty}\left[(-1)^n\frac{2^2 5^2\cdot\cdots\cdot(3n-1)^2 x^{3n+1}}{(3n+1)!}\right]$$

12. **(a)** Let $y(x)=\Sigma_{n=0}^{\infty}c_n x^n$. Then $x^2y''(x)=\Sigma_{n=2}^{\infty}n(n-1)c_n x^n=\Sigma_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^{n+2}$,
$xy'(x)=\Sigma_{n=1}^{\infty}nc_n x^n=\Sigma_{n=-1}^{\infty}(n+2)c_{n+2}x^{n+2}=c_1x+\Sigma_{n=0}^{\infty}(n+2)c_{n+2}x^{n+2}$, and the equation
$x^2y''+xy'+x^2y=0$ becomes $c_1x+\Sigma_{n=0}^{\infty}\left\{[(n+2)(n+1)+(n+2)]c_{n+2}+c_n\right\}x^{n+2}=0$. So $c_1=0$ and the recursion relation is $c_{n+2}=-\dfrac{c_n}{(n+2)^2}$, $n=0,1,2$, … . But $c_1=y'(0)=0$ so $c_{2n+1}=0$ for $n=0,1,2,\ldots$. Also,
$c_0=y(0)=1$, so $c_2=-\dfrac{1}{2^2}$, $c_4=-\dfrac{c_2}{4^2}=(-1)^2\dfrac{1}{4^2 2^2}=(-1)^2\dfrac{1}{2^4(2!)^2}$, $c_6=-\dfrac{c_4}{6^2}=(-1)^3\dfrac{1}{2^6(3!)^2}$, … ,
$c_{2n}=(-1)^n\dfrac{1}{2^{2n}(n!)^2}$. The solution is

$$y(x)=\Sigma_{n=0}^{\infty}c_n x^n=\Sigma_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^{2n}(n!)^2}$$

(b) The Taylor polynomials T_0 to T_{12} are shown in the graph. Because T_{10} and T_{12} are close together throughout the interval $[-5,5]$, it is reasonable to assume that T_{12} is a good approximation to the Bessel function on that interval.


5
-5
5
-5
T_4
T_0
T_8
T_{12}
T_{10}
T_6
T_2